Variational Methods and Optimal Control
Class Exercise 3 solutions

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1. Find the extremals of the following functionals

(a) $F\{y\} = \int_{-1}^{0} \sqrt{y(1+y'^2)}\, dx$ with end-points $y(-1) = 0$ and $y(0) = 0$.

Solution: The problem is autonomous, so look at the Hamiltonian

$$H = y' \frac{\partial f}{\partial y'} - f(y, y') = \frac{yy'^2}{\sqrt{y(1+y'^2)}} - \sqrt{y(1+y'^2)} = const$$

Putting everything over the same denominator we get

$$H = \frac{yy'^2 - y(1+y'^2)}{\sqrt{y(1+y'^2)}} = const$$

or more simply

$$H = \frac{-y}{\sqrt{y(1+y'^2)}} = const$$

Multiplying both sides by $\sqrt{y(1+y'^2)}$ and squaring we get

$$y^2 = c^2 y(1+y'^2).$$

There are two cases here:

i. $y(x) = 0$: Given that this solution fits the end points $y(-1) = 0$ and $y(0) = 0$, this provides a valid extremal. Note that this makes sense, because the integrand in non-negative, and so $F\{y\}$ has a lower bound at zero, which is achieved by this solution.

ii. In the second case

$$\begin{aligned} \int \frac{1}{\sqrt{y/c^2 - 1}}\, dy &= \int dx \\ 2c^2\sqrt{y/c^2 - 1} &= x + b \\ y &= \left(\frac{x+b}{2c}\right)^2 + c^2. \end{aligned}$$

However, note that $y(0) = 0$ and this function has minimum c^2 for real constants, so it cannot satisfy the end-point constraints.

Extra notes: In fact there are some subtle issues in this problem. Implicitly, the functional requires $y \geq 0$ to ensure it is real. This limits the type of variations we could consider, and one could easily doubt whether the Euler-Lagrange equations apply here. However, these can be resolved if one rewrites the function in terms of a new variable u such that $y = u^2$. The functional becomes

$$F\{u\} = \int_{-1}^{0} \sqrt{u^2(1+4u^2u'^2)}\, dx,$$

and the Hamiltonian reduces to

$$H = \frac{-u}{\sqrt{1+4u^2u'^2}},$$

leading to two possible solutions corresponding to the two above.

(b) [3 marks] $F\{y\} = \int_{0}^{1} (y'^2 - y^2 - y)e^{2x}\, dx$ with end-points $y(0) = 0$ and $y(1) = e^{-1}$.

Solution: The Euler-Lagrange equation

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx} 2y'e^{2x} + (2y+1)e^{2x} = 2y''e^{2x} + 4y'e^{2x} + (2y+1)e^{2x} = 0$$

Dividing by the two times (non-zero) exponential

$$y'' + 2y' + y = -1/2$$

Solving the homogenous DE $y'' + 2y' + y = 0$ we get

$$y = Ae^{-x} + Bxe^{-x}$$

The particular solution to the inhomogenous equation is

$$y = -1/2,$$

So the final solution is

$$y = Ae^{-x} + Bxe^{-x} - 1/2.$$

Now $y(0) = 0$ so $A = 1/2$, and

$$y = e^{-x}/2 + Bxe^{-x} - 1/2.$$

When $x = 1$ we have $y(1) = e^{-1}$ so

$$y = e^{-1}/2 + Be^{-1} - 1/2 = e^{-1}$$

Hence

$$B = e/2 + 1/2$$

so the final solution is

$$y = \frac{1}{2}\left[e^{-x} + (e+1)xe^{x} - 1\right].$$

2. Find the extremals of the following functionals

(a) [3 marks] $F\{y(x), z(x)\} = \int_{x_1}^{x_2} (2yz - 2y^2 + y'^2 - z'^2)\, dx$

Solution:

There are two dependent variables in this problem, and so the extremals must satisfy the two E-L equations

$$\begin{aligned} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} &= 0 \\ \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) - \frac{\partial f}{\partial z} &= 0 \end{aligned}$$

where $f(y, z, y', z') = 2yz - 2y^2 + y'^2 - z'^2$. Taking the appropriate derivatives, the E-L equations become

$$\begin{aligned} \tfrac{d}{dx}(2y') - 2z + 4y &= 2y'' + 4y - 2z = 0 \\ \tfrac{d}{dx}(-2z') - 2y &= -2z'' - 2y = 0 \end{aligned}$$

Divide both equations by 2, and differentiate the first equation twice

$$y^{(4)} + 2y'' - z'' = 0$$

Combine with the second equation to eliminate z''

$$y^{(4)} + 2y'' + y = 0$$

which is a 4th order homogenous linear ODE. The solutions will be of the form $e^{\lambda x}$, where λ satisfies the characteristic equation

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

and so $\lambda = \pm i, \pm i$, giving solutions

$$y(x) = (c_1 + c_2 x)\sin x + (c_3 + c_4 x)\cos(x)$$

where the constants c_1, c_2, c_3 and c_4 are determined by the end-points, and $z = y'' + 2y$.

(b) $F\{\mathbf{q}(t)\} = \int_{t_1}^{t_2} \dot{q}_1\dot{q}_2 + \dot{q}_2 q_3 + \dot{q}_3 q_1 \, dt$

Solution: We get one Euler-Lagrange equation for each of the dependent variables q_i so

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{q}_1}\right) - \frac{\partial f}{\partial q_1} &= 0 \\
\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{q}_2}\right) - \frac{\partial f}{\partial q_2} &= 0 \\
\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{q}_3}\right) - \frac{\partial f}{\partial q_3} &= 0
\end{aligned}$$

which give, respectively,

$$\begin{aligned}
\frac{d}{dx}\dot{q}_2 - \dot{q}_3 &= 0 \\
\frac{d}{dx}(\dot{q}_1 + q_3) &= 0 \\
\frac{d}{dx}q_1 - \dot{q}_2 &= 0
\end{aligned}$$

or

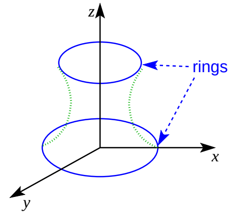
$$\begin{aligned}
\ddot{q}_2 - \dot{q}_3 &= 0 \\
\ddot{q}_1 + \dot{q}_3 &= 0 \\
\dot{q}_1 - \dot{q}_2 &= 0
\end{aligned}$$

Integrating these three equations we get

$$\begin{aligned}
\dot{q}_2 - q_3 &= c_1 \\
\dot{q}_1 + q_3 &= c_2 \\
q_1 - q_2 &= c_3
\end{aligned}$$

Solve homogenous equations (where the RHS is zero) by taking $q_1 = q_2$, and then noting that $q_3 = \pm\dot{q}_2$ so that q_3 must be zero, as must be the derivative of q_1 and q_2, so that only leaves the possible solution where q_1 and q_2 are constants, and $q_3 = 0$.

3. 2 marks **Surface of minimum area:** Consider a soap bubble suspended between two parallel concentric, but displaced rings of radius r_0 and r_1 (see the figure for a clearer view). Ignoring gravity and other external forces, the shape of the soap bubble will minimize the surface area. Use the Calculus of Variations to explain the shape this bubble will take.



[Hint: rotational symmetry can be used to reduce this to problem with one dependent variable.]

Solution: Note that the problem is circularly symmetric about the z axis, so we shall describe the problem in cylindrical co-ordinates

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= u
\end{aligned}$$

Under symmetry, the soap film will also be circularly symmetric about the z axis*, so we can describe it as a surface of revolution, i.e., it can be described by its radius profile $r(t)$ which is then rotated about the z axis to get the surface. The area of such a surface is simply the perimeter of a circle of radius r, multiplied by the small distance $\sqrt{1 + r'^2}dz$ (as for geodesic problems), resulting in an area integral[1]

$$A\{r\} = \int_{z_0}^{z_1} 2\pi r\sqrt{1 + r'^2}\, dz$$

By observation, this functional is almost the same as the functional describing a hanging chain (only the constants differ). As a result, $r(z)$ will be shaped like a catenoid (a catenary of revolution), i.e.,

$$r(z) = b\cosh\left(\frac{x - c}{b}\right)$$

where b and c are chosen to satisfy the end points $r(z_i) = r_i$ for $i = 0, 1$.

Note that, as for catanery, there can be multiple, or zero solutions. If the rings are too far apart, the bubble bursts!

Notes on circular symmetry: In the above we have deduced from the form of problem that it has circular symmetry, and therefore assumed that the solution must also possess this symmetry. This is a natural assumption, but in the best form should perhaps be justified, which we can do as follows.

[1] In more detail, consider the surface to be made up of a series of frustums of cones with small height given by dz, and radius at the top and bottom of $r(z)$ and $r(z + dz)$ respectively. The surface area of the frustum of a cone (e.g. see `http://mathworld.wolfram.com/ConicalFrustum.html`) is

$$dA = \pi(r(z) + r(z + dz))\sqrt{dz^2 + (r(z + dz) - r(z))^2} = 2\pi r(z)\sqrt{1 + r'^2}dz$$

where we use the first order Taylor approximation $r(z + dz) = r(z) + r'(z)dz$, and note that the dz^2 term will be negligable for small dz.

Alternatively, we can arrive at the integral by direct consideration of the surface integral in cylindrical co-ordinates, i.e., the surface area will be

$$A = \iint r(s)\, d\theta\, ds = 2\pi \int r(s)\, ds = 2\pi \int r(z)\sqrt{1 + r'^2}\, dz.$$

where we remove the integral with respect to θ because there is no dependence on angle, and the second step follows the same derivation for computing arc lengths in geodesics that we used in class.

We can construct a vertical plane through the z-axis, so that it bisects each of the two rings. Now consider the problem of the minimal surface on the RHS of the plane and the LHS. The problems are simply reflected versions of each other, and so the minimal surface on the LHS and RHS must be reflections of each other. However, we can construct a infinite number of such planes at any angle relative to the x-axis, and the only way the solution can preserve symmetry about all of them is for the solution to also have circular symmetry.

4. Ritz's Method: Use Ritz's method to find an approximate, non-trivial solution to the differential equation

$$y'' + \frac{1}{x} y' + \lambda y = 0,$$

in the domain $x \in [0, 1]$ where $y(1) = 1$, and hence determine an approximate value of λ that has a solution. [*Hints:* note that the equation can be written in the form

$$\frac{d}{dx}(xy') + \lambda xy = 0,$$

and find the corresponding integral for which this is the Euler-Lagrange equation. Once you have a variational problem, write use the trial function

$$y_{\text{trial}} = a + bx^2 + cx^4,$$

which we have chosen because the solution is expected to be an even function.]

Solution: Note that the Euler-Lagrange equation could arise from a variational problem of the form: find the extremals of

$$F\{y\} = \int_0^1 xy'^2 - \lambda xy^2 \, dx.$$

(Note that the λ might arise from a isoperimetric constraint of the form $\int_0^1 xy^2 = G$.) The end-point condition $y(1) = 1$ means that the trial function must have $a + b + c = 1$. Substituting the trial function we get

$$\begin{aligned}
F\{y\} &= \int_0^1 x(2bx + 4cx^3)^2 + \lambda x(a + bx^2 + cx^4)^2 \, dx \\
&= \int_0^1 4b^2x^3 + 16bcx^5 + 16c^2c^7 - \lambda(a^2x + b^2x^5 + c^2x^9 + 2abx^3 + 2bcx^7 + 2acx^5) \, dx \\
&= \left[b^2 + \frac{8}{3}bc + 2c^2 - \lambda\left(\frac{a^2}{2} + \frac{b^2}{6} + \frac{c^2}{10} + \frac{ab}{2} + \frac{bc}{4} + \frac{ac}{3}\right)\right] \\
&= b^2 + \frac{8}{3}bc + 2c^2 - \lambda\left(\frac{1}{6}b^2 + \frac{5}{12}bc + \frac{4}{15}c^2\right)
\end{aligned}$$

using $a = -b - c$. We can differentiate with respect to b and b to get

$$\begin{aligned}
\frac{\partial F}{\partial b} &= 2b + \frac{8}{3}c - \lambda\left(\frac{1}{3}b + \frac{5}{12}c\right) = \left(2 - \frac{\lambda}{3}\right) b + \left(\frac{8}{3} - \frac{5\lambda}{12}\right) c \\
\frac{\partial F}{\partial c} &= 4c + \frac{8}{3}b - \lambda\left(\frac{5}{12}b + \frac{8}{15}c\right) = \left(\frac{8}{3} - \frac{5\lambda}{12}\right) b + \left(4 - \frac{8\lambda}{15}\right) c
\end{aligned}$$

and each of these must be zero to find an extremal. Mutiply the top equation by $\left(\frac{8}{3} - \frac{5\lambda}{12}\right)$ and the bottom by $\left(2 - \frac{\lambda}{3}\right)$ and subtract the second from the first, and we remove b from the equation. The result is

$$c\left[\left(\frac{8}{3} - \frac{5\lambda}{12}\right)^2 - \left(4 - \frac{8\lambda}{15}\right)\left(2 - \frac{\lambda}{3}\right)\right] = 0,$$

where either $c = 0$ or

$$3\lambda^2 - 128\lambda + 640 = 0,$$

which has two solutions

$$\lambda = \frac{64 \pm \sqrt{2176}}{3} \simeq 5.7841 \text{ or } 36.8825.$$

Note that this is actually an approximate eigenvalue of the Sturm-Liouville problem defined at the start. The true eigenvalue is 5.7832 so we have done reasonably well using this approach.

5. 2 marks **Higher order derivatives:** find the extremal of the following functional

$$J\{y\} = \int_0^1 y''^2 - 240xy\,dx,$$

subject to $y(0) = 0$, $y'(0) = 1/2$, $y(1) = 1$ and $y'(1) = 1/2$.

Solution: The Euler-Poisson equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 2x + 2\frac{d^2}{dx^2}y'' = 0,$$

so the DE is

$$y^{(4)} = 120x,$$

which has solution

$$y(x) = x^5 + c_3x^3 + c_2x^2 + c_1x + c_0.$$

We determine the constants to fit the end point conditions

- $y(0) = 0$ implies that $c_0 = 0$
- $y'(0) = 1/2$ implies that $c_1 = 1/2$
- $y(1) = 1$ implies $1 + c_3 + c_2 + 1/2 = 1$
- $y'(1) = 1$ implies $5 + 3c_3 + 2c_2 + 1/2 = 1/2$

The last two equations give

$$\begin{aligned} c_3 + c_2 &= -1/2 \\ 3c_3 + 2c_2 &= -5 \end{aligned}$$

Solving we get

$$\begin{aligned} c_0 &= 0 \\ c_1 &= 1/2 \\ c_2 &= 7/2 \\ c_3 &= -4 \end{aligned}$$

So

$$y = x^5 - 4x^3 + \frac{7}{2}x^2 + \frac{1}{2}x.$$

The results are plotted in the following figure.