Variational Methods & Optimal Control

lecture 06

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Special case 2: autonomous problems continued

$$H(y,y') = y'\frac{\partial f}{\partial y'} - f(y,y') = const$$

We will see H again later – it often turns out to be a conserved quantity like energy, and so arises naturally in computing the shape of the brachystochrone.

Euler-Lagrange equation

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \left\{ y \in C^2[x_0, x_1] \,\middle|\, y(x_0) = y_0 \text{ and } y(x_1) = y_1 \right\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F, then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0}$$

Autonomous case

The autonomous case is where f has no explicit dependence on x, so $\partial f/\partial x = 0$.

Theorem 2.3.1: Let J be a functional of the form

$$J\{y\} = \int_{x_1}^{x_2} f(y,y')dx$$

and define the function H by

$$H(y,y') = y'\frac{\partial f}{\partial y'} - f(y,y')$$

Then H is constant along any extremal of y.

Example: Brachystochrone

he time taken is

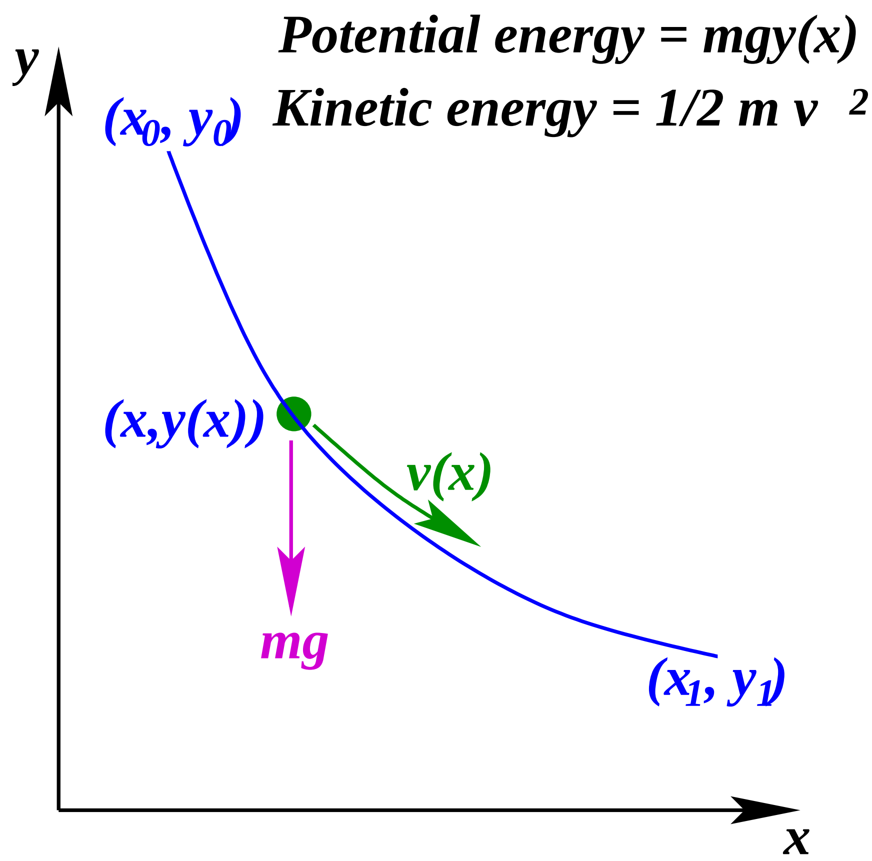
$$T\{y\} = \int_0^L \frac{ds}{v(s)}$$

he energy of a body is the sum of po-
ntial and kinetic energy

$$E = \frac{1}{2} m v(x)^2 + mgy(x)$$

nd a simple conservation law says this
constant, so

$$v(x) = \sqrt{\frac{2E}{m} - 2gy(x)}$$



Example: Brachystochrone (ii)

As for the geodesic in the plane

$$ds = \sqrt{1 + y'^2}dx$$

So the functional of interest (the time taken) is

$$T\{y\} = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{\frac{2E}{m} - 2gy(x)}}dx$$

We can perform a substitution

$$w(x) = \frac{1}{2g}\left(\frac{2E}{m} - 2gy(x)\right)$$

And note that $w'^2 = y'^2$, so (ignoring the constant factor of $-1/2g$) we look for extremals of

Example: Brachystochrone (iii)

Look for extremals of

$$T\{w\} = \int_{x_0}^{x_1} \sqrt{\frac{1+w'^2}{w}} dx$$

which does not contain x explicitly.

$$\begin{aligned} H(w,w') = w'\frac{\partial f}{\partial w'} - f &= \frac{w'^2}{w}\left(\frac{1+w'^2}{w}\right)^{-1/2} - \sqrt{\frac{1+w'^2}{w}} \\ &= \frac{w'^2}{\sqrt{w(1+w'^2)}} - \sqrt{\frac{1+w'^2}{w}} \\ &= \frac{-1}{\sqrt{w(1+w'^2)}} \end{aligned}$$

Example: Brachystochrone (iv)

$$H(w, w') = const$$

So we can write

$$w(1 + w'^2) = c_1$$

Let $w' = \tan\phi$, then $1 + w'^2 = \sec^2\phi$ and for $\kappa_1 = c_1/2$

$$w = \frac{c_1}{\sec^2\phi} = c_1 \cos^2\phi = \kappa_1 \left[1 + \cos(2\phi)\right]$$

$$\frac{dw}{d\phi} = -2\kappa_1 \sin(2\phi) = -4\kappa_1 \cos(\phi) \sin(\phi)$$

Example: Brachystochrone (v)

Also $dw/dx = \tan\phi$, which means

$$\frac{dx}{dw} = \frac{1}{\tan\phi} = \cot\phi$$

Also

$$\frac{dx}{d\phi} = \frac{dx}{dw}\frac{dw}{d\phi} = -4\kappa_1 \cos^2\phi = -2\kappa_1(1 + \cos(2\phi))$$

Integrating

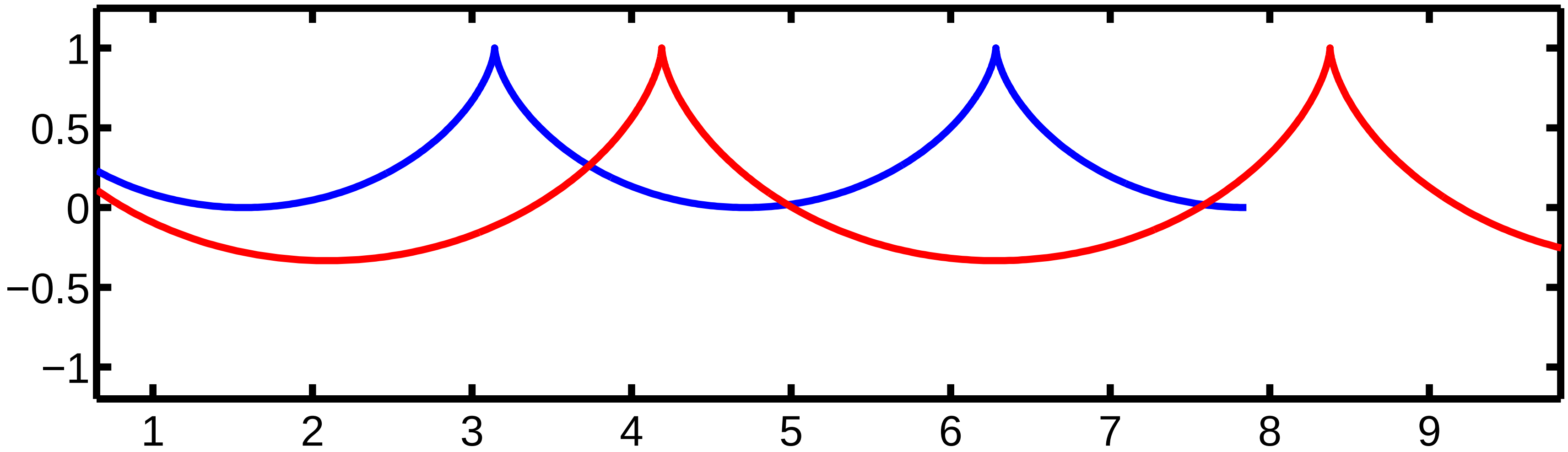
$$x = \kappa_2 - \kappa_1(2\phi + \sin(2\phi))$$

Along with

$$w = \kappa_1 [1 + \cos(2\phi)]$$

we have a parametric form of the solution.

Cycloids



Example: Brachystochrone solution

Take $\theta + \pi = 2\phi$ and we get

$$\begin{aligned} x &= \kappa_2 + \kappa_1(\theta - \sin(\theta)) \\ w &= \kappa_1 \left[1 - \cos(\theta)\right] \end{aligned}$$

Lets change back to y, remembering $w(x) = \frac{1}{2g}\left(\frac{2E}{m} - 2gy(x)\right)$, and that $E = \frac{1}{2}mv^2 + mgy = const$ and $v(x_0) = 0$, so that $E = mgy_0$, hence

$$y = y_0 - w$$

Note that $y(x)$ doesn't depend on g or m!
Now $y(x_0) = y_0$ and so $w(\theta_0) = 0$, which we get when $\theta_0 = 0$.
Now $x(\theta_0) = x_0$ and so $\kappa_2 = x_0$, so the solution is

Example: Brachystochrone solution

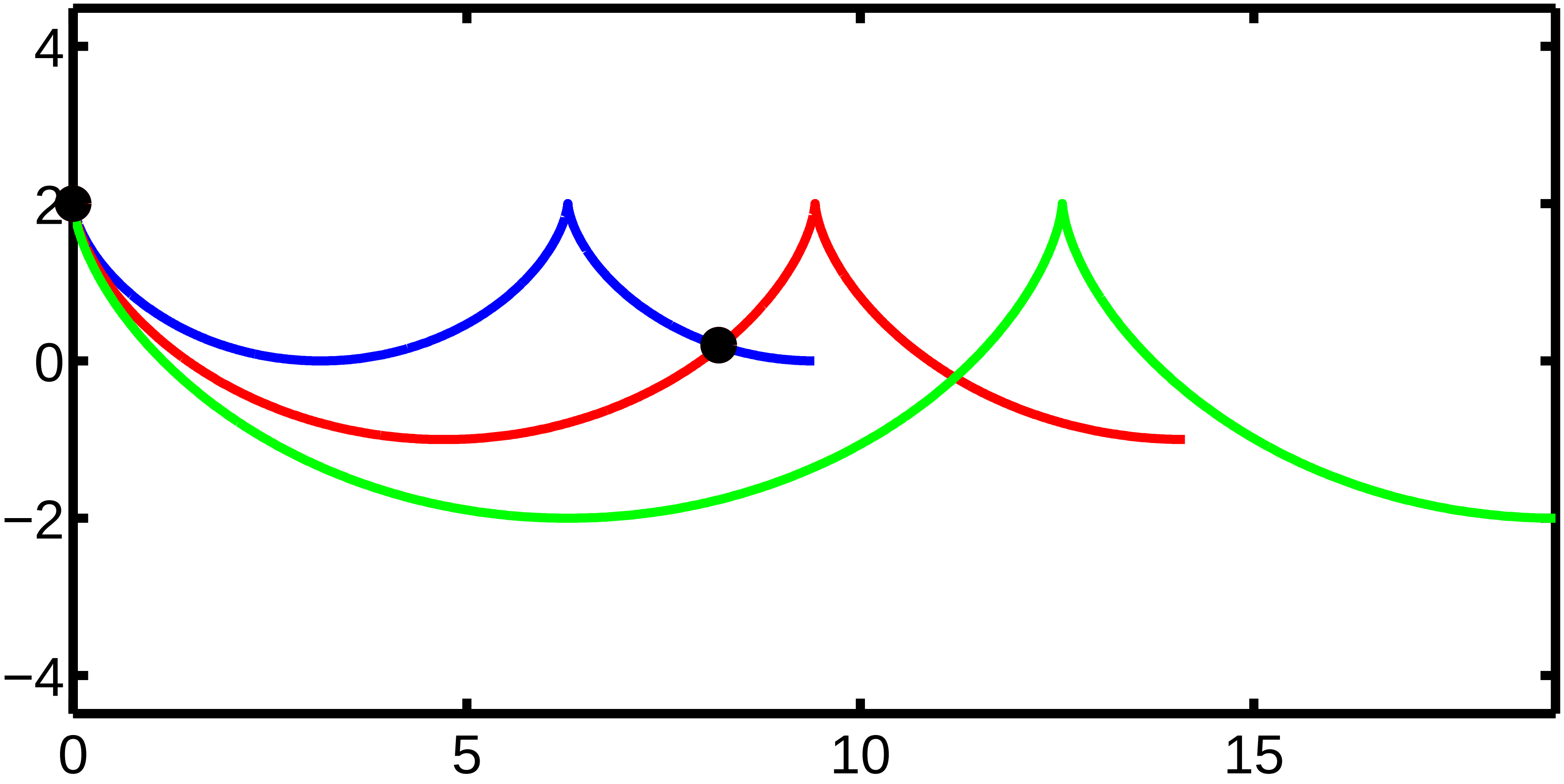
Take $\theta + \pi = 2\phi$ and we get

$$\boxed{\begin{array}{rcl} x &=& x_0 + \kappa_1(\theta - \sin(\theta)) \\ y &=& y_0 - \kappa_1[1 - \cos(\theta)] \end{array}}$$

Now, note that $y(x_1) = y_1$. We find θ_1 first by solving

$$\begin{aligned} y_1 &= y_0 - \kappa_1[1 - \cos(\theta_1)] \\ [1 - \cos(\theta_1)] &= \frac{y_0 - y_1}{\kappa_1} \\ \cos(\theta_1) &= 1 - \frac{y_0 - y_1}{\kappa_1} \\ \theta_1 &= \arccos\left(1 - \frac{y_0 - y_1}{\kappa_1}\right) \end{aligned}$$

Cycloids



More than one possible solution!!

We need to find the fastest one!

Meaning of H

- H is a **conserved** quantity.
- In physics often see such, e.g. the energy
  H is not energy in Brachystochrone problem
- Can derive conservation laws mathematically.
  rather than deriving them as physical laws
- later on we consider Noether's theorem

Newton’s aerodynamic problem

“If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameter move with equal velocities in the direction of the axis of the cylinder, the resistance of the globe will be half as great as that of the cylinder ... I reckon that this proposition will be not without application in the building of ships”.

Isaac Newton, Principia Mathematica

Newton's aerodynamical problem

Consider finding the optimal shape of a rocket's nose cone in order that it creates the least resistance when passing through air.
Assumptions:

- Air is thin, and composed of perfectly elastic particles:
  - particles will bounce off the nose cone with equal speed, and equal angle of reflection and incidence.
  - We ignore tangential friction.
  - We ignore "non-Newtonian" affects such as those from compression of the air.

Realistic for high-altitude, supersonic flight

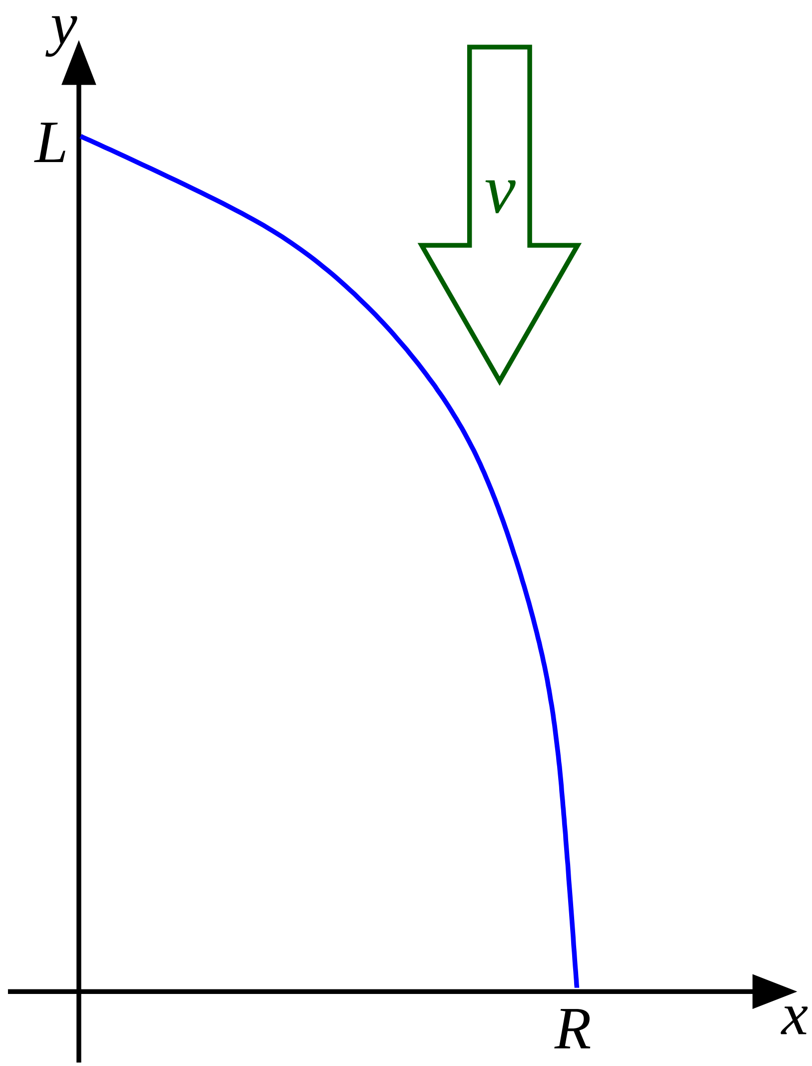
Newton’s aerodynamical problem

Consider finding the optimal shape of a rocket’s nose cone in order that it creates the least resistance when passing through air.
Assumptions:

- As the rocket may rotate along its length, the nose cone must be circularly symmetric, and so we reduce the problem to one of determining the optimal profile of the nose cone.
- The rocket’s nose cone must have radius R at its base, and length L, and its shape should be convex
    - its profile must be concave and non-increasing
    - ratio $L/2R$ is called the **fineness ratio**
    - bigger is better, though little gain for $> 5:1$

Newton's aerodynamical problem

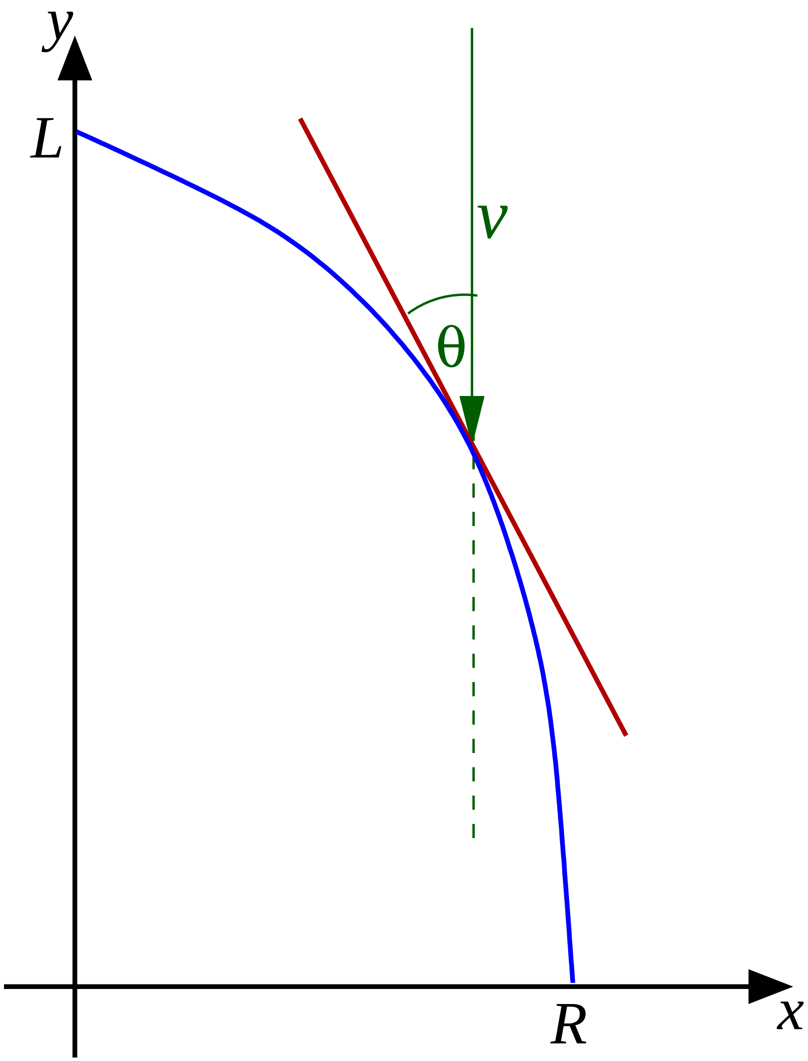
It is irrelevant whether we move the object, or the medium, so assume the latter for convenience.



Newton's aerodynamical problem

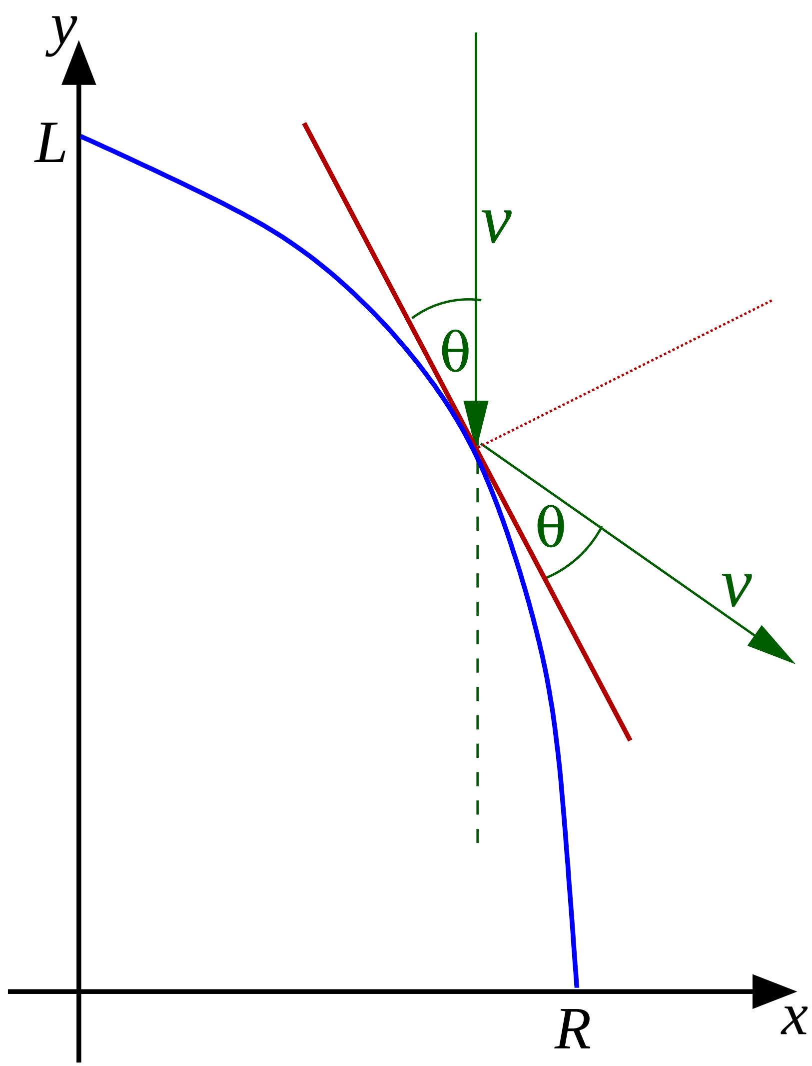
We can calculate the angle between the incident particle and the tangent to the surface by simple trig

$$\cot\theta = \tan(\pi/2 - \theta) = -y'.$$



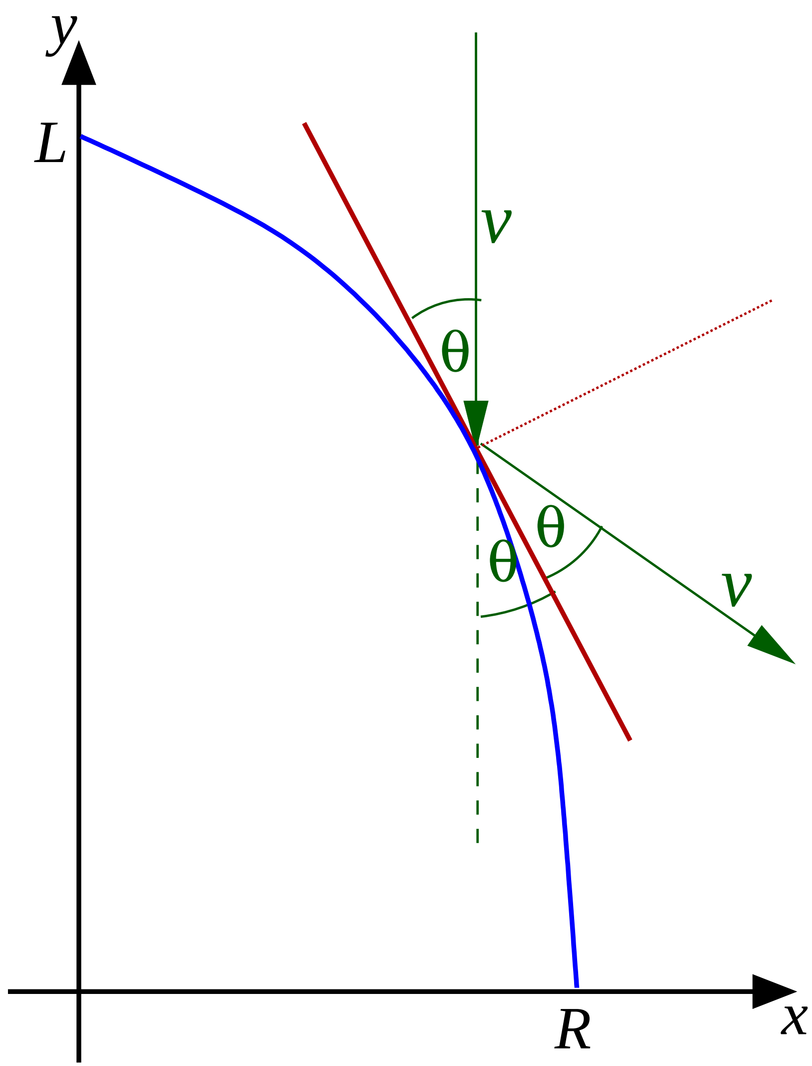
Newton’s aerodynamical problem

The angle of incident equals the angle of reflection.



Newton’s aerodynamical problem

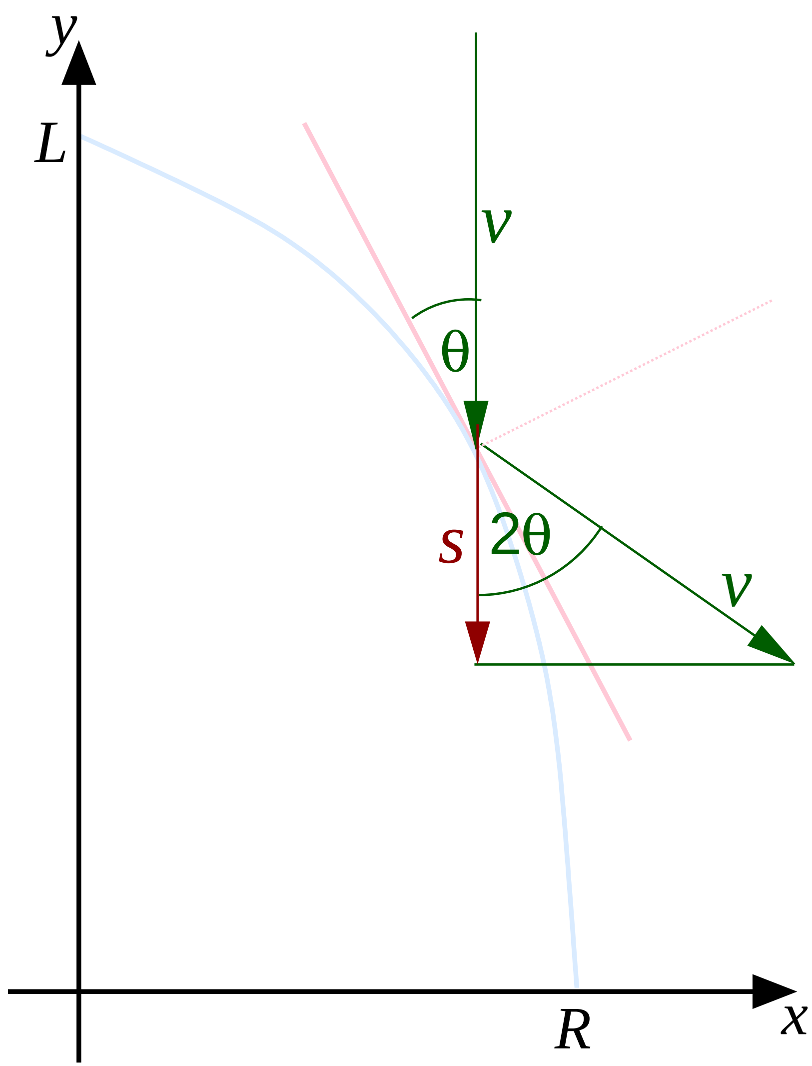
The angle between the reflected particle and the vertical is 2θ.



Newton’s aerodynamical problem

The velocity in the vertical direction after the collision is

$$s = v\cos(2\theta) = v\left(1 - 2\sin^2\theta\right).$$



Newton's aerodynamical problem

$$Force = ma$$

- $m = $ mass
- $a = $ acceleration = change in velocity

$$a = v - s = 2v\sin^2\theta.$$

Scale constants so that

$$2vm = 1,$$

and then

$$Force = \sin^2\theta = \frac{1}{1+\cot^2\theta} = \frac{1}{1+y'^2}.$$

Newton's aerodynamical problem

- Previous calculation gives force per particle $= 1/(1+y'^2)$
- Need to integrate over surface area
- Surface area at radius x is
$$2\pi x\, dx.$$
- Scaling to remove irrelevant constants, the functional describing the resistance
$$F\{y\} = \int_0^R \frac{x}{1+y'^2}\, dx,$$
- subject to $y(0) = L$ and $y(R) = 0$ and
$y' \leq 0$ and $y'' \geq 0$

Newton's aerodynamical problem

The Euler-Lagrange equations are

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{2xy'}{(1+y'^2)^2} = 0$$

So for a given constant c, we get

$$\frac{2xy'}{(1+y'^2)^2} = c.$$

Rearranging we get

$$2xy' \quad = \quad c(1+y'^2)^2$$

Newton's aerodynamical problem

We'll solve this when we get to optimal control.
For now here is the parametric solution without explanation

$$
\begin{aligned}
x(u) &= c\left(\frac{1}{u} + 2u + u^3\right) = \frac{c}{u}(1+u^2)^2 \\
y(u) &= L - c\left(-\ln u - \frac{7}{4} + u^2 + \frac{3}{4}u^4\right)
\end{aligned}
$$

But notice that

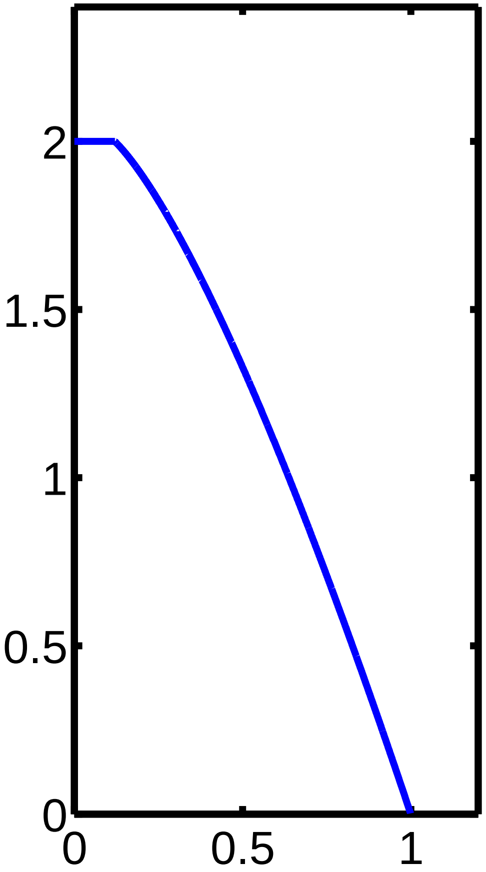
$$
\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du} \Big/ \frac{dx}{du} = -u
$$

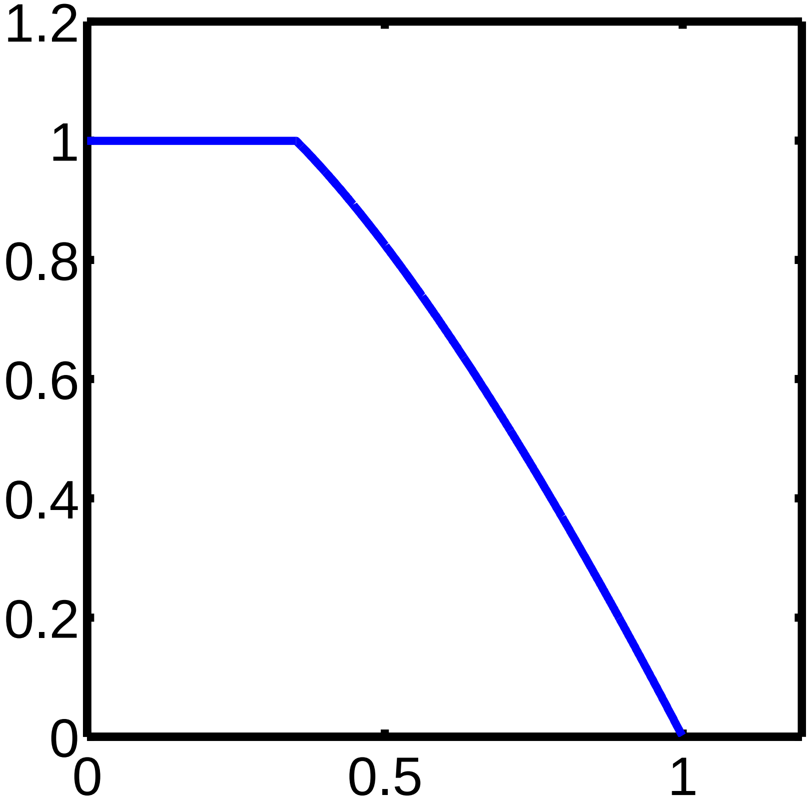
from which it is relatively clear that this is a solution.

Newton’s aerodynamical problem

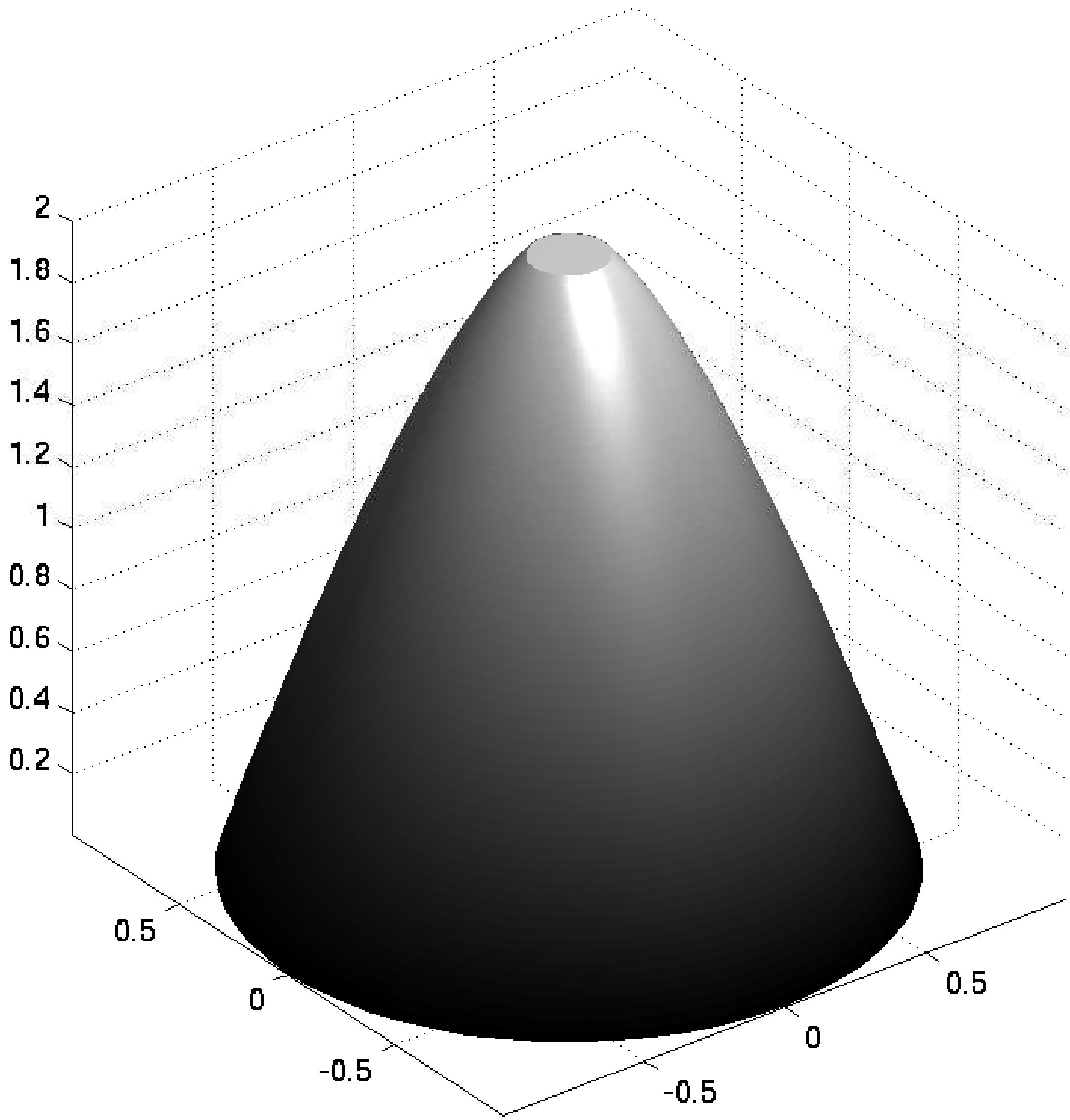
Solution looks almost like a blunted cone

- perhaps that seems counter-intuitive?





Newton’s aerodynamical problem



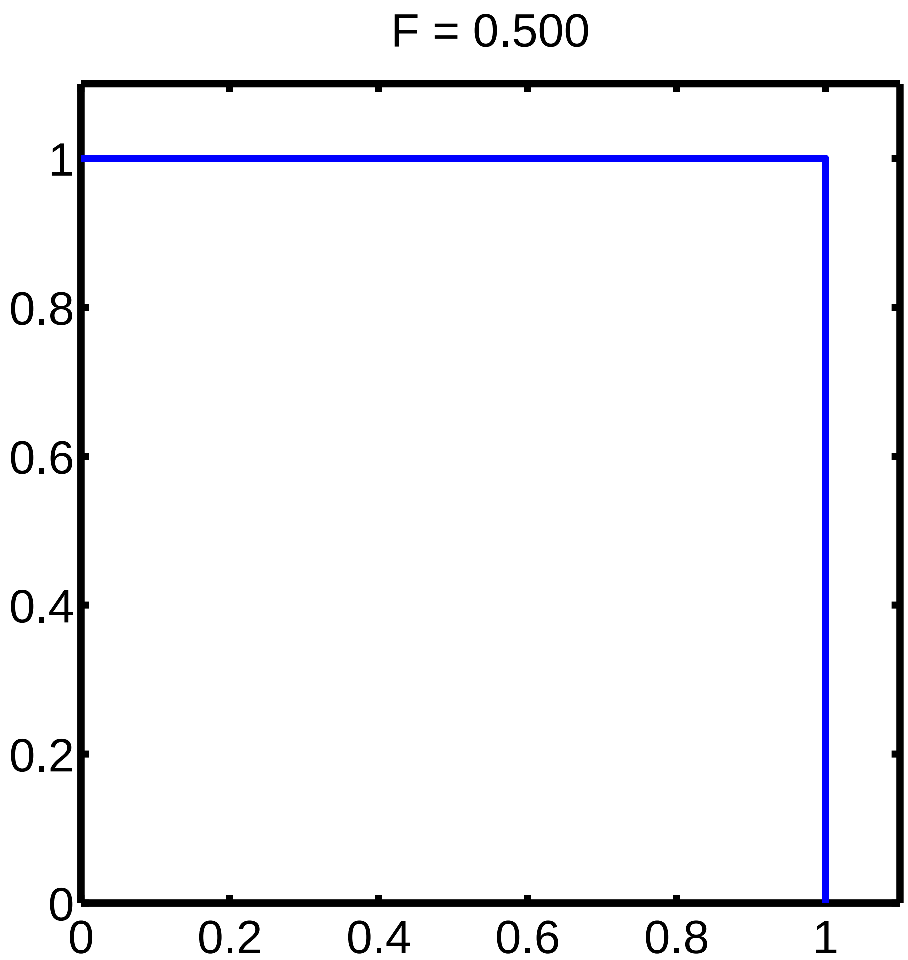
Alternatives: cylinder

Cylinder:

$$
\begin{aligned}
y' &= 0 \\
F\{y\} &= \int_0^R x\,dx \\
&= \frac{R^2}{2}
\end{aligned}
$$

For $R = 1$

$$F = 1/2$$



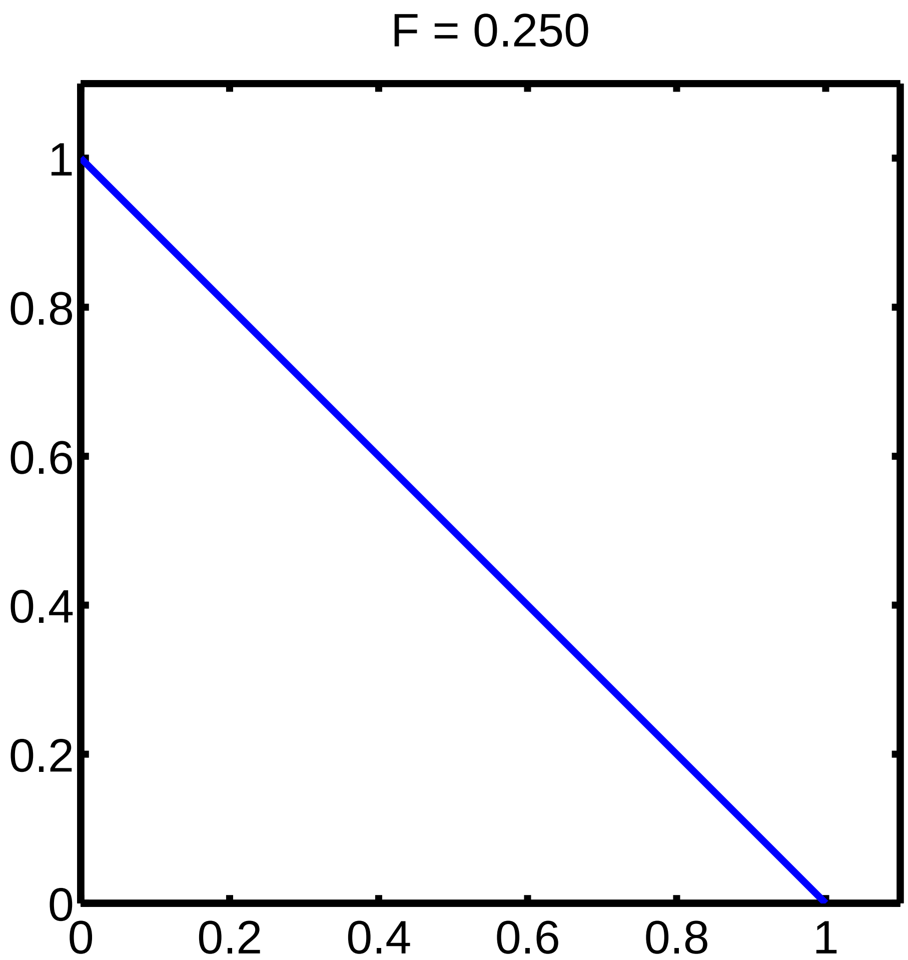
Alternatives: cone

Cone:

$$
\begin{aligned}
y' &= -L/R \\
F\{y\} &= \int_0^R \frac{x}{1+(L/R)^2}\,dx \\
&= \frac{R^2}{2(1+(L/R)^2)}
\end{aligned}
$$

For $R = L = 1$

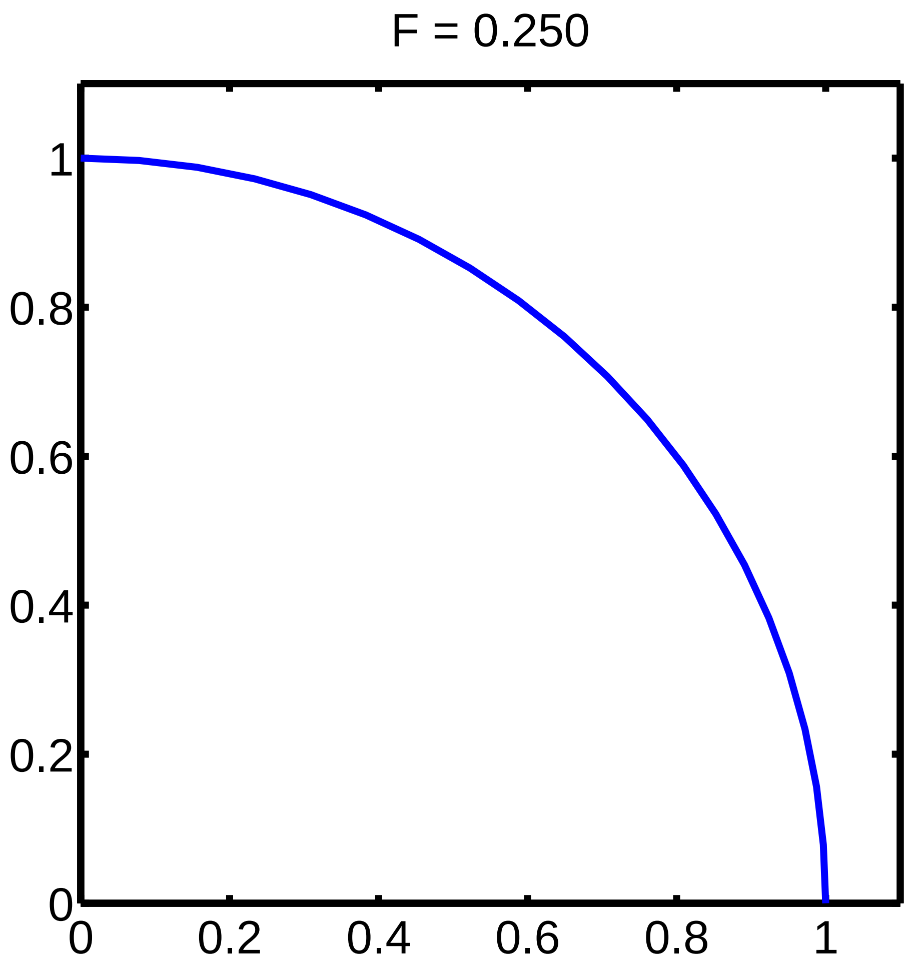
$$F = 1/4$$



Alternatives: sphere

Sphere: $R = L = 1$

$$
\begin{aligned}
x^2 + y^2 &= 1 \\
y' &= -x/y \\
&= -x/\sqrt{1-x^2} \\
F\{y\} &= \int_0^1 \frac{x}{1+y'^2}\,dx \\
&= \int_0^1 x(1-x^2)\,dx \\
&= \frac{1}{4}
\end{aligned}
$$



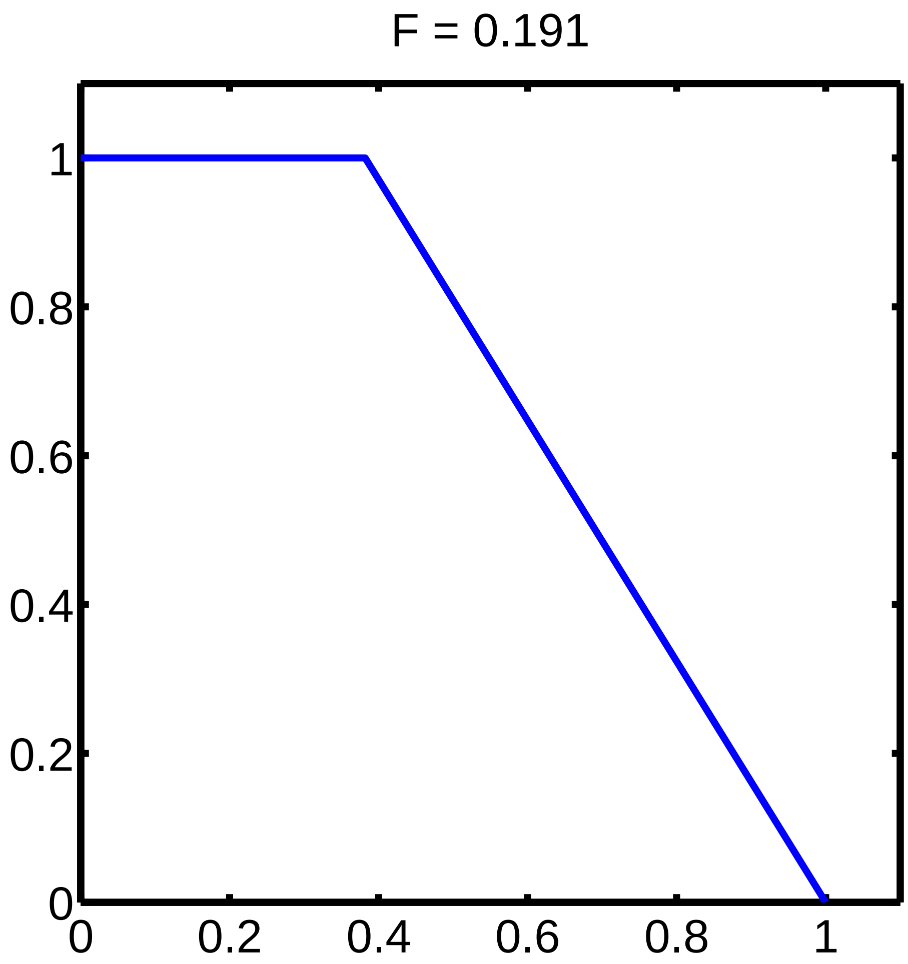
Alternatives: frustum of cone

Frustum of cone: corner at a

$$\begin{aligned} y' &= \begin{cases} 0 & x \leq a \\ -L/(R-a) & x \geq a \end{cases} \\ F\{y\} &= \int_0^a x\,dx + \int_a^R \frac{x}{1+y'^2}\,dx \\ &= \frac{a^2L^2 + R^2(R-a)^2}{2(L^2+(R-a)^2)} \end{aligned}$$

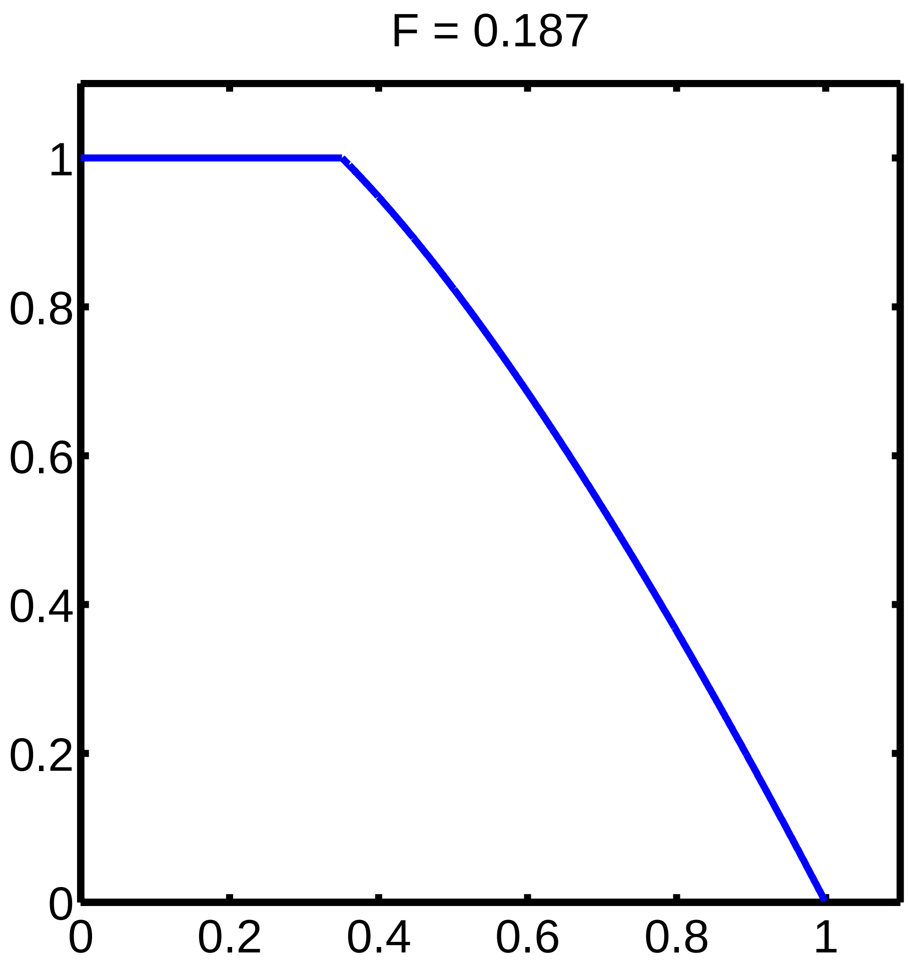
Optimal value of a:

$$a = \frac{(L^2+2R^2) - L\sqrt{L^2+4R^2}}{2R}$$



Alternatives: optimal

Optimal profile:
F computed numerically

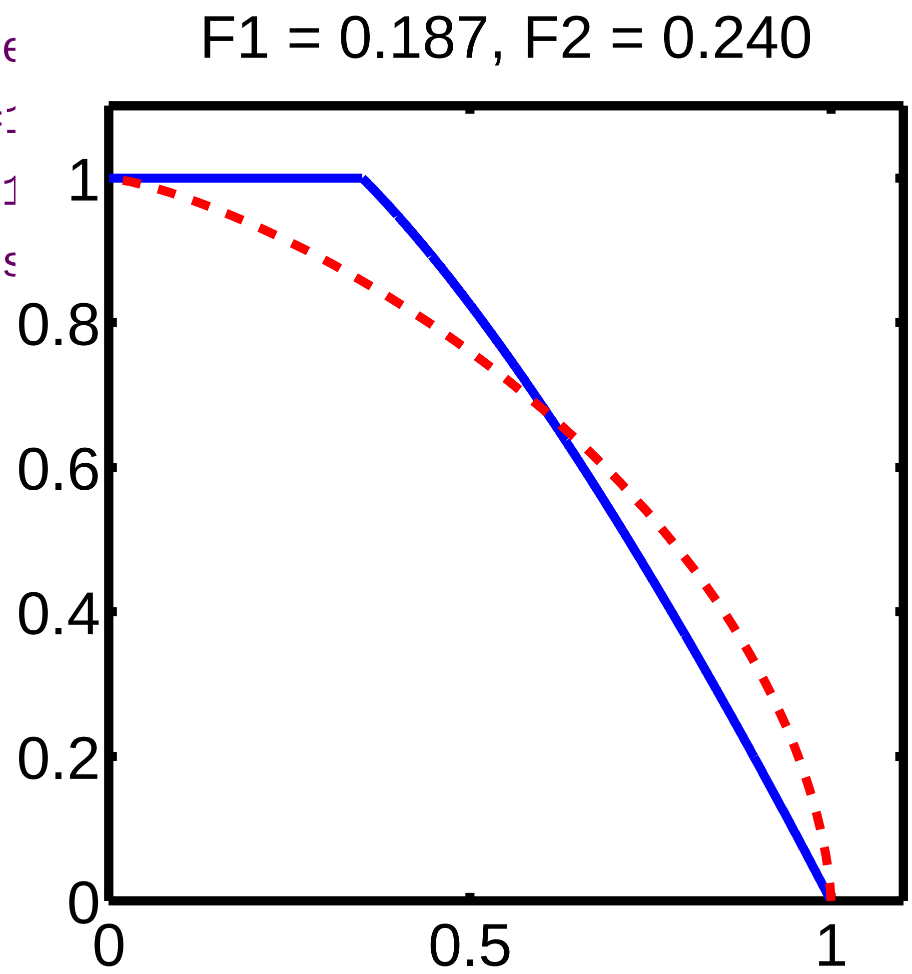


Alternatives: Haack series

Haack series:

`http://en.wikipedia.org/wiki/Nose_cone`
`http://www.info-central.org/?article=1`
`http://mcfisher.0catch.com/other/mach1`
`http://www.if.sc.usp.br/~projetosulfos`

F computed numerically



Typical shapes

- Note that the frustum of a cone isn't much worse than the optimal shape.
- other shapes: ogive, Haack, ...
- In the context of bullets a flattened end is called a **meplat**.
  - typically justified by
    - making all bullets precise
      - tips are hard to get just right
    - impact damage
  - but they wouldn't do it if it wasn't working

Bullets