

Variational Methods & Optimal Control

lecture 09

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Extension 1: higher-order derivatives

When f includes higher-order derivatives then the E-L equations can be extended, e.g., if the function includes a y'' term, i.e., $f(x, y, y', y'')$, then

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 0$$

but now we now need extra edge conditions. A simple example we will consider is the shape of a bent bar.

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Extensions

Now we consider extensions to the simple E-L equations presented so far:

- ▶ when f includes higher-order derivatives, e.g., $f(x, y, y', y'')$, e.g., the shape of a bent bar.
- ▶ when there are several dependent variables (i.e., y is a vector), e.g., calculating a particles trajectory.
- ▶ when there are several independent variables (i.e., x is a vector), e.g. calculating extremal surface.

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Standard Euler-Lagrange equation

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \rightarrow \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y') dx,$$

where f has continuous partial derivatives of second order with respect to x , y , and y' , and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_0 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F , then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0}$$

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Higher-order derivatives

Let $F : C^2[x_0, x_1] \rightarrow \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', y'') dx,$$

where f has continuous partial derivatives of second order with respect to $x, y, y',$ and y'' , and $x_0 < x_1$. As before, the necessary condition for the extremum is that the first variation be zero, e.g.

$$\delta F(\eta, y) = 0$$

Taylor's theorem

As before we perturb y to get $\hat{y} = y + \varepsilon\eta$
Once again we apply Taylor's theorem to derive

$$\begin{aligned} f(x, y + \varepsilon\eta, y' + \varepsilon\eta', y'' + \varepsilon\eta'') &= \\ f(x, y, y', y'') + \varepsilon \left[\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} + \eta'' \frac{\partial f}{\partial y''} \right] + O(\varepsilon^2) \end{aligned}$$

and hence that

$$F\{y + \varepsilon\eta\} = \int_{x_0}^{x_1} f(x, y, y', y'') + \varepsilon \left[\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} + \eta'' \frac{\partial f}{\partial y''} \right] dx + O(\varepsilon^2)$$

First Variation

So, now the first variation will be given by

$$\begin{aligned} \delta F(\eta, y) &= \lim_{\varepsilon \rightarrow 0} \frac{F\{y + \varepsilon\eta\} - F\{y\}}{\varepsilon} \\ &= \int_{x_0}^{x_1} \left[\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} + \eta'' \frac{\partial f}{\partial y''} \right] dx \\ &= \left[\eta \frac{\partial f}{\partial y'} \right]_{x_0}^{x_1} + \left[\eta' \frac{\partial f}{\partial y''} \right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \left[\eta \frac{\partial f}{\partial y} - \eta \frac{d}{dx} \frac{\partial f}{\partial y'} - \eta' \frac{d}{dx} \frac{\partial f}{\partial y''} \right] dx \\ &= \left[\eta \frac{\partial f}{\partial y'} \right]_{x_0}^{x_1} + \left[\eta' \frac{\partial f}{\partial y''} \right]_{x_0}^{x_1} - \left[\eta \frac{d}{dx} \frac{\partial f}{\partial y''} \right]_{x_0}^{x_1} \\ &\quad + \int_{x_0}^{x_1} \left[\eta \frac{\partial f}{\partial y} - \eta \frac{d}{dx} \frac{\partial f}{\partial y'} + \eta \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} \right] dx \end{aligned}$$

New boundary conditions

We require new fixed-end point conditions

$$\begin{aligned} y(x_0) &= y_0 & y(x_1) &= y_1 \\ y'(x_0) &= y'_0 & y'(x_1) &= y'_1 \end{aligned}$$

which implies that

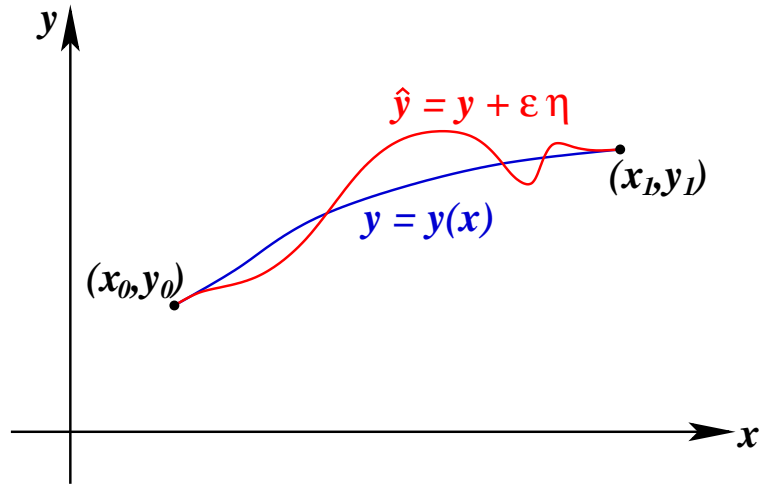
$$\begin{aligned} \eta(x_0) &= 0 & \eta(x_1) &= 0 \\ \eta'(x_0) &= 0 & \eta'(x_1) &= 0 \end{aligned}$$

Which gives

$$\delta F(\eta, y) = \int_{x_0}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} \right] dx$$

Fixing the end-points

We now fix the derivative and value of y at the end points.



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4th Order Euler-Lagrange equation

$\delta F(\eta, y) = 0$ for arbitrary η satisfying the boundary conditions, so the result is the 4th order Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 0$$

This is a 4th order differential equation.

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Generalization

Let $F : C^2[x_0, x_1] \rightarrow \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', \dots, y^{(n)}) dx,$$

where f has continuous partial derivatives of second order with respect to $x, y, y', \dots, y^{(n)}$, and $x_0 < x_1$, and the values of $y, y', \dots, y^{(n-1)}$ are fixed at the end-points, then the extremals satisfy the condition

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} + \dots + (-1)^n \frac{d^n}{dx^n} \frac{\partial f}{\partial y^{(n)}} = 0$$

This is sometimes called the **Euler-Poisson Equation**.

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Example 1

$$F\{y\} = \int_0^1 (1 + y'^2) dx$$

subject to $y(0) = 0, y(1) = 1, y'(0) = 1, y'(1) = 1$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

$$\frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = \frac{d^2}{dx^2} 2y'' = 2 \frac{d^4 y}{dx^4}$$

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Example 1 (cont)

The E-P equation gives

$$\frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 2 \frac{d^4 y}{dx^4} = 0$$

The solution is

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

Given the end-points

$$y(0) = 0 \Rightarrow c_1 = 0$$

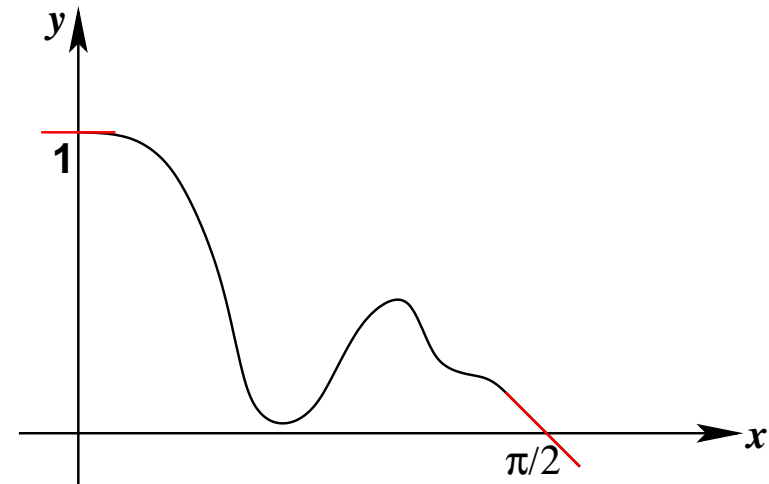
$$y'(0) = 1 \Rightarrow c_2 = 1$$

$$y(1) = 1 \Rightarrow c_2 + c_3 + c_4 = 1$$

$$y'(1) = 1 \Rightarrow c_2 + 2c_3 + 3c_4 = 1$$

Final solution is $y(x) = x$

Example 2 (cont)



Example 2

$$F\{y\} = \int_0^{\pi/2} (y''^2 - y^2 + x^2) dx$$

subject to $y(0) = 1, y(\pi/2) = 0, y'(0) = 0, y'(\pi/2) = -1$

$$\frac{\partial f}{\partial y} = -2y$$

$$\frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

$$\frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 2 \frac{d^4 y}{dx^4}$$

Notice the x^2 doesn't influence the form of extremal!

Example 2 (cont)

The E-P equation gives

$$\frac{\partial f}{\partial y} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = -2y + 2 \frac{d^4 y}{dx^4} = 0$$

The solution is

$$y(x) = Ae^x + Be^{-x} + C \sin x + D \cos x$$

Given the end-points

$$y(0) = 1 \Rightarrow A + B + D = 1$$

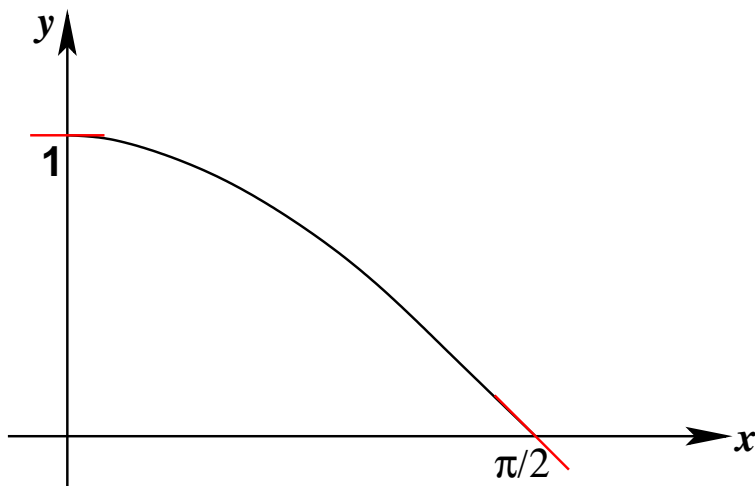
$$y'(0) = 0 \Rightarrow A - B + C = 0$$

$$y(\pi/2) = 0 \Rightarrow Ae^{\pi/2} + Be^{-\pi/2} + C = 0$$

$$y'(\pi/2) = -1 \Rightarrow Ae^{\pi/2} - Be^{-\pi/2} - D = -1$$

Example 2 (solution)

$$y(x) = \cos(x)$$



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Example 3

Let $y : [0, d] \rightarrow \mathbb{R}$ describe the shape of the beam, and $\rho : [0, d] \rightarrow \mathbb{R}$ be the load per unit length on the beam.

For a bent elastic beam the potential energy from elastic forces is

$$V_1 = \frac{\kappa}{2} \int_0^d y'^2 dx, \quad \kappa = \text{flexural rigidity}$$

The potential energy is

$$V_2 = - \int_0^d \rho(x)y(x) dx$$

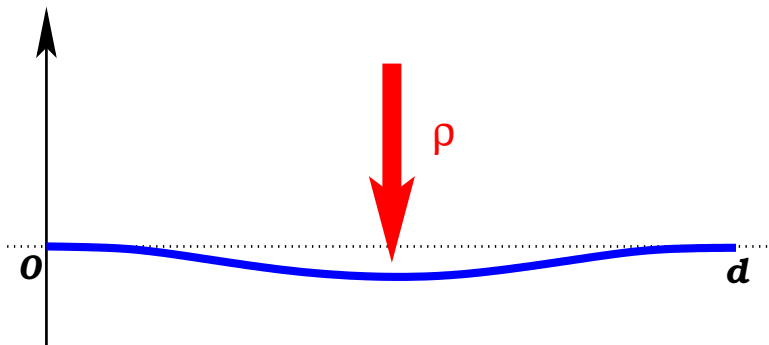
Thus the total potential energy is

$$V = \int_0^d \frac{\kappa y'^2}{2} - \rho(x)y(x) dx$$

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Example 3

Bent elastic beam.



Two end-points are fixed, and clamped so that they are level, e.g.

$$y(0) = 0, \quad y'(0) = 0, \quad \text{and} \quad y(d) = 0 \quad \text{and} \quad y'(d) = 0.$$

The load (per unit length) on the beam is given by a function $\rho(x)$.

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Example 3

The Euler-Poisson equation is

$$\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} &= 0 \\ -\rho(x) + \kappa y^{(4)} &= 0 \\ y^{(4)} &= \frac{\rho(x)}{\kappa} \end{aligned}$$

This DE has solution

$$y(x) = P(x) + c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

where the c_k 's are the constants of integration, and $P(x)$ is a particular solution to $P^{(4)}(x) = \rho(x)/\kappa$.

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Example 3: uniform load

If the beam is uniformly loaded, then $\rho(x) = \rho$ and so

$$y(x) = \frac{\rho x^4}{4! \kappa} + c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

The end-conditions imply

$$y(0) = 0 \Rightarrow c_0 = 0$$

$$y'(0) = 0 \Rightarrow c_1 = 0$$

$$y(d) = 0 \Rightarrow \frac{\rho d^4}{4! \kappa} + c_0 + c_1 d + c_2 d^2 + c_3 d^3 = 0$$

$$y'(d) = 0 \Rightarrow \frac{\rho d^3}{3! \kappa} + c_1 + 2c_2 d + 3c_3 d^2 = 0$$

Example 3: uniform load

$$\tilde{y}(x) = -\frac{\rho(d-x)^2 x^2}{24 \kappa}$$

Maximum displacement occurs at $x = d/2$, and is given by

$$\tilde{y}(d/2) = -\frac{\rho d^4}{384 \kappa}$$

Contrast this with the catenary.

$$\tilde{y}(x) = c_1 \cosh\left(\frac{x - c_2}{c_1}\right)$$

where c_1 and c_2 are determined by the end-points (there are no physical values such as m or g in the solution).

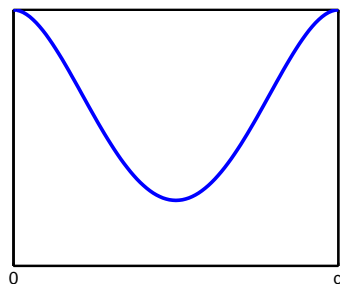
Example 3: uniform load

Choose a solution of the form

$$y(x) = \frac{\rho(d-x)^2 x^2}{24 \kappa}$$

Then the derivative

$$y'(x) = \frac{2\rho(d-x)x^2}{12\kappa} + \frac{\rho(d-x)^2 x}{12\kappa}$$



We can see that the constraints are satisfied

$$y(0) = 0$$

$$y'(0) = 0$$

$$y(d) = 0$$

$$y'(d) = 0$$