ADELAIDE UNIVERSITY AUSTRALIA

Examination in School of Mathematical Sciences

Semester 2, 2008

006128	Variational Methods and Optimal Control
	APP MATH 3010

Official Reading Time:	10 mins
Writing Time:	180 mins
Total Duration:	190 mins

NUMBER OF QUESTIONS: 5 TOTAL MARKS: 60

Instructions

- Answer ALL questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue books are provided.
- Calculators are NOT permitted.
- 2 double sided pages of handwritten notes are allowed.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO

1. (a) Find the extremal of

$$F\{y\} = \int_1^2 y'(1 + x^2 y')\, dx$$

for $y(1) = 2$ and $y(2) = 1$.

(b) Find the extremal of

$$F\{y\} = \int_0^{\pi/8} \left[y'^2 + 2yy' - 16y^2\right]\, dx$$

for $y(0) = 2$ and $y(\pi/8) = 0$.

[12 marks]

Please turn over for page 3

2. We are going to accelerate some a craft of (unloaded) mass M_0 by firing its rocket. The rocket burn will expend fuel, reducing the mass of the craft as we go (and hence potential changing its acceleration). Ignoring gravitational influences on the craft, calculate the rate of burn profile that will provide the greatest total change in velocity $\Delta v = v(t_1) - v(t_0)$, given the fixed amount of fuel F_0 we have available. Calculate the maximum total change in velocity. Interpret your results, and give the Δv of the craft.

   Note that as we burn fuel, the mass $M(t)$ of the craft will change. You may use the following formulae:

$$\begin{aligned}
v(t) &= \text{velocity at time } t \\
F(t) &= \text{fuel at time } t \\
M(t) &= \text{Mass at time } t \\
&= M_0 + F(t) \\
\text{Force} &= c \times \text{ burn rate of fuel} \\
&= c\frac{dM}{dt} \\
\dot{v} &= \frac{\text{Force}}{M(t)}
\end{aligned}$$

   where c is the exhaust velocity of the rocket.

[6 marks]

3. (a) We wish to minimize a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} A(x,y)\sqrt{1+y'^2}\,dx$$

where $A(x,y) > 0$ for all x and y. Show (using the natural boundary conditions for a free end-point) that the transversal between a point, and the curve Γ, which minimizes this functional, must have a tangent at the point of contact with Γ that is at right-angles to the tangent to the curve Γ.

(b) Use part (b) above to find the extremals of

$$F\{y\} = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{y}\,dx$$

which connect the origin to the vertical line $x = 1$. Describe (simply) the shape of this curve.

[16 marks]

4. A patient waiting to undergo an operation has an infection, which it is desirable to reduce before the operation is performed. However, the operation is important, and the surgeon does not want to delay the operation too much. Left to itself, the infection would die away, but this could take a long time. We can speed things up by administering an antibiotic. We end up with conflicting objectives: operate quickly, but preferably with a reduced level of infection, though this causes a delay.

   We measure the level of infection at time t by $x(t)$. The control $u(t)$ is the amount of drug prescribed to reduce the infection. The maximum permitted dose is $u = 1$. The cost function has three components, one (a terminal cost) related to reducing the infection $x(t_1)$ at the time of the operation t_1, a cost related to the time t_1 we have to wait before the operations, and a third cost related simply to the amount of medication we use, e.g. the total cost is

   $$F\{x, u\} = \frac{1}{2}x(t_1)^2 + \int_0^{t_1} 2 + au \; dt,$$

   where $a > 0$ is constant giving the cost of the medication.

   The state equation is

   $$\dot{x} = -x - u$$

   Note that the initial state $x(0) = x_0 > 0$ is fixed, and the end time t_1 and end-state are free.

   (a) Write the Hamiltonian of the system.

   (b) State a simple condition this Hamiltonian must satisfy, and why.

   (c) Is the Hamiltonian linear in the control u, and if so, identify the switching function, and the type of control we will apply.

   (d) Using the Pontryagin Maximum Principle, determine the optimal form of the optimal control u for the problem.

   (e) Given the problem also has free end value $x(t_1)$, what is the natural boundary condition at t_1? What implication does this have for the sign of p, and hence for the switching function?

   (f) From above how would one go about calculating the control, and thence the time of the operation.

   [18 marks]

5. (a) Given a functional of the form

$$F\{y\} = \int_a^b p_n(y')\,dx$$

where $p_n(\cdot)$ is a polynomial of degree n, show that the extremals will always be straight lines (where they exist).

(b) Hence comment on the types of functions that can form extremals of a functional

$$F\{y\} = \int_a^b f(y')\,dx$$

where f is a smooth function (i.e., all of its derivatives exist and are continuous).

[8 marks]