Variational Methods and Optimal Control
Class Exercise 1 solutions

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

1. 0 marks Use the technique of Lagrange multipliers to maximize $V = xyz$ for $x, y, z \geq 0$ subject to the pair of constraints

$$\begin{aligned} xy + yz + zx &= 1 \\ x + y + z &= 3 \end{aligned}$$

Solution: Introduce slack variables α, β, γ to express the inequalities $x, y, z \geq 0$ as $x - \alpha^2 = 0$, $y - \beta^2 = 0$, $z - \gamma^2 = 0$. We employ Lagrange multipliers to form the new objective function

$$\mathcal{L} = xyz + \lambda_1(xy + yz + zx - 1) + \lambda_2(x + y + z - 3) + \lambda_3(x - \alpha^2) + \lambda_4(y - \beta^2) + \lambda_5(z - \gamma^2).$$

First we deal with the last three terms. The equation $\partial\mathcal{L}/\partial\alpha = 0$ yields $\lambda_3(-2\alpha) = 0$, so either $\lambda_3 = 0$ or $\alpha = 0$. Similarly either $\lambda_4 = 0$ or $\beta = 0$, and either $\lambda_5 = 0$ or $\gamma = 0$. If any of λ_3, λ_4, λ_5 is nonzero, then at least one of α, β, γ is zero and therefore at least one of x, y, z is zero. Hence $V = 0$. Since $x,\ y,\ z \geq 0$, this corresponds to a global minimum. So we must take $\lambda_3 = \lambda_4 = \lambda_5 = 0$ for a maximum.

Note that if $x = y = z = v$, say, then the second constraint implies $v = 1$ and the first constraint $v^2 = 1/3$, a contradiction. Hence we can't have $x = y = z$.

The equation $\partial\mathcal{L}/\partial x = 0$ provides

$$yz + \lambda_1(y + z) + \lambda_2 = 0.$$

By symmetry we have also

$$zx + \lambda_1(z + x) + \lambda_2 = 0.$$

Subtraction yields

$$z(y - x) + \lambda_1(y - x) = 0 \quad \text{or} \quad (z + \lambda_1)(y - x) = 0.$$

Hence either

$$x = y \quad \text{or} \quad z = -\lambda_1.$$

By symmetry

$$y = z \quad \text{or} \quad x = -\lambda_1$$

and

$$z = x \quad \text{or} \quad y = -\lambda_1.$$

We can't have any two of $x = y$, $y = z$, $z = x$, for then $x = y = z$. Similarly we can't have all of $z = -\lambda_1$, $x = -\lambda_1$, $y = -\lambda_1$. Hence we must have

$$x = y,\ x = -\lambda_1,\ y = -\lambda_1, \quad \text{that is} \quad x = y = -\lambda_1,$$

or

$$y = z = -\lambda_1,$$

or

$$z = x = -\lambda_1.$$

With the first possibility, the second constraint gives $z = 3 - 2x$ and the first constraint $x^2 + 2xz = 1$, so

$$x^2 + 2x(3 - 2x) = 1 \quad \text{or} \quad 3x^2 - 6x + 1 = 0.$$

Hence

$$x = 1 \pm \frac{\sqrt{6}}{3} \quad \text{and therefore} \quad y = 1 \pm \frac{\sqrt{6}}{3} \quad \text{and} \quad z = 1 \mp \frac{2\sqrt{6}}{3}.$$

Now $1 - 2\sqrt{6}/3 < 0$, so to get $V > 0$ we must choose

$$x = y = 1 - \frac{\sqrt{6}}{3}, \quad z = 1 + \frac{2\sqrt{6}}{3}$$

which leads to

$$V = xyz = -1 + \frac{4\sqrt{6}}{9}.$$

The same value arises from the cyclic permutations

$$y = z = 1 - \frac{\sqrt{6}}{3}, \quad x = 1 + \frac{2\sqrt{6}}{3}$$

and

$$z = x = 1 - \frac{\sqrt{6}}{3}, \quad y = 1 + \frac{2\sqrt{6}}{3}.$$

2. 0 marks Maximize $V = x^2 + 2y^2 - z^2$ subject to

$$x^2 + y^2 + z^2 \leq 1$$

Solution: Use a slack variable u to express the constraint as an equality

$$x^2 + y^2 + z^2 + u^2 = 1.$$

This leads to a new objective function

$$\mathcal{L} = V + \lambda\left(x^2 + y^2 + z^2 + u^2 - 1\right).$$

The condition $\partial L/\partial x = 0$ gives

$$2x + \lambda.2x = 0, \quad \text{so} \quad 2x(1 + \lambda) = 0 \quad \text{and} \quad x = 0 \quad \text{or} \quad \lambda = -1.$$

Partial differential with respect to y, z, u in turn give similarly

$$y = 0 \quad \text{or} \quad \lambda = -2;$$

$$z = 0 \quad \text{or} \quad \lambda = 1;$$

$$u = 0 \quad \text{or} \quad \lambda = 0.$$

By the constraint, we can't have all of x, y, z, u equal to zero. Hence we must have either

$$\lambda = -1,\ y = z = u = 0, \quad \text{or}$$

$$\lambda = -2,\ x = z = u = 0, \quad \text{or}$$

$$\lambda = 1,\ x = y = u = 0, \quad \text{or}$$

$$\lambda = 0,\ x = y = z = 0.$$

In the first case, the constraint gives $x^2 = 1$ and $V = 1$. Similarly the other three cases in turn yield $y^2 = 1$ and $V = 2$; $z^2 = 1$ and $V = -1$; $u^2 = 1$ and $V = 0$. Thus the maximum is $V = 2$, and this arises when $x = z = 0$ and $y = \pm 1$.

3. 5 marks Which of the following are functionals of the function $y(x)$ (label yes or no).
Solution:

(a)	$y(0)+4$	yes
(b)	$\left.\frac{dy}{dx}\right\|_0$	yes (assuming the derivative exists)
(c)	$\min\{y(x)\|0 \leq x \leq 1\}$	yes (assuming the minimum exists)
(d)	$\int_0^1 y\,dx$	yes
(e)	$\int_0^\pi \left[\frac{d^n y}{dx^n}\right]^3 f(x)\,dx$	yes (assuming the derivatives exist)

4. 1 mark Given the L^2-norm $||f||_2 = \sqrt{\int_0^1 f(x)^2\,dx}$ on the vector space $L^2[0,1]$, describe (in one sentence) the ε-neigbourhood of the function $y = x$.

Solution:

The ε-neigbourhood of the function $y = x$ is the set of functions within distance ε of $y = x$, where distance is defined using the L^2 norm of the difference between two functions.

5. 4 marks Find an upper bound for the minimum of the functional

$$J\{y\} = \int_0^1 y^2 y'^2\,dx,$$

subject to $y(0) = 0$ and $y(1) = 1$ using the trial functions

$$y_\varepsilon(x) = x^\varepsilon,$$

with $\varepsilon > 1/4$. Justify your argument.

Solution:

$$\begin{aligned} y_\varepsilon &= x^\varepsilon \\ y'_\varepsilon &= \varepsilon x^{\varepsilon-1} \end{aligned}$$

and so

$$\begin{aligned} J\{y_\varepsilon\} &= \int_0^1 y^2 y'^2\,dx \\ &= \varepsilon^2 \int_0^1 x^{4\varepsilon-2}\,dx \\ &= \varepsilon^2 \left[\frac{x^{4\varepsilon-1}}{4\varepsilon-1}\right]_0^1 \\ &= \frac{\varepsilon^2}{4\varepsilon-1} \\ \frac{dJ\{y_\varepsilon\}}{d\varepsilon} &= \frac{2\varepsilon(4\varepsilon-1)-4\varepsilon^2}{(4\varepsilon-1)^2} \end{aligned}$$

At a stationary point the derivative is zero and so we require the denumerator of $dJ/d\varepsilon$ to be zero, i.e.,

$$2\varepsilon^2 - \varepsilon = 2\varepsilon(2\varepsilon - 1) = 0,$$

so $\varepsilon = 0$ or $1/2$, but only the latter solution is greater than $1/4$, and so this is a stationary point.

We can see it is a minimum by taking the second derivative with respect to ε to get

$$\begin{aligned} \frac{d^2 J\{y_\varepsilon\}}{d\varepsilon^2} &= \frac{d}{d\varepsilon}\frac{4\varepsilon^2 - 2\varepsilon}{(4\varepsilon-1)^2} \\ &= \frac{2}{(4\varepsilon-1)^3} \end{aligned}$$

which is positive for $\varepsilon > 1/4$.

Calculating $J\{y_\varepsilon\} = \frac{\varepsilon^2}{4\varepsilon-1}$ at the minimum we get $J\{y^*\} = 1/4$, which is an upper bound on the true minimum of the functional, because the functional applies over a wider class of possible functions y, and we know that there may be a better one.