Variational Methods and Optimal Control
Class Exercise 4: due before lecture, on, on Thursday 4th October, 2012

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

1. Consider the functional

$$I\{y, z\} = \int_{x_0}^{x_1} y^2 + z^2 \, dx$$

subject to the constraint

$$y' = z - y$$

- What type of constraint do we have?
- Write down the form of the problem including a Lagrange multiplier in the integral.
- Determine the Euler-Lagrange equations for y and z.
- Solve the equations to find the form of the extremal curve of I under the constraint.

2. Given that

$$F\{y(x)\} = \int_0^1 (y'^2 - y^2) \, dx,$$

with the constraint on $y(x)$ that

$$\int_0^1 \sqrt{1 + y'^2} \, dx = \sqrt{2},$$

and the end conditions $y(0) = 0$ and $y(1) = 1$, prove that $F\{y(x)\}$ achieves its minimum value for $y = x$.

3. The maximum entropy principle is an extension of Laplace's principle of insufficient reason, which in essence says we should not assume things that are not supported by evidence. For instance, in probability, unless we have reason to suspect otherwise, we would assume events are equally likely, e.g., the probability of heads coming up on a coin toss is 1/2.

Maximum entropy extends this by noting that if we maximize the (Shannon) entropy of a probability distribution constrained by the facts we know about the distribution, we will derive the estimate of that distribution which makes the least assumptions about the distribution that aren't supported by the data.

The Shannon entropy of a distribution with two variables is defined to be

$$H\{p\} = -\iint p(x, y) \ln p(x, y) \, dx \, dy,$$

where $p(x, y)$ is the probability density function (assume that $p(x, y) > 0$ over the region of integration). Note that all probability density functions satisfy the constraint that

$$\iint p(x, y) \, dx \, dy = 1,$$

because probabilities must always add to one. Given only the information that the variables (x, y) lie in the unit square $[0, 1] \times [0, 1]$, derive the maximum entropy distribution.

4. In Newton's aerodynamical problem we minimized resistance

$$F\{y\} = \int_0^R \frac{x}{1 + y'^2}\, dx,$$

subject to $y(0) = L$ and $y(R) = 0$ (and $y' \leq 0$ and $y'' \geq 0$).

In nose-cone design this is sometimes approximated by assuming that the nose-cone will be long and thin, so y' will be large (and negative in our formulation). In that case, we may approximate $1 + y'^2$ by y'^2 and simplify the problem as shown in tutorial 2.

Now, using this approximation, consider an alternative formulation of the problem where we don't specify the length of the nose-cone, we specify the maximum surface area (often called the "wetted area") of the nose-cone. Using the approximation, derive the optimal shape of the nose-cone under this constraints.