

Variational Methods & Optimal Control

lecture 07

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Special case 3

When f has no explicit dependence on y the E-L equations simplify to give

$$\frac{\partial f}{\partial y'} = \text{const}$$

An example where we might use this is in calculating geodesics on non-planar objects such as the sphere.

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Euler-Lagrange equation

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \rightarrow \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y') dx,$$

where f has continuous partial derivatives of second order with respect to x , y , and y' , and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_0 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F , then for all $x \in [x_0, x_1]$

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0$$

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No explicit y dependence

Suppose the function is of the form

$$J\{y\} = \int_{x_0}^{x_1} f(x, y') dx$$

where y does not appear explicitly.

The Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial y'} = c_1$$

where c_1 is a constant.

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Solving

$\frac{\partial f}{\partial y'}$ is a known function of x and y' ,
so this is a first order DE for y .

In principle for $\frac{\partial^2 f}{\partial y'^2} \neq 0$ can recast

$$\frac{\partial f}{\partial y'} = c_1 \quad \text{as} \quad y' = g(x, c_1)$$

for some g .

Geodesics on the unit sphere

Find the shortest path between two points on the unit sphere.

Spherical co-ordinates

Define

λ = latitude

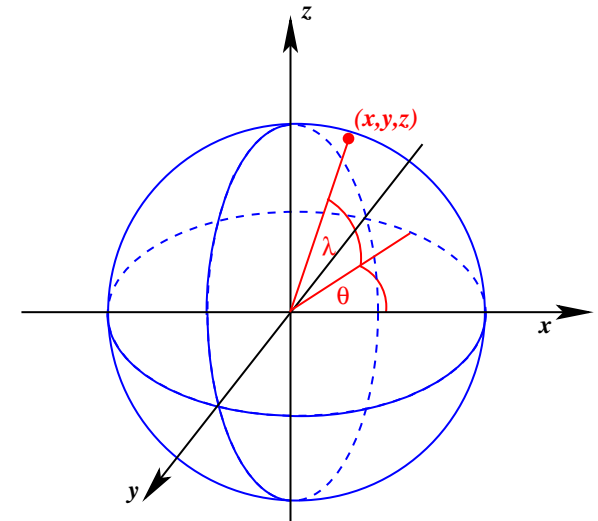
θ = longitude

Cartesian co-ordinates (x, y, z)

$x = \cos(\theta) \cos(\lambda)$

$y = \sin(\theta) \cos(\lambda)$

$z = \sin(\lambda)$



Transformation to spherical co-ord.

$x = \cos(\theta) \cos(\lambda)$

$y = \sin(\theta) \cos(\lambda)$

$z = \sin(\lambda)$

Chain rule

$$dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \lambda} d\lambda = -\sin(\theta) \cos(\lambda) d\theta - \cos(\theta) \sin(\lambda) d\lambda$$

$$dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \lambda} d\lambda = \cos(\theta) \cos(\lambda) d\theta - \sin(\theta) \sin(\lambda) d\lambda$$

$$dz = \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \lambda} d\lambda = \cos(\lambda) d\lambda$$

$$ds^2 = dx^2 + dy^2 + dz^2 = d\lambda^2 + \cos^2(\lambda) d\theta^2$$

Geodesics on the unit sphere

$$\int_{(x(s_0), y(s_0), z(s_0))}^{(x(s_1), y(s_1), z(s_1))} 1 ds = \int_{\lambda_0}^{\lambda_1} \left[1 + \cos^2(\lambda) \left(\frac{d\theta}{d\lambda} \right)^2 \right]^{\frac{1}{2}} d\lambda$$

θ is like y , λ is like x , $\frac{d\theta}{d\lambda} = \theta'$ is like y' , hence EL eqn:

$$\begin{aligned} \frac{\partial}{\partial \theta'} [1 + \cos^2(\lambda) \theta'^2]^{\frac{1}{2}} &= c_1 \\ \frac{\cos^2(\lambda) \theta'}{[1 + \cos^2(\lambda) \theta'^2]^{\frac{1}{2}}} &= c_1 \\ \frac{\cos^4(\lambda) \theta'^2}{1 + \cos^2(\lambda) \theta'^2} &= c_1^2 \end{aligned}$$

Geodesics on the unit sphere

Re-arrange

$$\cos^4(\lambda) \theta'^2 = c_1^2 (1 + \cos^2(\lambda) \theta'^2)$$

Re-arrange some more

$$\begin{aligned} \theta'^2 &= \frac{c_1^2}{\cos^4(\lambda) - c_1^2 \cos^2(\lambda)} \\ \theta' &= \left\{ \frac{c_1^2}{\cos^2(\lambda) (\cos^2(\lambda) - c_1^2)} \right\}^{\frac{1}{2}} \\ \theta' &= g(\lambda, c_1) \end{aligned}$$

Analogous to $y' = g(x, c_1)$.

The constant

$$\frac{\cos^4(\lambda) \theta'^2}{1 + \cos^2(\lambda) \theta'^2} = c_1^2$$

Now

$$\theta'^2 \cos^4(\lambda) \leq \theta'^2 \cos^2(\lambda) \leq 1 + \theta'^2 \cos^2(\lambda)$$

So

$$c_1 \in [-1, 1]$$

So we can replace c_1 with

$$c_1 = \cos(\alpha)$$

Solving the DE

Insert $c_1 = \cos(\alpha)$

$$\begin{aligned} \theta' &= \frac{\cos(\alpha)}{\cos(\lambda) [\cos^2(\lambda) - \cos^2(\alpha)]^{\frac{1}{2}}} \\ \theta &= \int \frac{\cos(\alpha)}{\cos(\lambda) [\cos^2(\lambda) - \cos^2(\alpha)]^{\frac{1}{2}}} d\lambda \\ &= \int \frac{\sec^2(\lambda)}{[\sec^2(\alpha) - \sec^2(\lambda)]^{\frac{1}{2}}} d\lambda \\ &= \int \frac{\sec^2(\lambda)}{[\tan^2(\alpha) - \tan^2(\lambda)]^{\frac{1}{2}}} d\lambda \quad \text{as } \sec^2 x = 1 + \tan^2 x \end{aligned}$$

Solving the DE (part ii)

$$\theta = \int \frac{\sec^2(\lambda)}{[\tan^2(\alpha) - \tan^2(\lambda)]^{\frac{1}{2}}} d\lambda = \frac{1}{\tan(\alpha)} \int \frac{\sec^2(\lambda)}{[1 - \tan^2(\lambda)/\tan^2(\alpha)]^{\frac{1}{2}}} d\lambda$$

Substitute $u = \tan(\lambda)/\tan(\alpha)$

Then $d\lambda = \frac{\tan(\alpha)}{\sec^2(\lambda)} du$

$$\begin{aligned} \theta &= \int \frac{1}{[1 - u^2]^{\frac{1}{2}}} du \\ &= \sin^{-1} \left(\frac{\tan(\lambda)}{\tan(\alpha)} \right) - \beta \end{aligned}$$

$$\text{As } \frac{d}{du} \sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}}$$

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The solution

$$\sin(\theta + \beta) = \frac{\tan(\lambda)}{\tan(\alpha)}$$

Note we can write this

$$\sin(\theta + \beta) = \frac{1}{\tan(\alpha)} \frac{\sin(\lambda)}{\cos(\lambda)}$$

$$\tan(\alpha) \cos(\lambda) \sin(\theta + \beta) = \sin(\lambda)$$

$$\tan(\alpha) \cos(\lambda) [\sin(\theta) \cos(\beta) + \cos(\theta) \sin(\beta)] = \sin(\lambda)$$

Convert back to Cartesian co-ordinates,

$$\tan(\alpha) \sin(\beta)x + \tan(\alpha) \cos(\beta)y = z$$

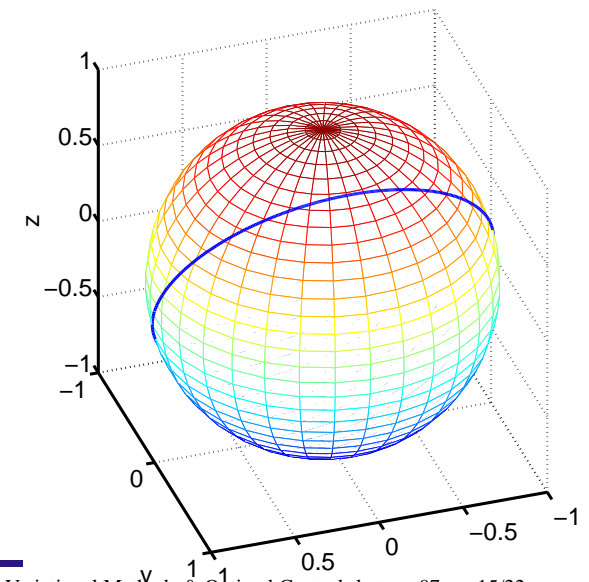
which is the equation of a plane, through the origin.

Hence, solution is a **great circle**, the intersection of plane (through the origin) and the sphere.

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Example

We can find the solution because three points (the origin plus the start and end point of the curve) define a plane, and therefore the solution is the intersection of this plane with the sphere.



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Co-ordinate transformation

More generally, spherical co-ordinates

$$x = r \cos(\theta) \cos(\lambda)$$

$$y = r \sin(\theta) \cos(\lambda)$$

$$z = r \sin(\lambda)$$

And

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = J \begin{pmatrix} d\theta \\ d\lambda \\ dr \end{pmatrix}, \quad J = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \lambda} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \lambda} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \lambda} & \frac{\partial z}{\partial r} \end{pmatrix}$$

Where J is the Jacobian matrix

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Jacobians

If

$$\mathbf{y} = \begin{pmatrix} y_1(\mathbf{x}) \\ y_2(\mathbf{x}) \\ \vdots \\ y_n(\mathbf{x}) \end{pmatrix}$$

Then the Jacobian matrix is

$$J(\mathbf{x}) = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$

The Jacobian determinant

Then the determinant of the Jacobian matrix is also sometimes called the Jacobian

$$|J(\mathbf{x})| = \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right|$$

This gives the ratios of n -dimensional volumes between the two co-ord. systems, i.e.

$$d\mathbf{y} = |J(\mathbf{x})| d\mathbf{x}$$

Transforms and integrals

Substitution in 1D: $y = \phi(x)$

$$\int_{x_0}^{x_1} f(\phi(x)) \frac{d\phi}{dx} dx = \int_{\phi(x_0)}^{\phi(x_1)} f(y) dy$$

In 2D

$$\int_R f(x, y) dx dy = \int_{R^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Geodesics

Can we find a geodesic on other surfaces in \mathbb{R}^3 ?

Consider a surface parameterized by $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$, and minimize the arc length

$$L = \int ds = \int \sqrt{dx^2 + dy^2 + dz^2}$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dx^2 = \left(\frac{\partial x}{\partial u} \right)^2 du^2 + 2 \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} du dv + \left(\frac{\partial x}{\partial v} \right)^2 dv^2$$

and likewise for dy^2 and dz^2 .

Geodesics

So we can write the path length as

$$\begin{aligned}L &= \int \sqrt{P + 2Qv' + Rv'^2} du \\ &= \int \sqrt{Pu'^2 + 2Qu' + R} dv\end{aligned}$$

where $u' = du/dv$ and $v' = dv/du$ and

$$\begin{aligned}P &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ Q &= \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} \\ R &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\end{aligned}$$

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Geodesics

Then the Euler-Lagrange equations become

$$\frac{\frac{\partial P}{\partial v} + 2v' \frac{\partial Q}{\partial v} + v'^2 \frac{\partial R}{\partial v}}{2\sqrt{P + 2Qv' + Rv'^2}} - \frac{d}{du} \left(\frac{Q + Rv'}{\sqrt{P + 2Qv' + Rv'^2}} \right) = 0$$

References:

<http://mathworld.wolfram.com/GreatCircle.html>

<http://mathworld.wolfram.com/Geodesic.html>

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