ADELAIDE UNIVERSITY
AUSTRALIA

Examination in School of Mathematical Sciences

Semester 2, 2004

6128	VARIATIONAL METHODS AND OPTIMAL CONTROL III
	APP MATH 3010

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 3 TOTAL MARKS: 75

Instructions

- Answer ALL questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue books are provided.
- Calculators are NOT permitted.
- Closed book examination.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO

1. (a) Determine the extremal to

$$\int_0^{\pi/2} \left(y^2 - y'^2\right)\, dx$$

with fixed end-points $y(0) = 0$ and $y(\pi/2) = 1$.

(b) Show that x^3 is an extremal for

$$\int_{-1}^{1} \left(x^2 y'^2 + 12y^2\right)\, dx$$

joining the points $(-1, -1)$ and $(1, 1)$.

[25 marks]

2. (a) Determine an extremal for the functional

$$F\{x\} = \int_0^T \sqrt{1 + \dot{x}^2}\, dt$$

which starts at $x(0) = 2$ and which terminates on the curve

$$x = -4t + 5$$

Illustrate your solution with a sketch, and highlight the relationship between the extremal curve, and the terminating point.

(b) The potential energy present in a bent elastic beam (given minimal deflections) under uniform force ρ is given by

$$V\{y\} = \int_0^d \frac{\kappa}{2} y''^2 - \rho y\, dx$$

where $y(x)$ is the shape of the beam, and its end-points are at $x = 0$ and $x = d$. Under static equilibrium, the beam will take a shape that minimizes this potential energy.

1. Write out the Euler-Poisson equations that this static equilibrium solution must satisfy.
2. If the beam is clamped horizontal at $x = 0$, but free to move at $x = d$, write down the end-point constraints that apply to this problem.
3. Show that the following solution satisfies the Euler-Poisson equations, and the end-point constraints.

$$y(x) = \frac{\rho x^2 (6d^2 - 4dx + x^2)}{24\kappa}$$

4. From this solution, derive the end-point deflection of the beam.

[25 marks]

3. (a) By considering perturbations around an extremal curve $y(x)$ of the form $\hat{y}(x) = y(x) + \epsilon\eta(x)$, and using Taylor's theorem, derive the form of the First Variation for the fixed end point variational problem of finding extremals of

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx$$

subject to $y(x_0) = y_0$ and $y(x_1) = y_1$. Use integration by parts to put the First Variation in a form suitable for proving the Euler-Lagrange equations must be satisfied for an extremal curve.

(b) We wish to move a rocket from our solar system, to Alpha Centauri in minimal time. Ignoring local gravitational effects, and relativistic considerations, the rocket simply obeys Newton's laws of motion. The crew of the rocket will die if the rocket accelerates at more than 3 Earth gravities ($3g$) for any substantial period of time.

1. Explain what type of optimal control problem we face, and what type of optimal control would result.
2. Using state variables x_1 to represent distance from the solar system, x_2 to indicate velocity (towards Alpha Centauri), and given the mass of the rocket is m, and the force applied to the rocket is u, write the problem as an optimization problem.
3. Determine the Hamiltonian of the problem.
4. Use Hamilton's equations to find the co-state variables at time t.
5. Assuming the rocket starts at speed 0 (at the Solar System), at time 0, but that we do not constrain its speed at time T, when it reaches Alpha Centauri, what are the constraints we can add to the problem.
6. Use the additional constraints to determine the constants of integration in the co-state variables, and from this, determine the optimal control.
7. Given the optimal control, solve the state equations to determine the optimal path to Alpha Centauri.
8. Comment on reasons why this model would not be adequate for planning a mission to Alpha Centauri.

Note: Ignore relativistic effects such as the maximum physical speed of an object being limited by the speed of light.

[25 marks]