# Transform Methods & Signal Processing

## lecture 08

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

Discipline of Applied Mathematics
School of Mathematical Sciences
University of Adelaide

July 27, 2009

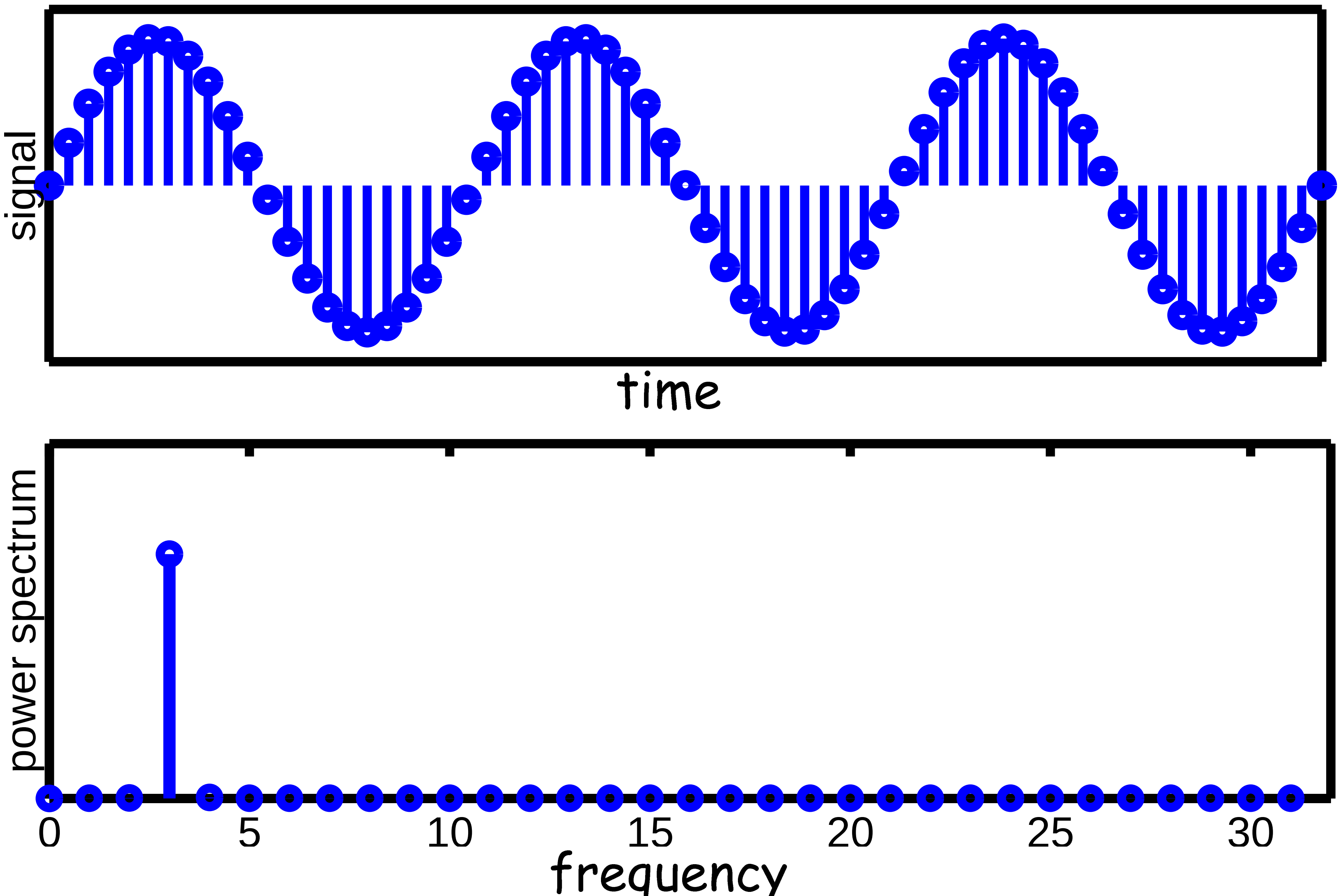
# Windows

No we don't mean the common operating system. Windows are a way of minimizing leakage when performing Fourier transforms, but they lead into a more sophisticated time-sensitive versions of the Fourier Transform called the Short-Time Fourier Transform.

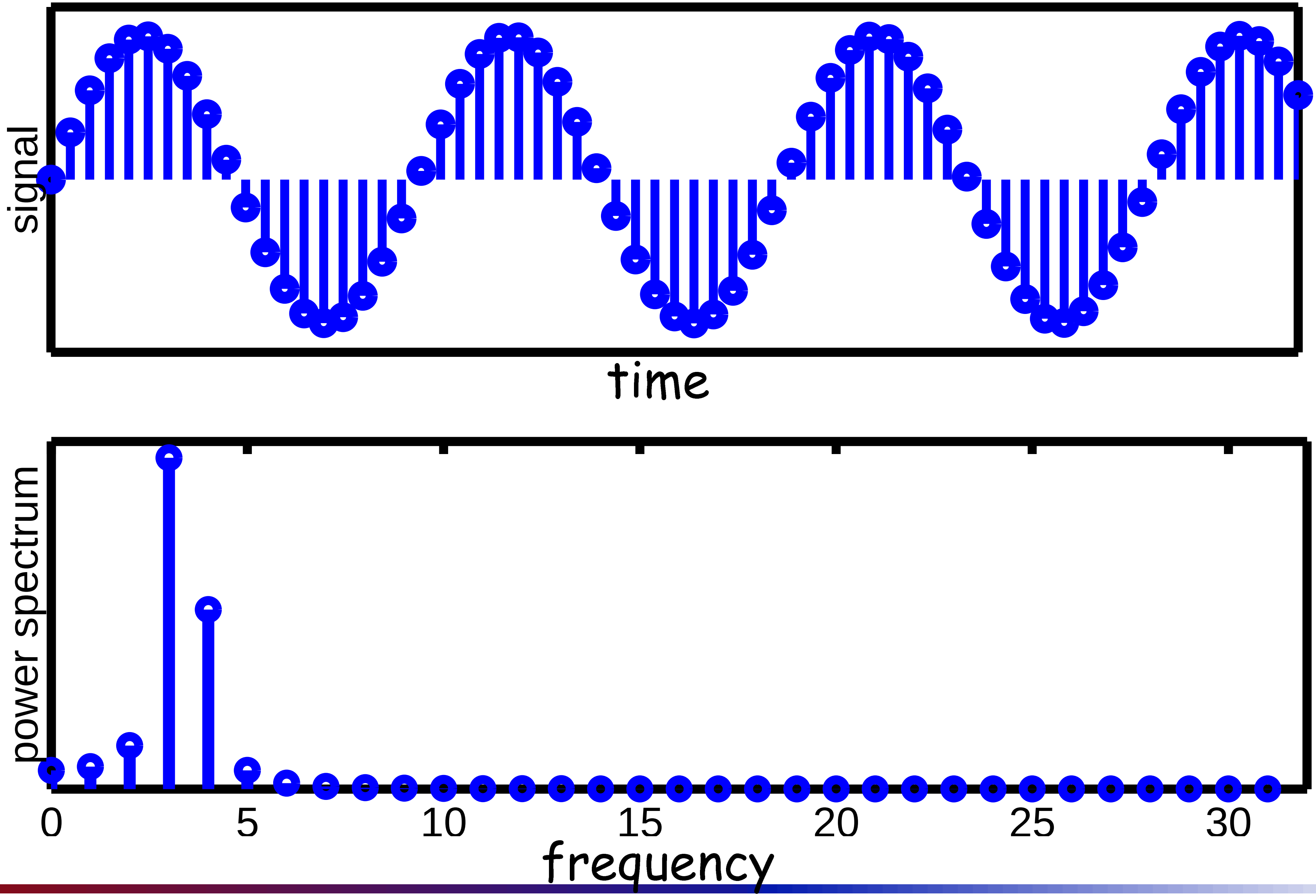
# Leakage

- always have finite signals
- implicit assumption in DFT is periodicity
  - we look at correlation of signal to sin's and cosines with periods that match the length of the data
- What if a signal is not periodic?
- What if the period is not the same as the length of the data?
- We get leakage

# Leakage example



# Leakage example



# What causes leakage

- The DFT uses a finite number of frequencies.
- Not all signals fit this mold exactly: what happens to sinusoids with non-integral frequencies?
- Their power is spread over a few frequencies.
- Note we are representing the signal by a series of numbers $X(k)$ which represent the correlation of the signal to a particular sinusoid with freq. $kf_s/N$,
- another way to understand, is to think of each element $X(k)$ of the DFT as a narrow bandpass filter, centered on frequency $kf_s/N$, but which have side lobes.

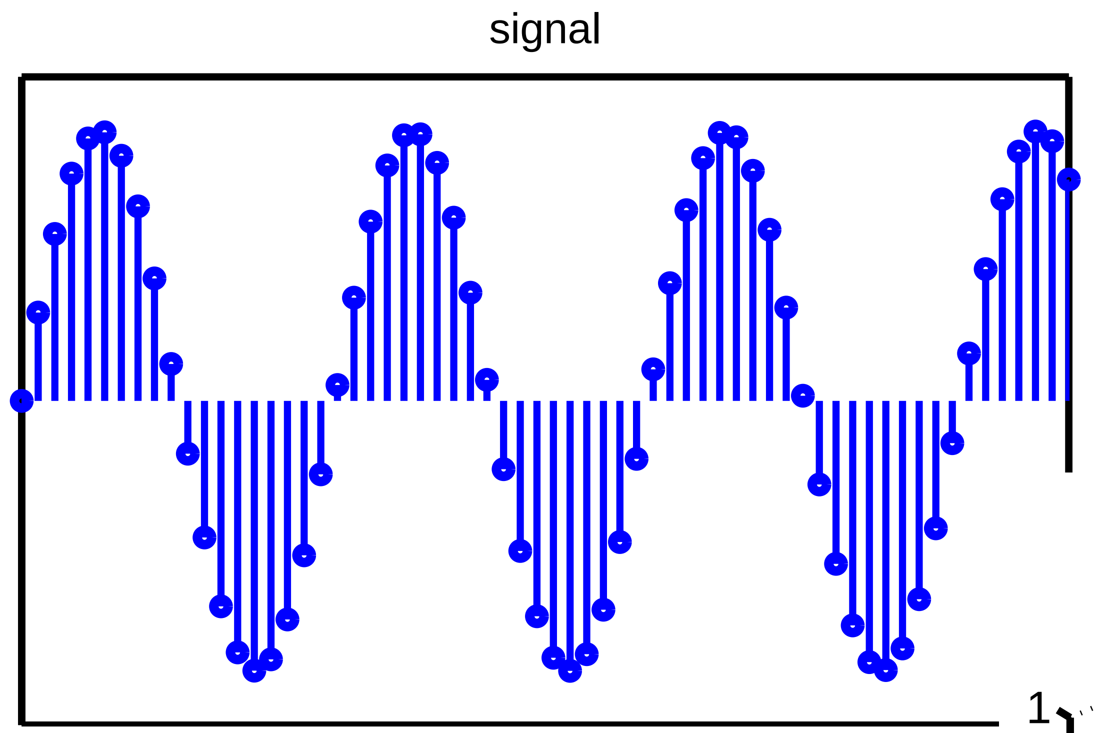
# Alternative view

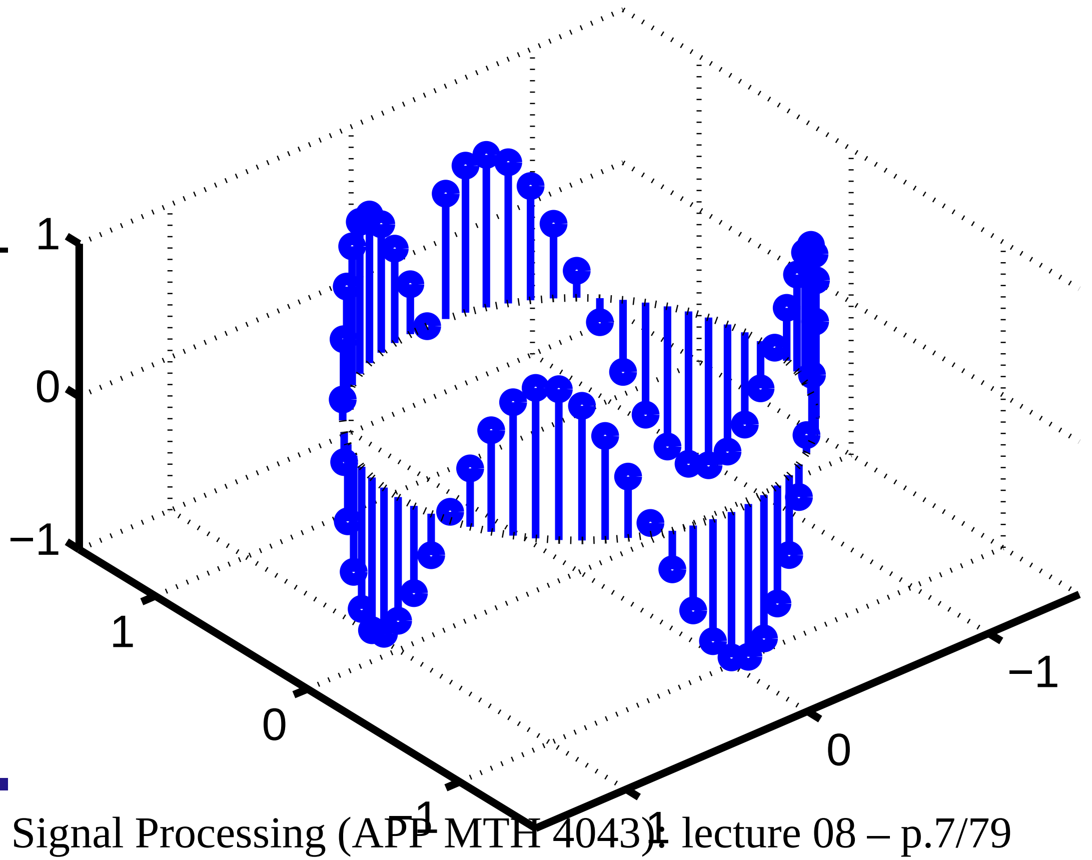
- alternative view: DFT truncated signal implicitly assumes signal is **periodic**, but it isn't, so what happens at the edges?
- Edges induce transients
- transients introduce extra frequency components

Why do we care?

- side lobes reduce **sensitivity**
- determine the smallest signal we can detect against a background of another signal

# Periodic signal view



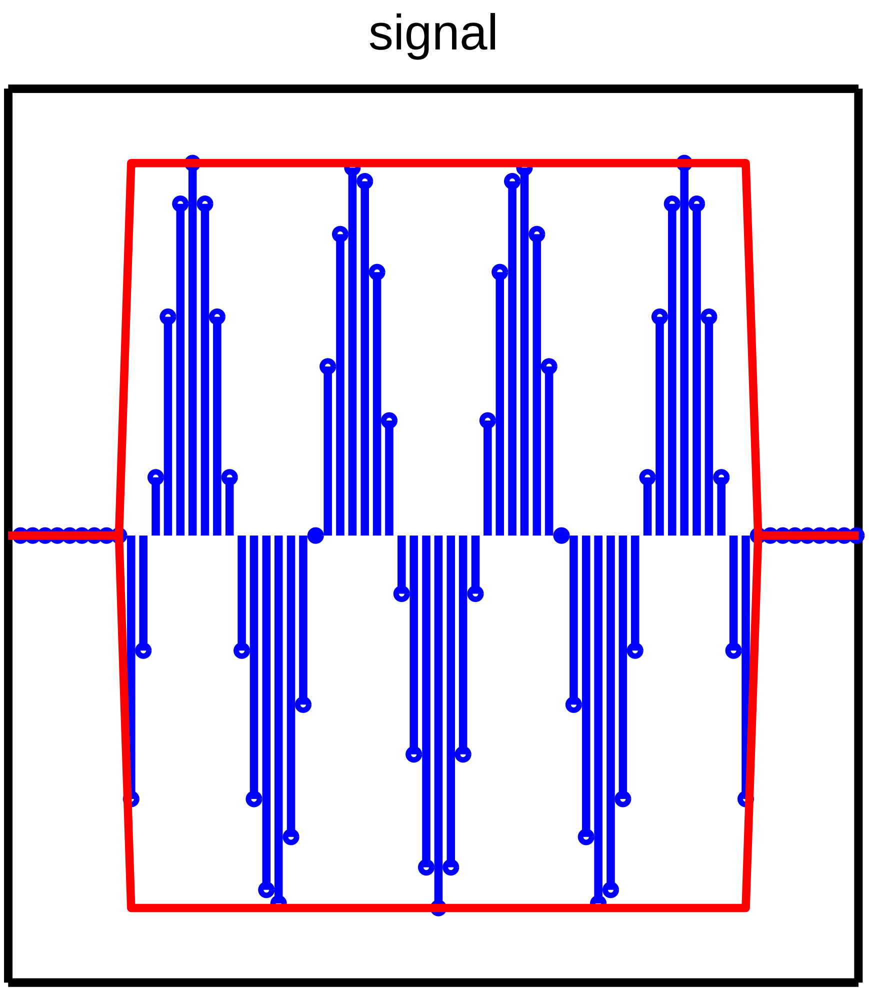
# Yet another view

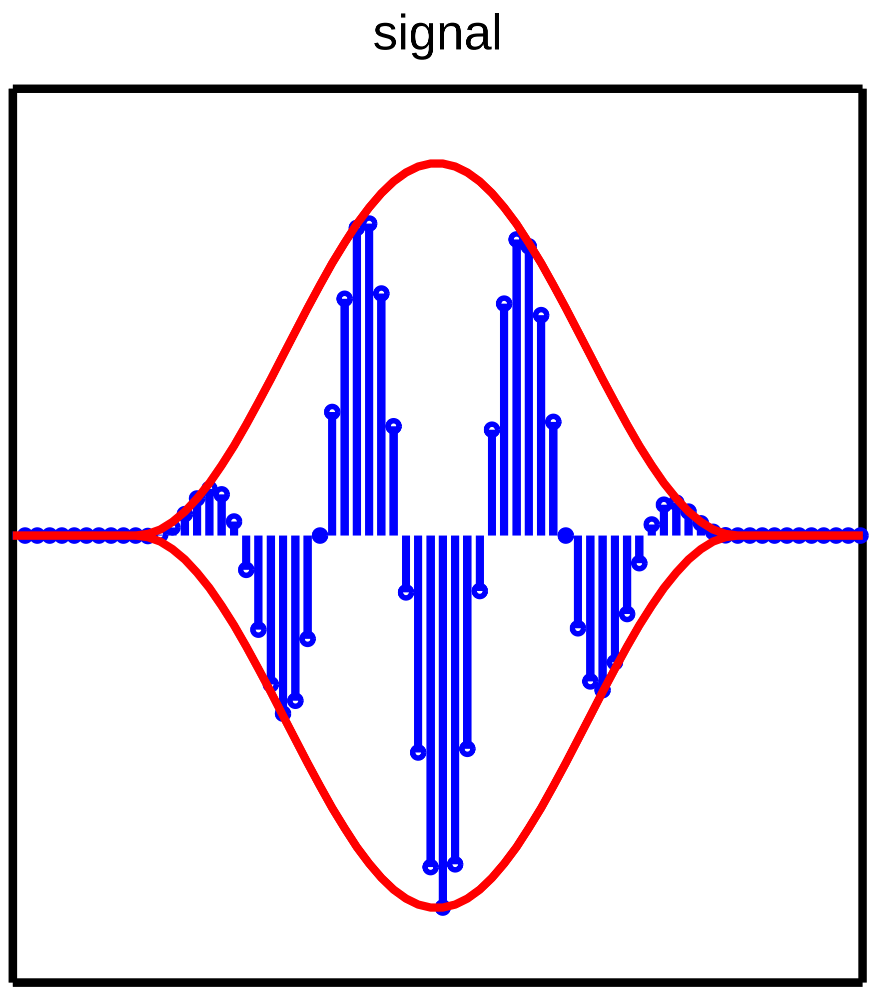
The signal can be thought of as an infinite duration signal, that has been truncated by rectangular window.

- input signal is the result of the product of an (infinite) signal with a rectangular window
  - convolution property
  - resulting FT is the convolution of the FT of the rectangle (a sinc), with the FT of the signal
  - FT of a rectangle is a sinc function
- what happens if we use a smoother window?

# Windows

Windows reduce the transient at the edges, but giving edge points less weight, e.g.



# Impact of Windows

Product of the signal with a window function

product in time domain = convolution in frequency domain

- just have to look at transfer functions of windows.
- want to reduce size of side-lobes
- we can choose our own window function!

Note that windows may drop the overall power of the signal so (by Rayleigh-Parseval) the power in the output signal drops. However, relative magnitudes are more important here than absolutes!

# Windows

All defined for $n = 0, 1, \ldots, N-1$

- Rectangular (default) $w_N(n) = 1$
- Bartlett (triangular) $w_N(n) = 1 - \left| \frac{n-N/2}{N/2} \right|$
- Welch (Riesz) $w_N(n) = 1 - \left( \frac{n-N/2}{N/2} \right)^2$
- Hanning $w_N(n) = 0.5 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right)$
- Hamming $w_N(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right)$
- Blackman $w_N(n) = 0.42 - 0.5 \cos\left(\frac{2\pi n}{N}\right) + 0.08 \cos\left(\frac{4\pi n}{N}\right)$
- Blackman-Harris (3 term)
  $w_N(n) = 0.42323 - 0.49755 \cos\left(\frac{2\pi n}{N}\right) + 0.07922 \cos\left(\frac{4\pi n}{N}\right)$
- Gaussian $w_N(n) = \exp\left[ -4.5 \left( \frac{n-N/2}{N/2} \right)^2 \right]$

# Basis for Comparison

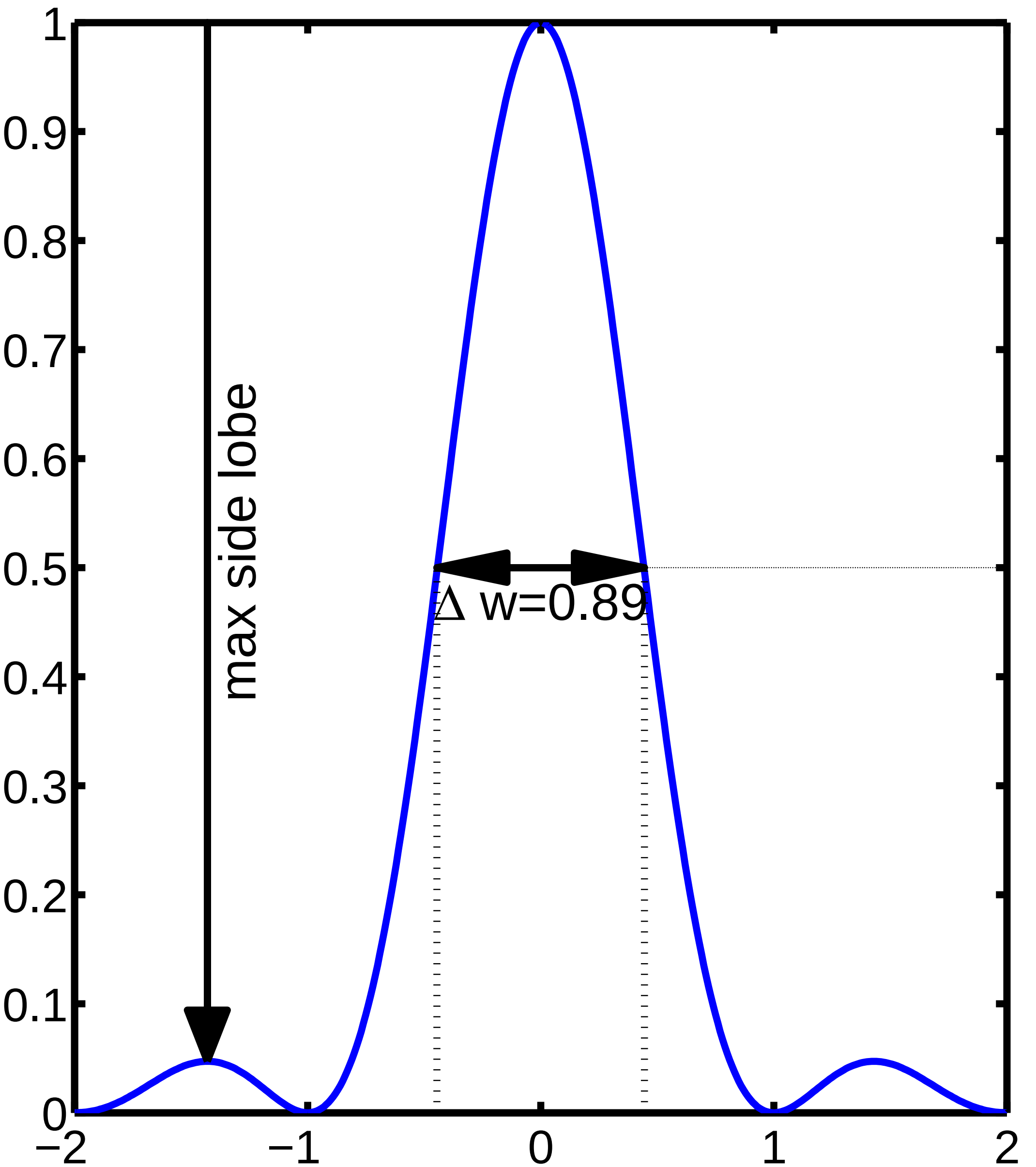
- measure drop to largest side-lobe
  - measure of sensitivity
- also measure “width” of the windows frequency response by looking at where the power drops off by a factor of a half, i.e. we find $\Delta\omega$ such that

$$\frac{|F(\Delta\omega/2)|^2}{|F(0)|^2} = \frac{1}{2}$$

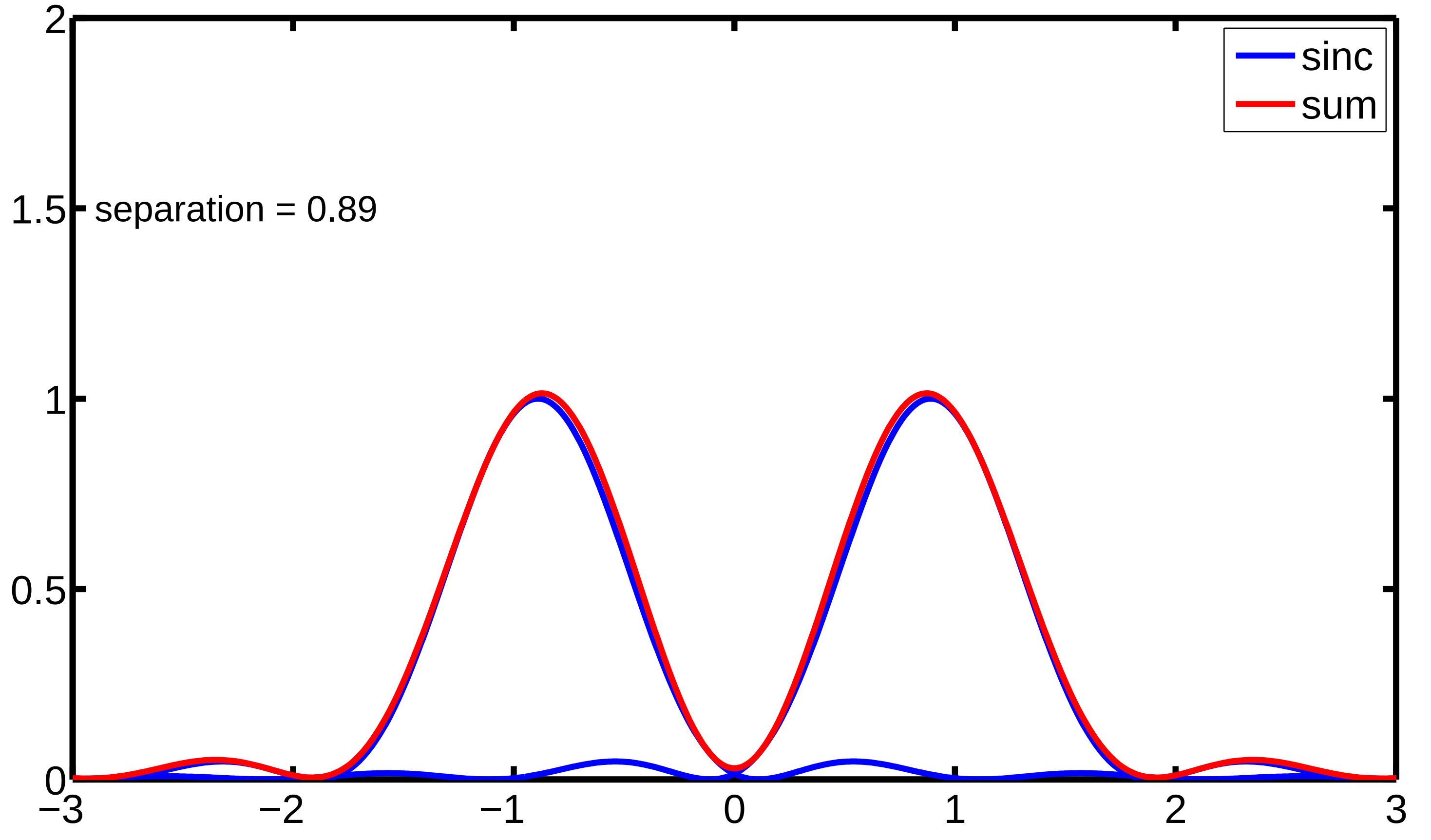
  - minimum resolution bandwidth
  - two peaks of same magnitude have to be at least this far apart to resolve them as separate
- we will also look at some other properties in a minute

# Rectangular Window

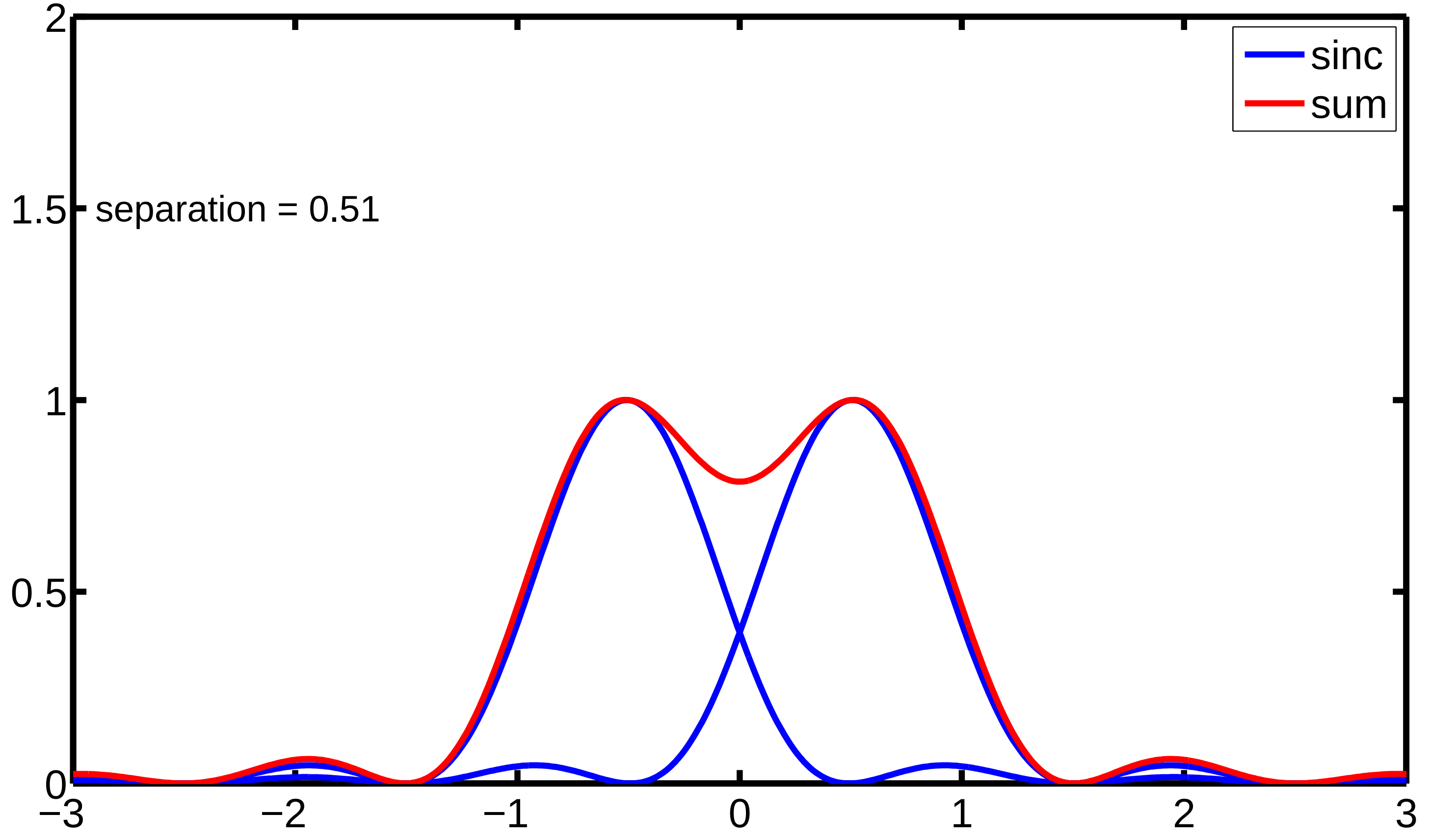
- calculate results for rectangular window
- $\mathcal{F}\{r(t)\} = \text{sinc}(s)$
- figure shows $\text{sinc}(s)^2$
- $\Delta w = 0.89$ is the width
- drop to the max side lobe is $-13$ dB

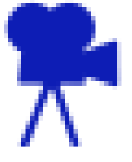


# Rectangular Window

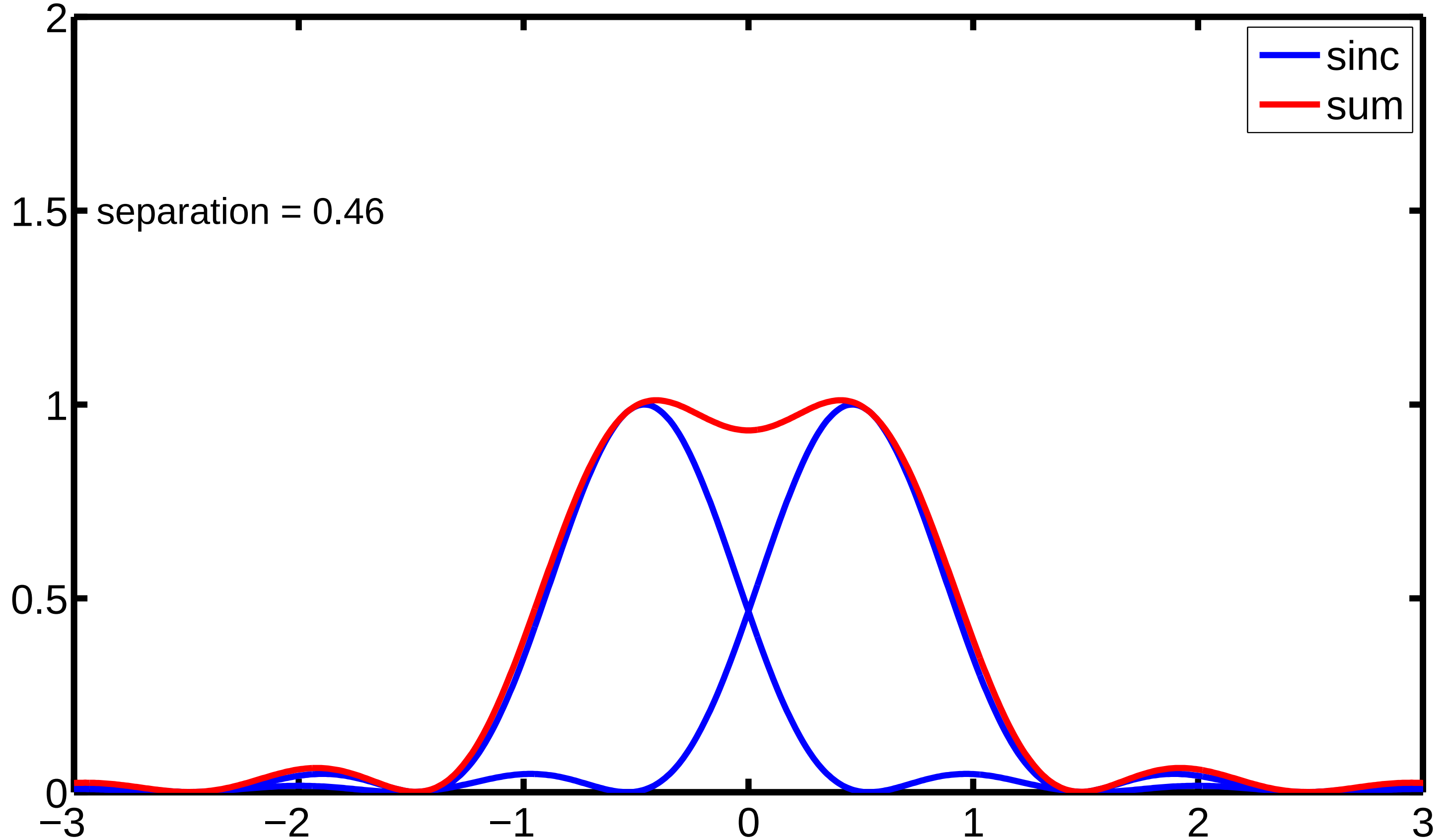


# Rectangular Window



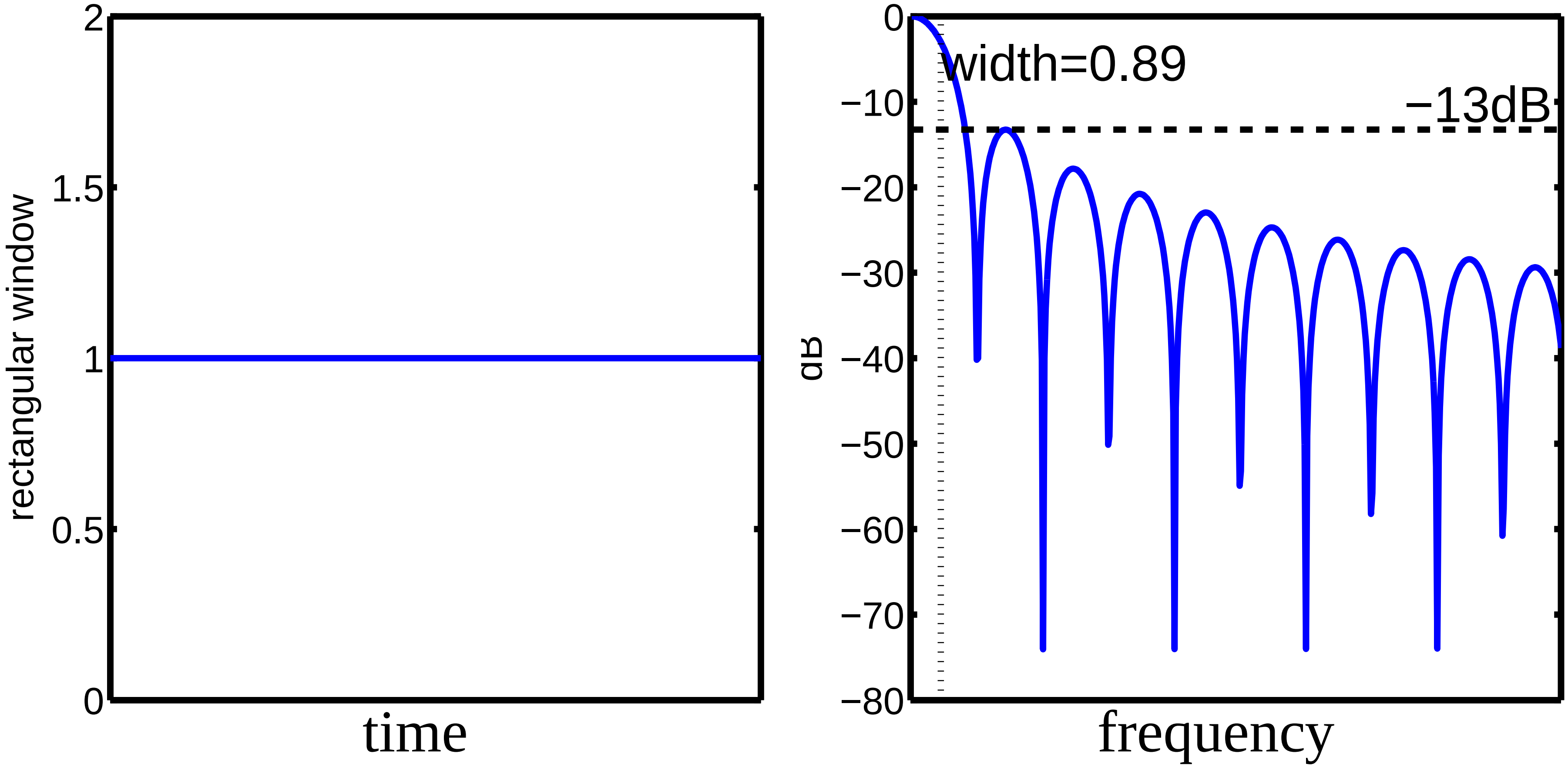



# Rectangular Window



# Rectangular Window

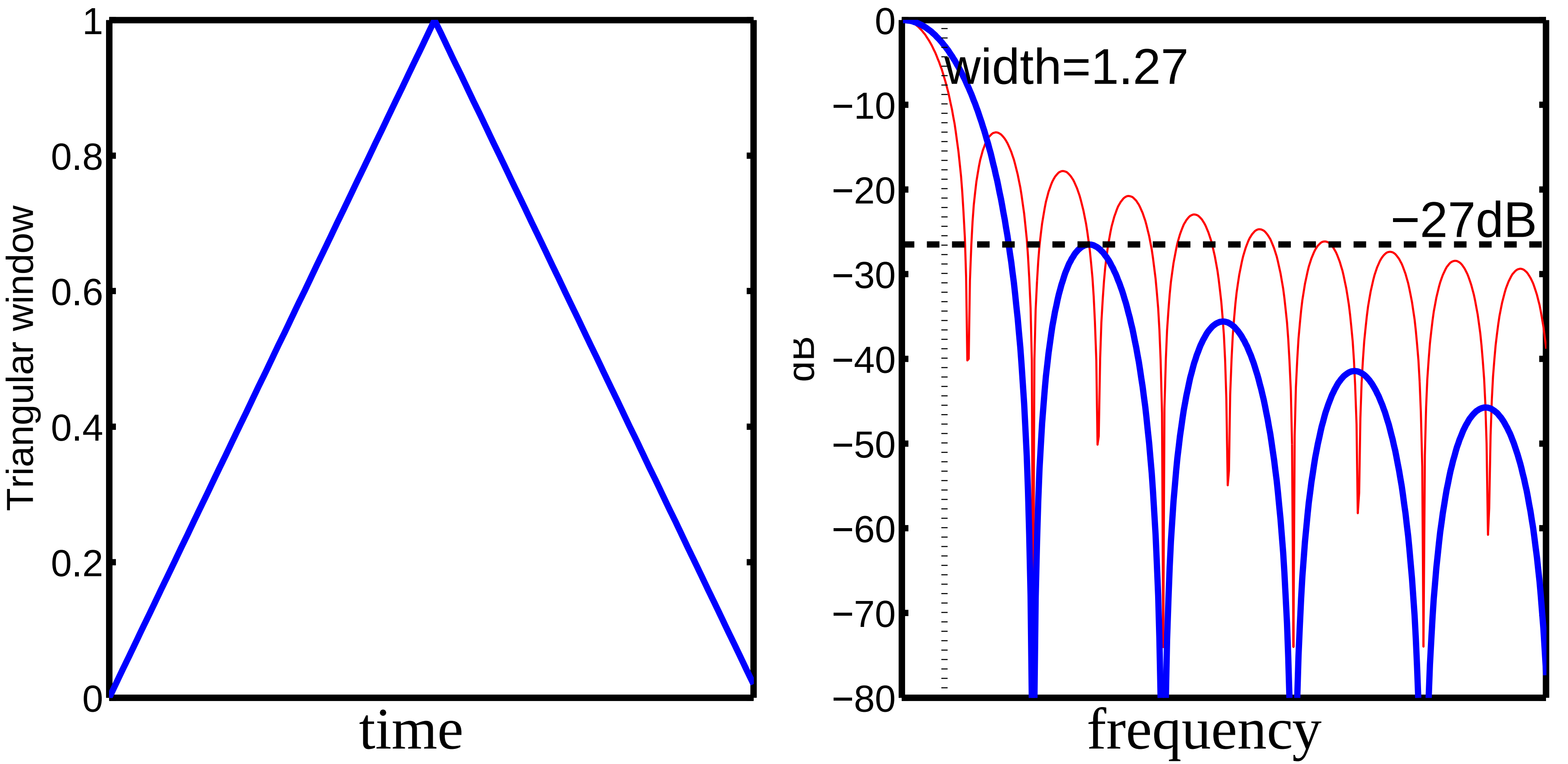
The window you have when you don't have a window




$$w_N(n) = 1$$

# Triangular Window

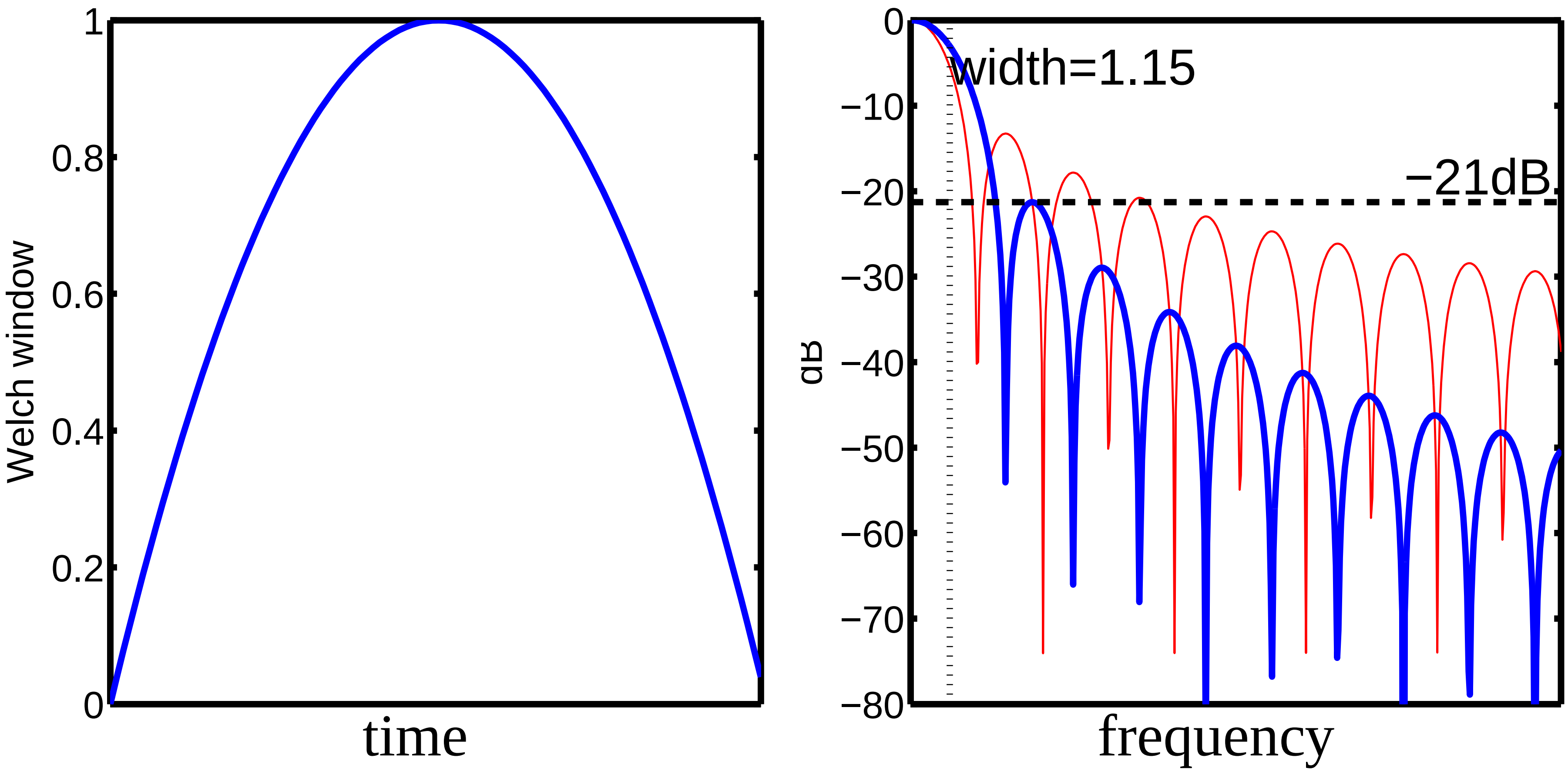
Reduce the size of the discontinuity




$$w_N(n) = 1 - \left| \frac{n - N/2}{N/2} \right|$$

# Welch Window

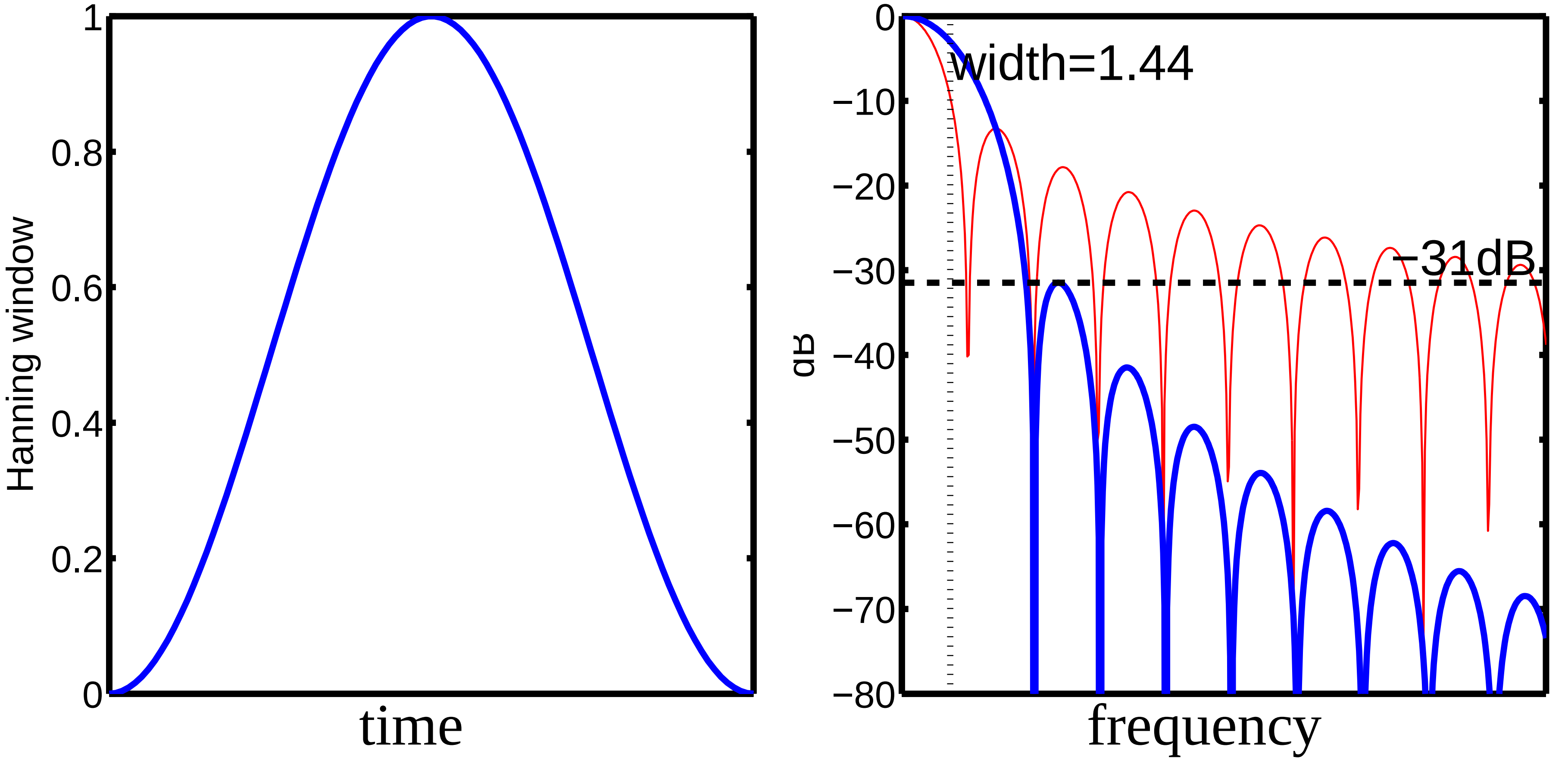
Reduce the size of the discontinuity, but keep power.




$$w_N(n) = 1 - \left( \frac{n - N/2}{N/2} \right)^2$$

# Hanning Window

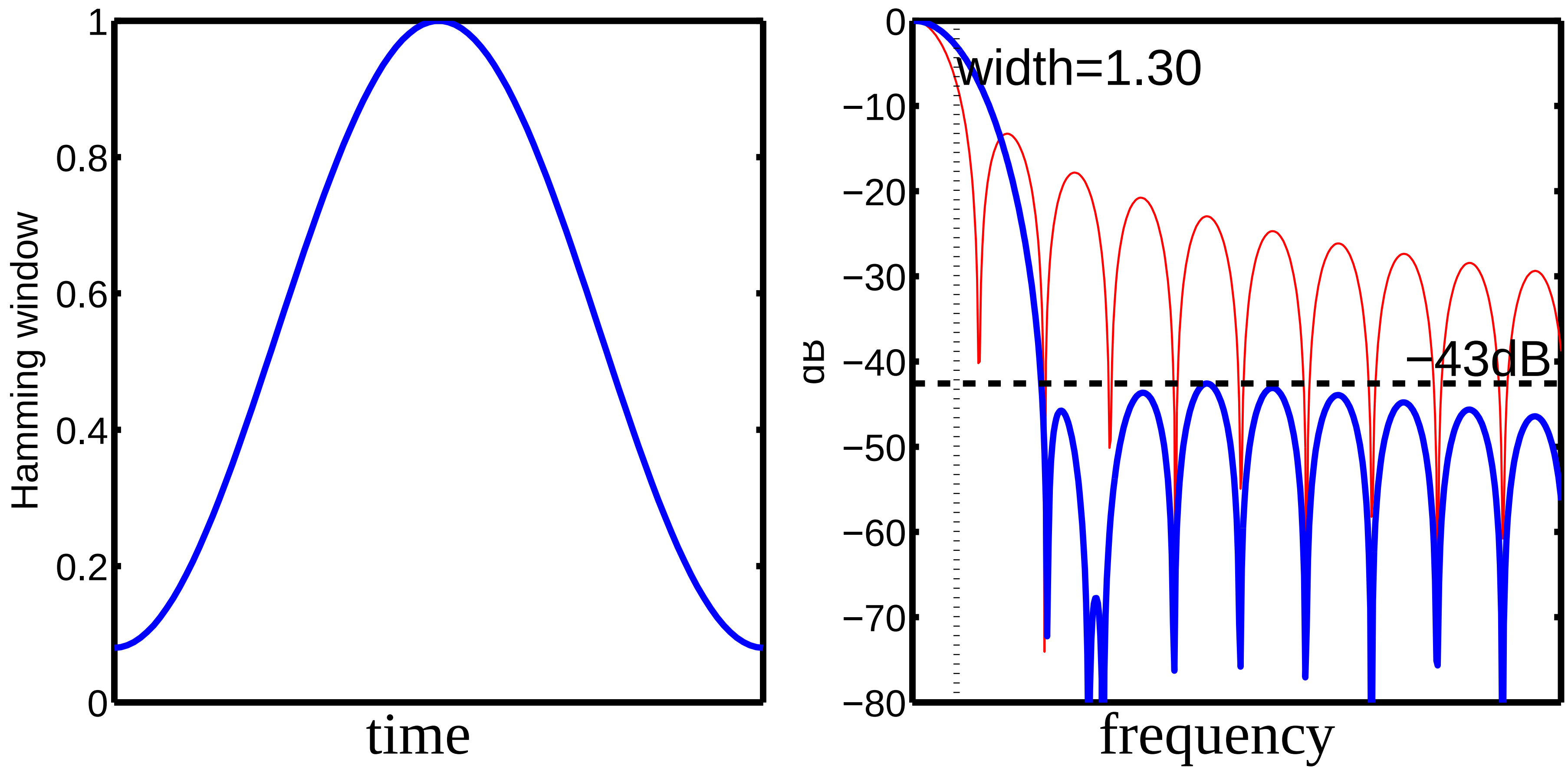
Make the discontinuity smooth, as well as small.




$$w_N(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)$$

# Hamming Window

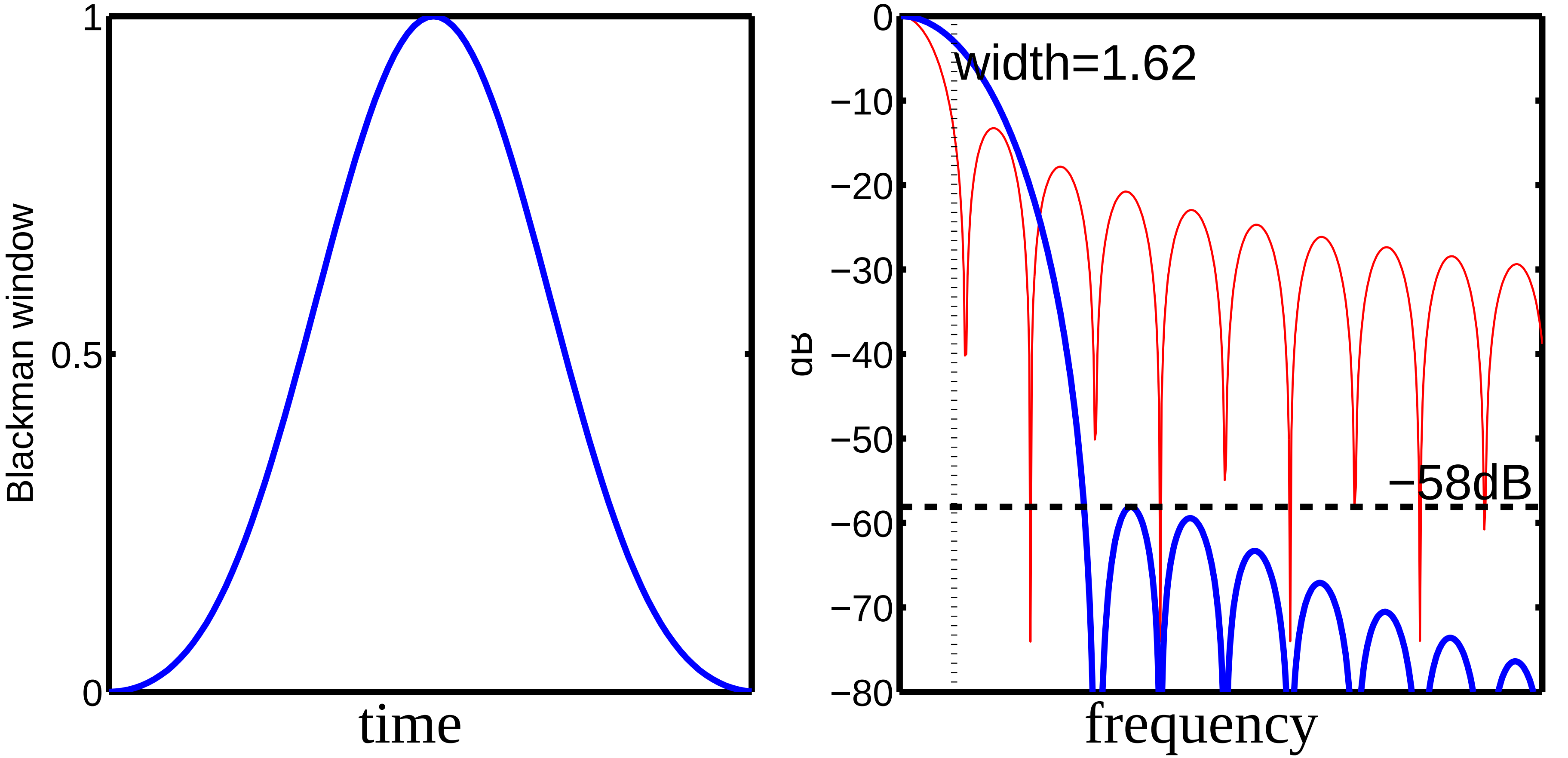
Side-lobe reduction, from $w_N(n) = \alpha + (1-\alpha)\cos\left(\frac{2\pi n}{N}\right)$




$$w_N(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$$

# Blackman Window

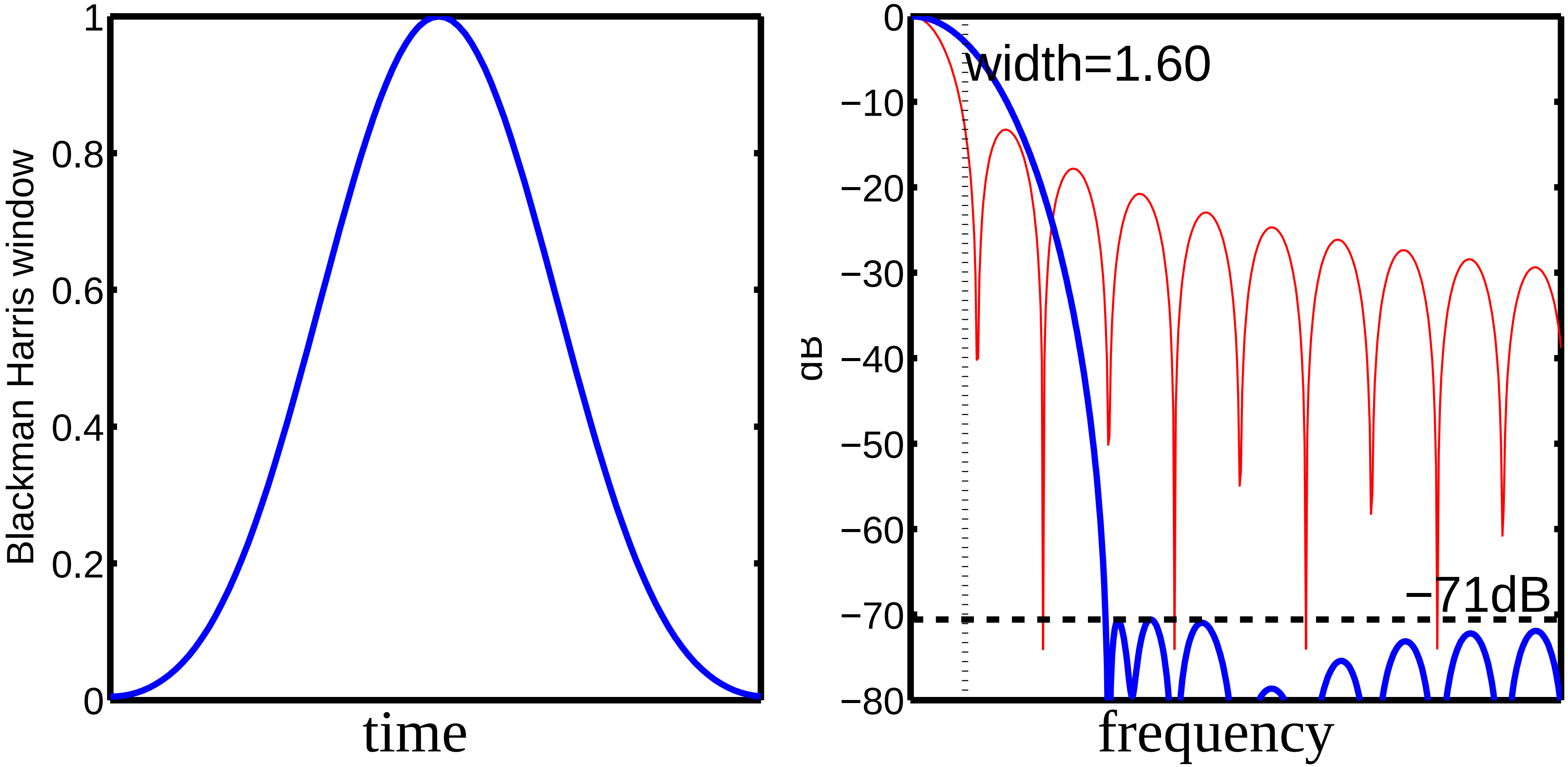
Side-lobe reduction through an extra term.




$$w_N(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N}\right) + 0.08\cos\left(\frac{4\pi n}{N}\right)$$

# Blackman-Harris Window (3 term)

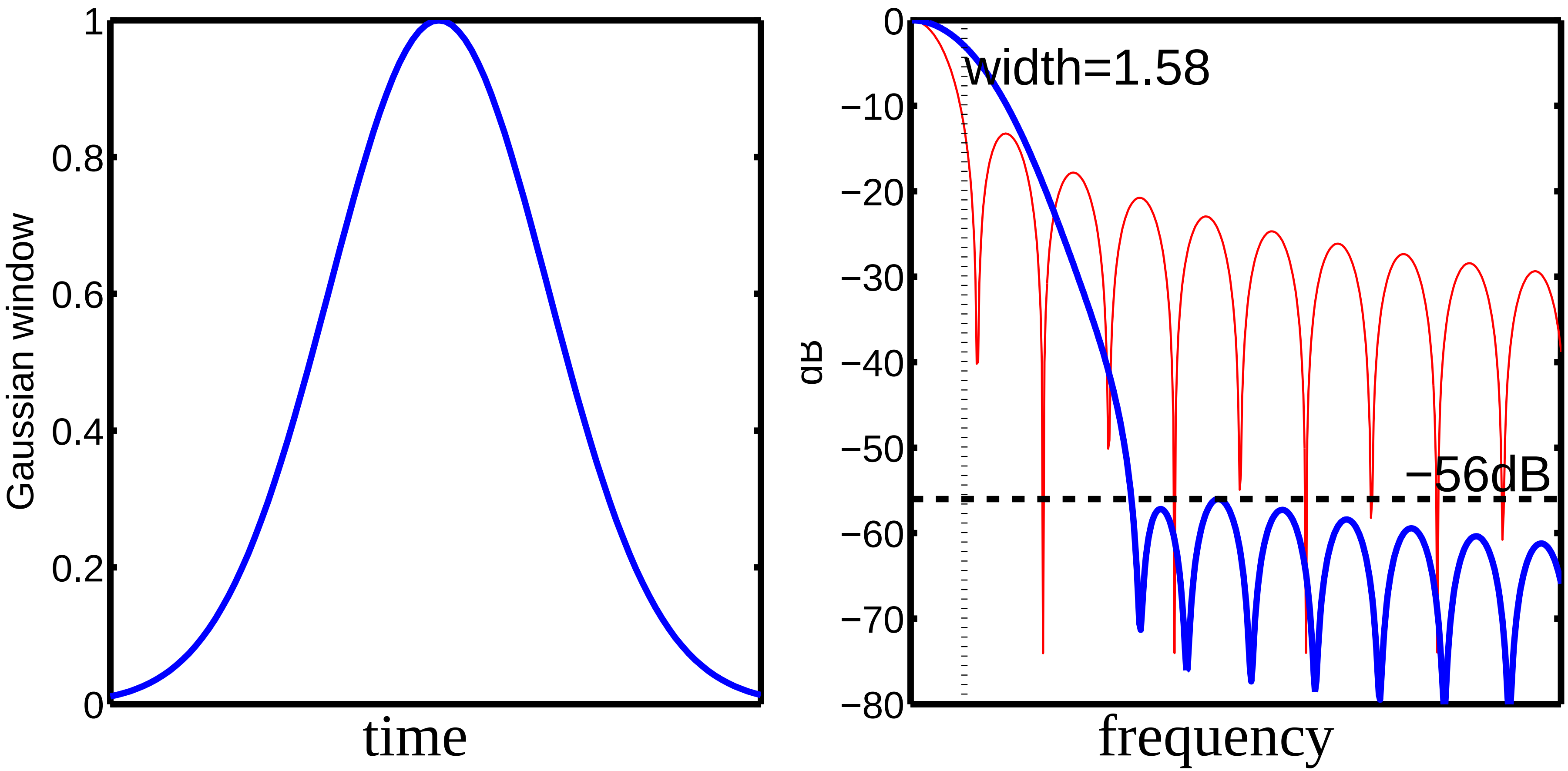
Optimize Blackman.




$$w_N(n) = 0.42323 - 0.49755 \cos\left(\frac{2\pi n}{N}\right) + 0.07922 \cos\left(\frac{4\pi n}{N}\right)$$

# Gaussian Window

Minimum uncertainty (see later)




$$w_N(n) = \exp\left[-4.5\left(\frac{n - N/2}{N/2}\right)^2\right]$$

# Windowing

There is a tradeoff between resolution and sensitivity!

- better sensitivity (lower side-lobes)
  $\Rightarrow$ less resolution
- better resolution (of frequencies)
  $\Rightarrow$ worse sensitivity

Another tradeoff in the roll-off of side lobes.

- smoother function
  $\Rightarrow$ steeper roll-off
  but less drop off in first side-lobe

Some windows have a parameter that can tune the tradeoffs

# Tunable tradeoffs

Windows with tunable tradeoffs

- Kaiser-Bessel

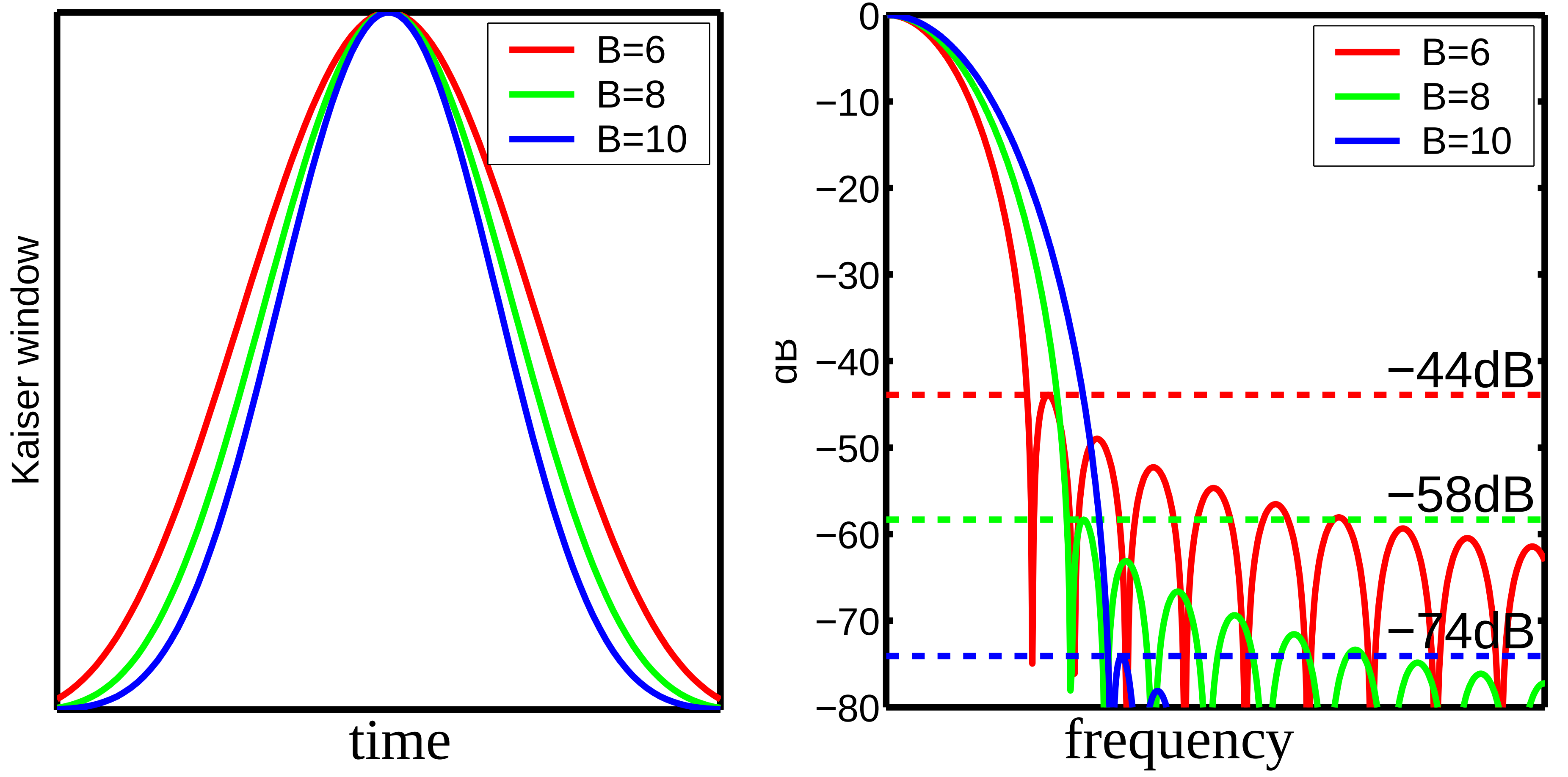
$$w_N(n) = \frac{I_0\left[B\sqrt{1-\left(\frac{n-p}{p}\right)^2}\right]}{I_0(B)}$$

where $p = \frac{N-1}{2}$, and $I_0$ is the zero order modified Bessel function of the first kind, given by

$I_0(x) = 1 + \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2}$

Choosing different values of $B$ tunes the tradeoff

# Kaiser Window




$$w_N(n) = I_0\left[B\sqrt{1-\left(\frac{n-p}{p}\right)^2}\right] / I_0(B)$$

# Tunable tradeoffs

Windows with tunable tradeoffs

- Chebyshev (Dolph-Chebyshev)
  $w_N(n) =$ the N-point inverse DFT of

$$\frac{\cos\left[N\cos^{-1}\left(\alpha\cos\left(\frac{\pi m}{N}\right)\right)\right]}{\cosh(N\cosh^{-1}(\alpha))}$$

where

$$\alpha = \cosh\left[\frac{1}{N}\cosh^{-1}\left(10^{\gamma}\right)\right]$$

not shown here

# Tunable tradeoffs

Windows with tunable tradeoffs

- Gaussian (tune the standard deviation)

$$w_N(n) = \exp\left[-\alpha\left(\frac{n-N/2}{N/2}\right)^2\right]$$

We find size of the discontinuity at the edge of the window by taking $n = 0$, e.g. it is $\exp(-\alpha)$. The side-lobes from such an edge will resemble the rectangular side-lobes, with their -13dB attenuation, and so the side-lobes of the Gaussian will be approximately

$$\text{side-lobe} = -13 + 20\log_{10} e^{-\alpha} = -13 - 20\alpha \log_{10} e$$

Actually they vary from this a little, but the relationship is useful, as we can also predict the width of the Gaussian window precisely as it is just a scaled version of itself.

# The Uncertainty Principle

We have seen there are basic tradeoffs in window choice. The uncertaintly principle shows that these tradeoffs are fundamental and unavoidable.

# Uncertainty principle

The tradeoff relates to a general principle: **uncertainty**

- We can't squeeze more information out of a sequence
- we can only change the way we see the information
- here we tradeoff sensitivity for resolution

Scaling property of FTs tells us something

$$f(at) \rightarrow \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

- if we make the window 'narrower' to exclude more of the transients (that cause leakage), then we make the FT 'wider'

# Uncertainty principle

Another way to understand

- frequency resolution depends on the number of data points in our dataset
- Windowing reduces the power from some data points
- a little like reducing the number of data points
- so we need a longer data sequence for a finer resolution

# Regularity and decay

We can extend the intuition from the above by looking at relationship between regularity of the function $f(t)$ and the decay rate of $|F(s)|$, e.g.

If there exists a constant $K$, and $\varepsilon > 0$ such that

$$|F(s)| \leq \frac{K}{1+|s|^{p+1+\varepsilon}}$$

Then $f$ has at least $p$ continuous derivatives.

Hence, if $F(s)$ has compact support then $f \in C^{\infty}$.

# Regularity and decay

**Proof:** By definition of the IFT

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{i2\pi st}\, ds$$

If $F \in L^1(\mathbb{R})$ then the above implies $f$ is continuous and bounded, because

$$|f(t)| \leq \int_{-\infty}^{\infty} |F(s) e^{i2\pi st}|\, ds = \int_{-\infty}^{\infty} |F(s)|\, ds$$

Take the $k$th order derivative, WRT to $t$, and we get

$$|f^{(k)}(t)| \leq \int_{-\infty}^{\infty} |(i2\pi s)^k F(s) e^{i2\pi st}|\, ds \leq (2\pi)^k \int_{-\infty}^{\infty} |s|^k\, |F(s)|\, ds$$

# Regularity and decay

**Proof:** Now, if

$$|F(s)| \leq \frac{K}{1+|s|^{p+1+\varepsilon}}$$

Then,

$$\int_{-\infty}^{\infty} |F(s)| \left(1+|s|^p\right) ds \leq \int_{-\infty}^{\infty} \frac{K(1+|s|^p)}{1+|s|^{p+1+\varepsilon}} \, ds < \infty$$

which also implies that

$$\int_{-\infty}^{\infty} |F(s)| \, |s|^k \, ds < \infty$$

for all $k \leq p$, so the derivative $f^{(k)}(t)$ exists and is bounded.

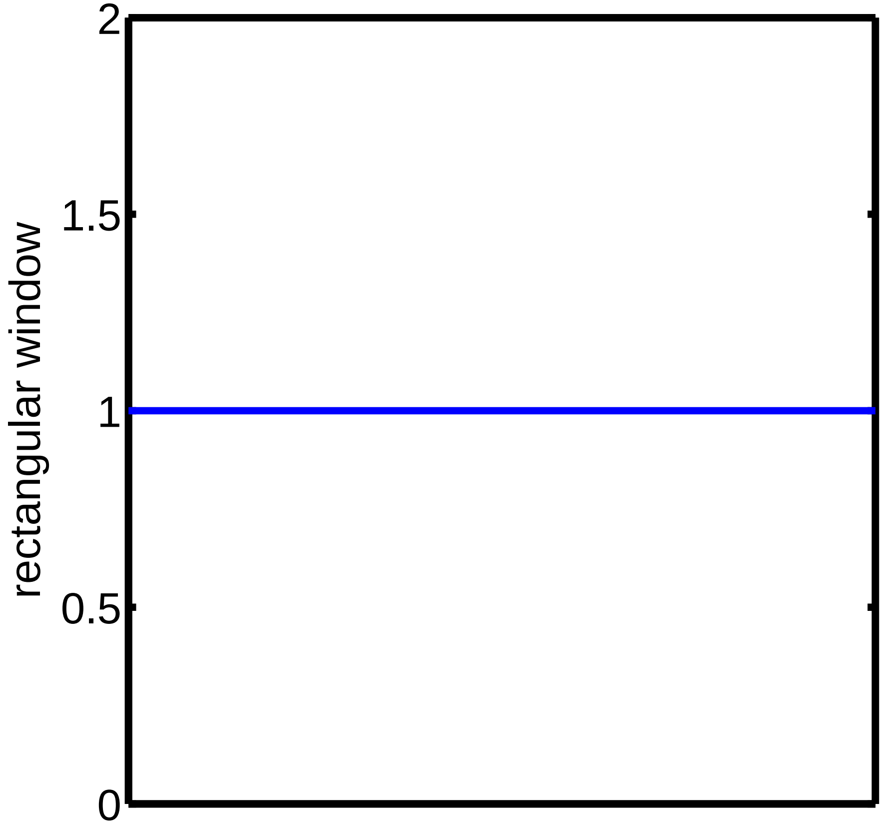
# Regularity and decay

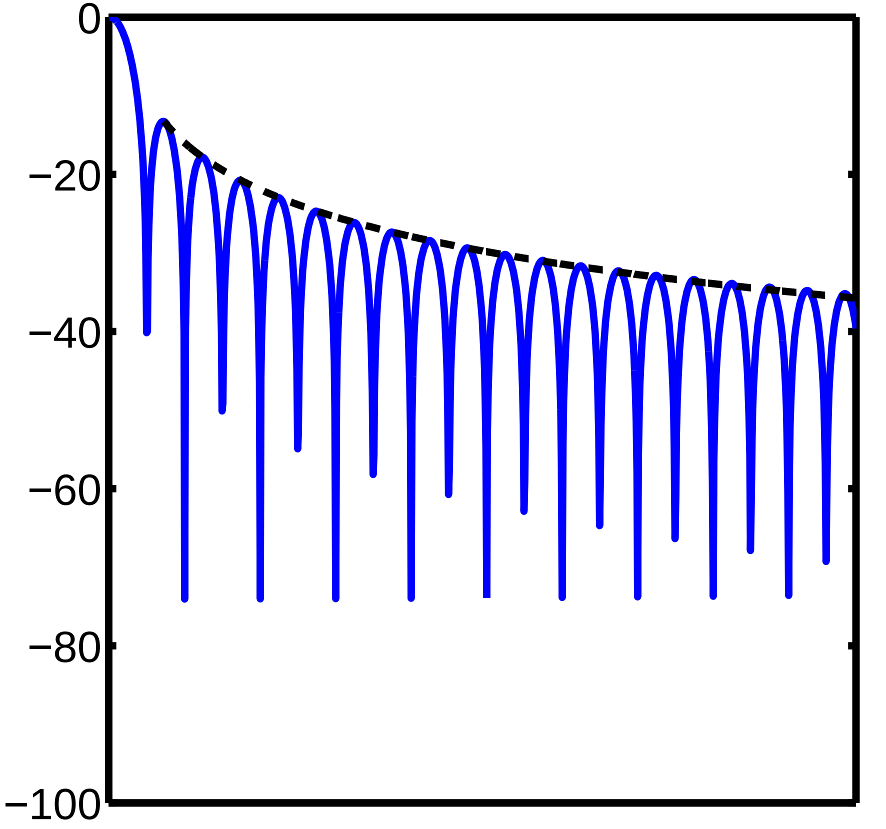
Windowing examples: consider 5 windowing functions

- **Rectangular:** $w_N(n) = 1$
  This has a discontinuity.
- **Triangular:** $w_N(n) = 1 - \left| \frac{n-N/2}{N/2} \right|$
  This has a discontinuity in the first derivative.
- **Welch:** $w_N(n) = 1 - \left( \frac{n-N/2}{N/2} \right)^2$
  This has a discontinuity in the first derivative.
- **Hanning:** $w_N(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)$
  This has a discontinuity in the 2nd derivative.
- **Hamming:** $w_N(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$
  This has a discontinuity, but of smaller size than for the rectangular window.

# Rectangular Window

Discontinuous

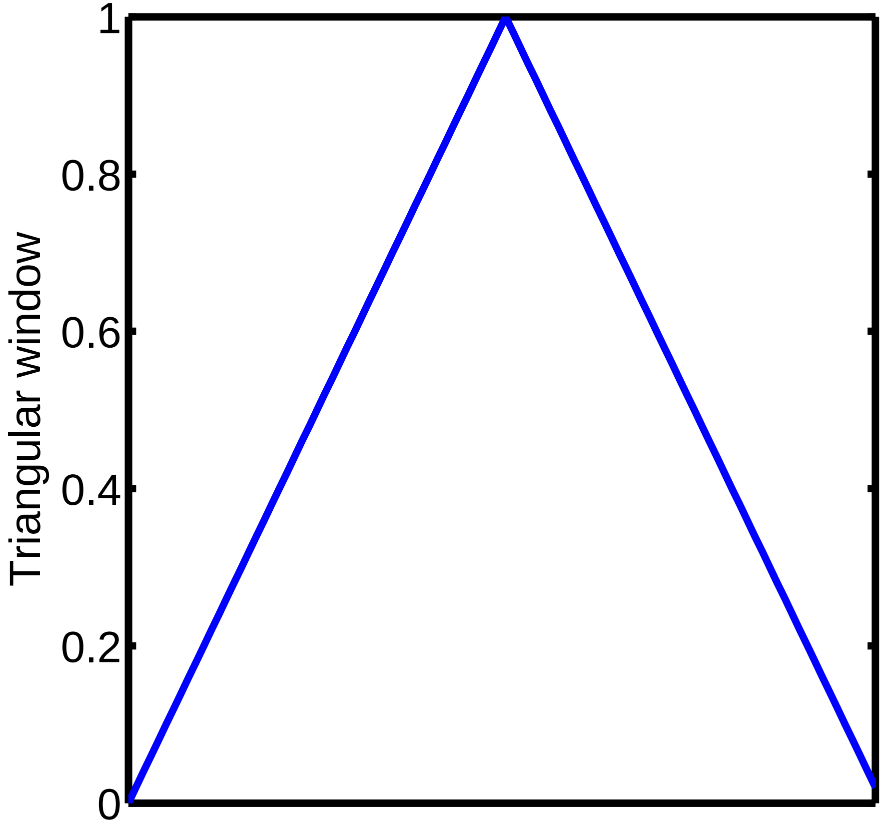


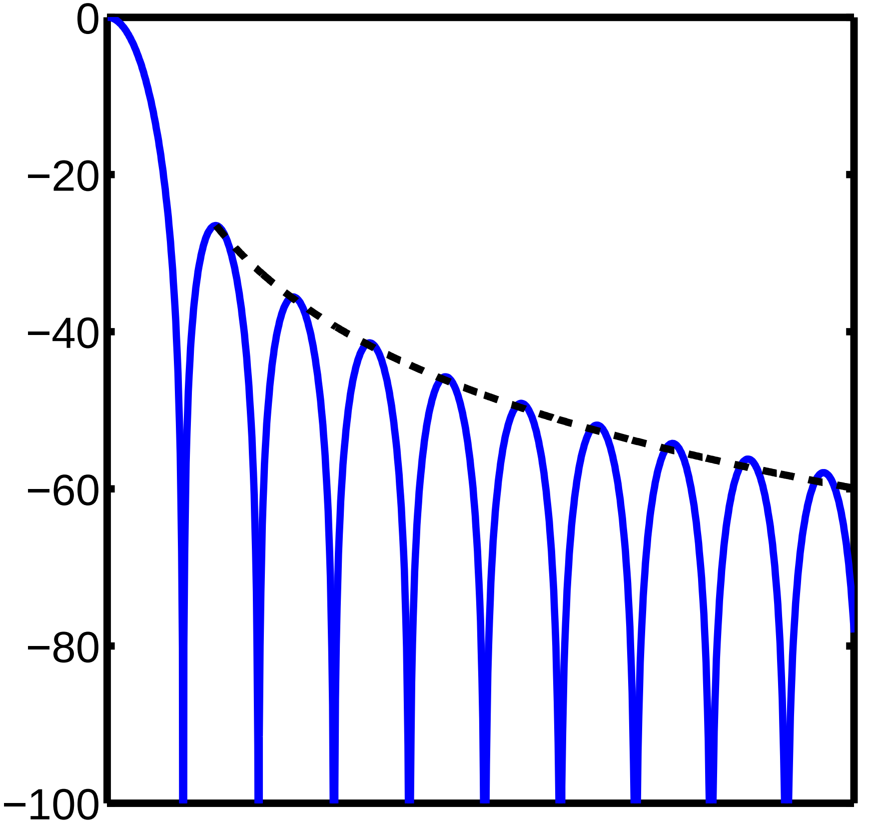




Decay $F(s) \sim \dfrac{K}{1+|s|}$

# Triangular Window

Discontinuity in the first derivative

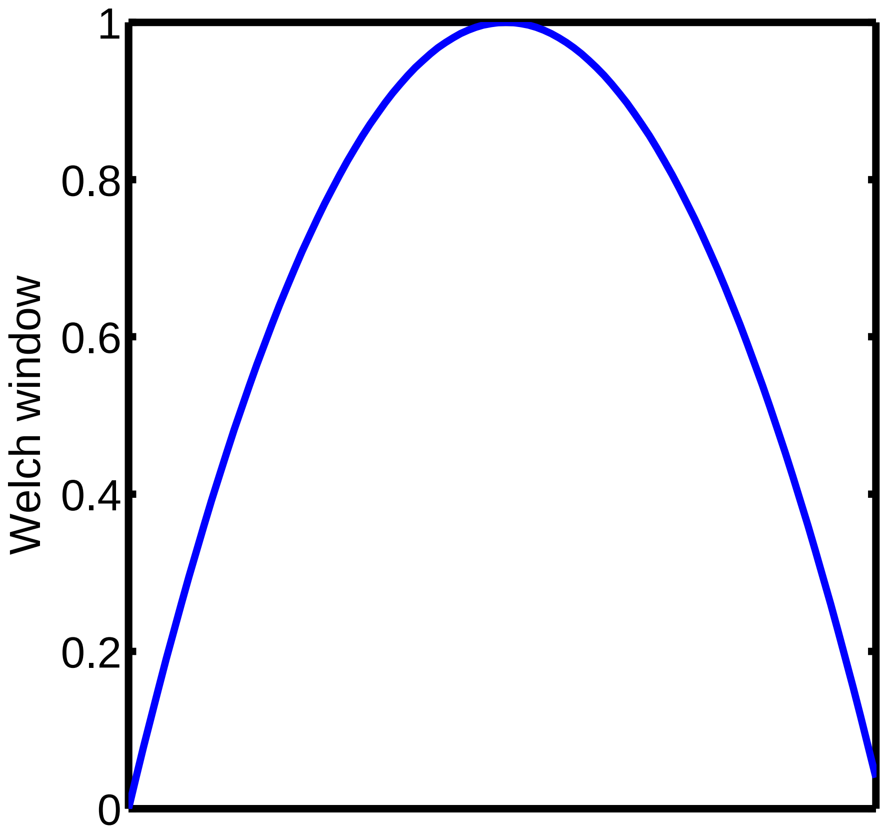


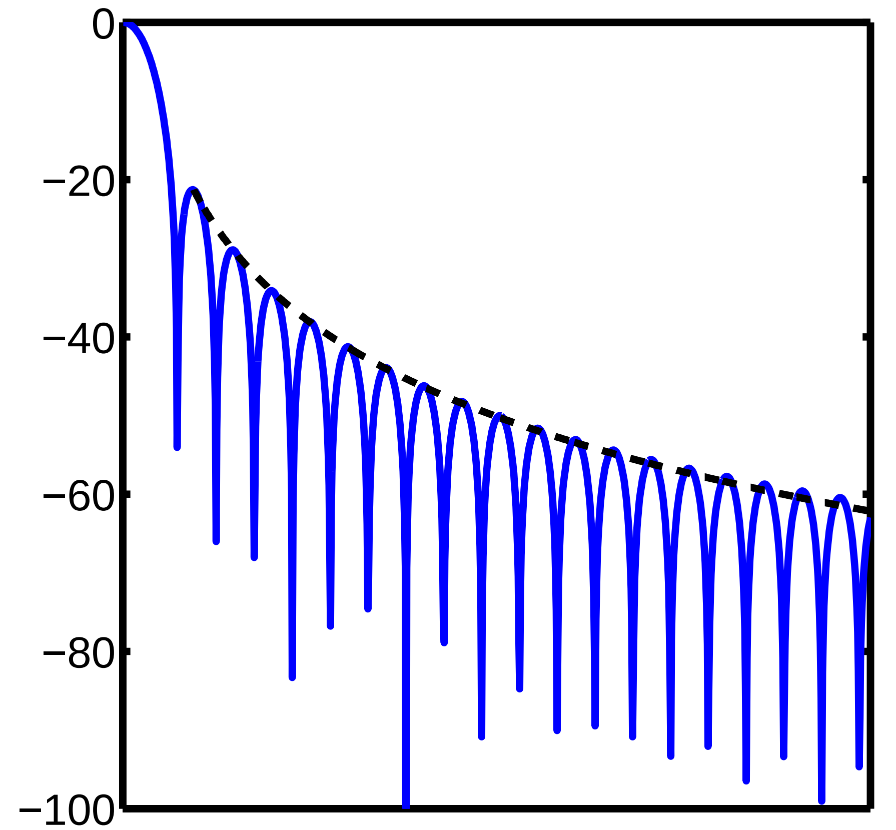




Decay $F(s) \sim \dfrac{K}{1+|s|^2}$

# Welch Window

Discontinuity in the first derivative

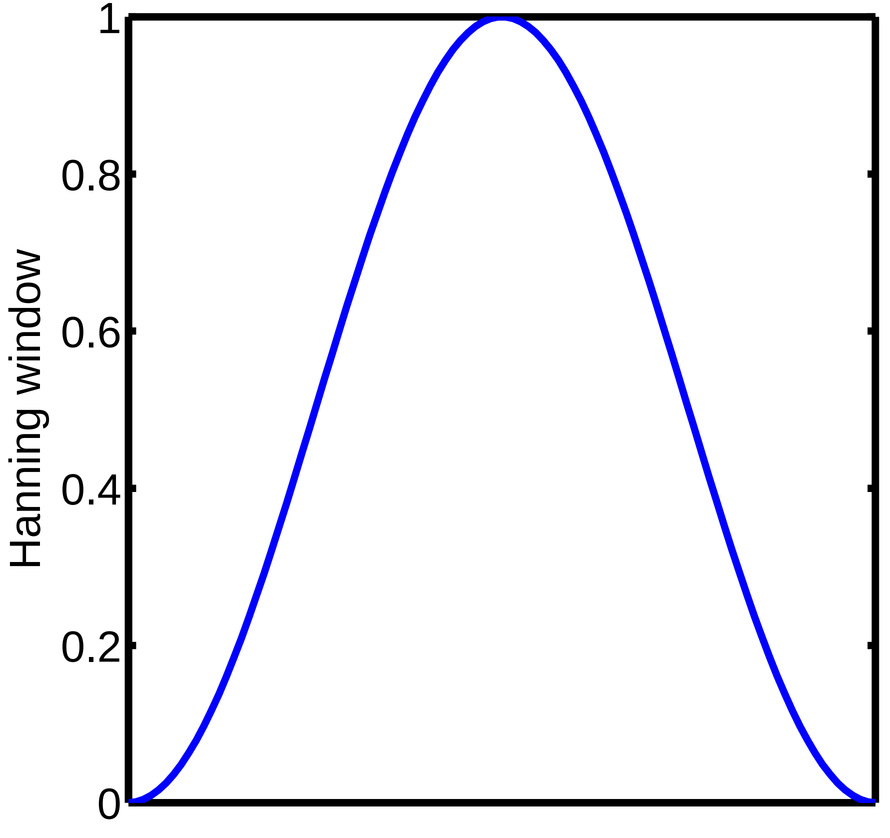


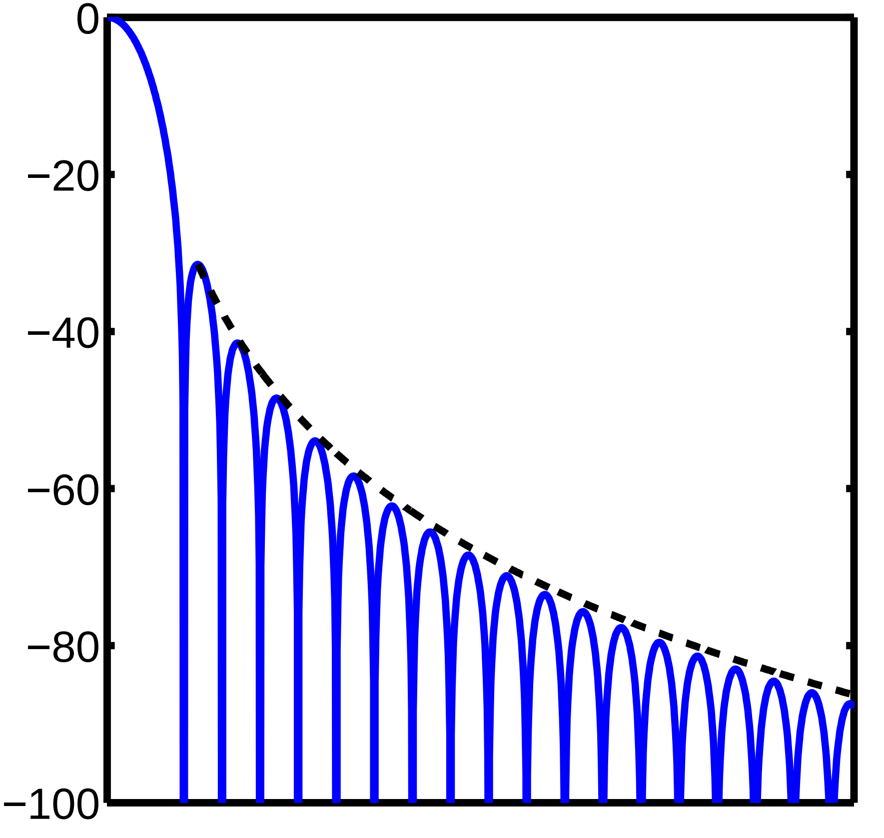




Decay $F(s) \sim \frac{K}{1+|s|^2}$

# Hanning Window

Discontinuity in the 2nd derivative

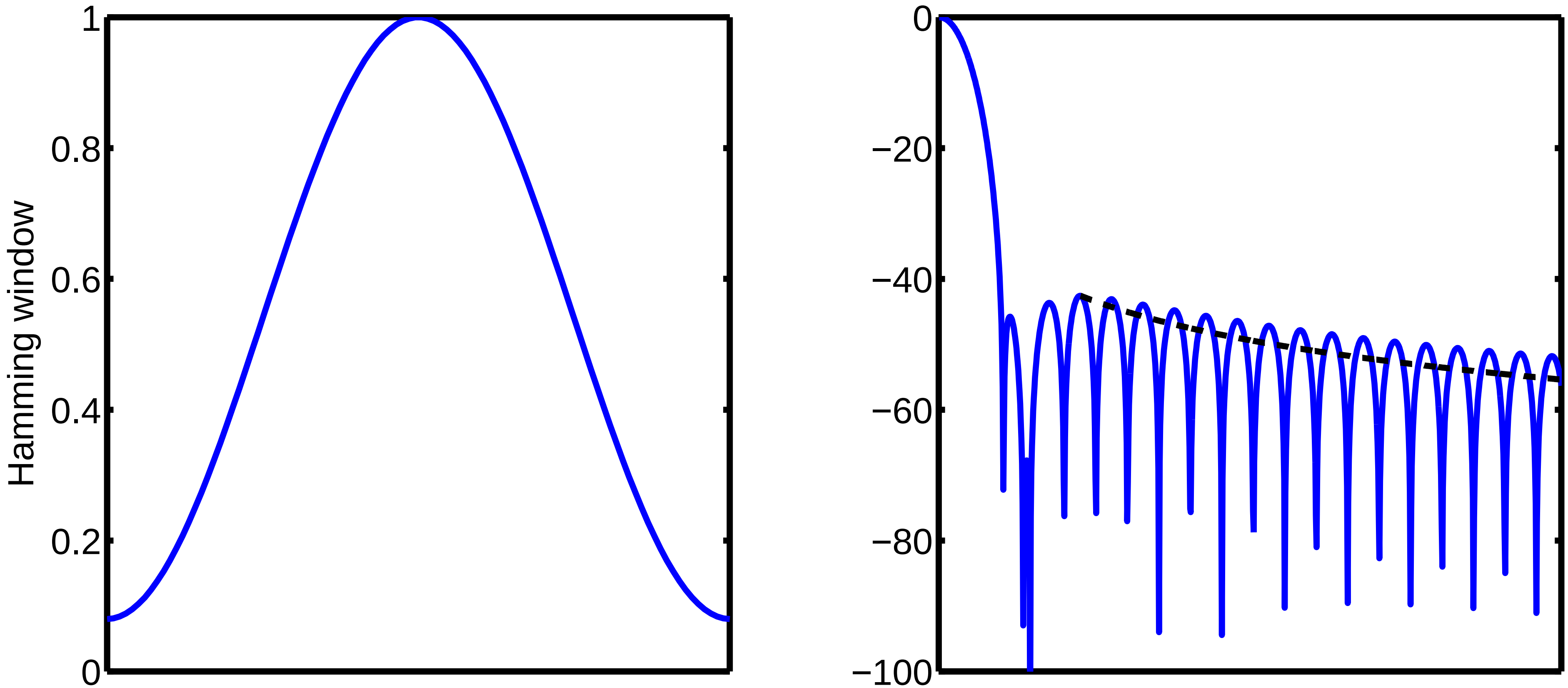



Decay $F(s) \sim \frac{K}{1+|s|^3}$

# Hamming Window

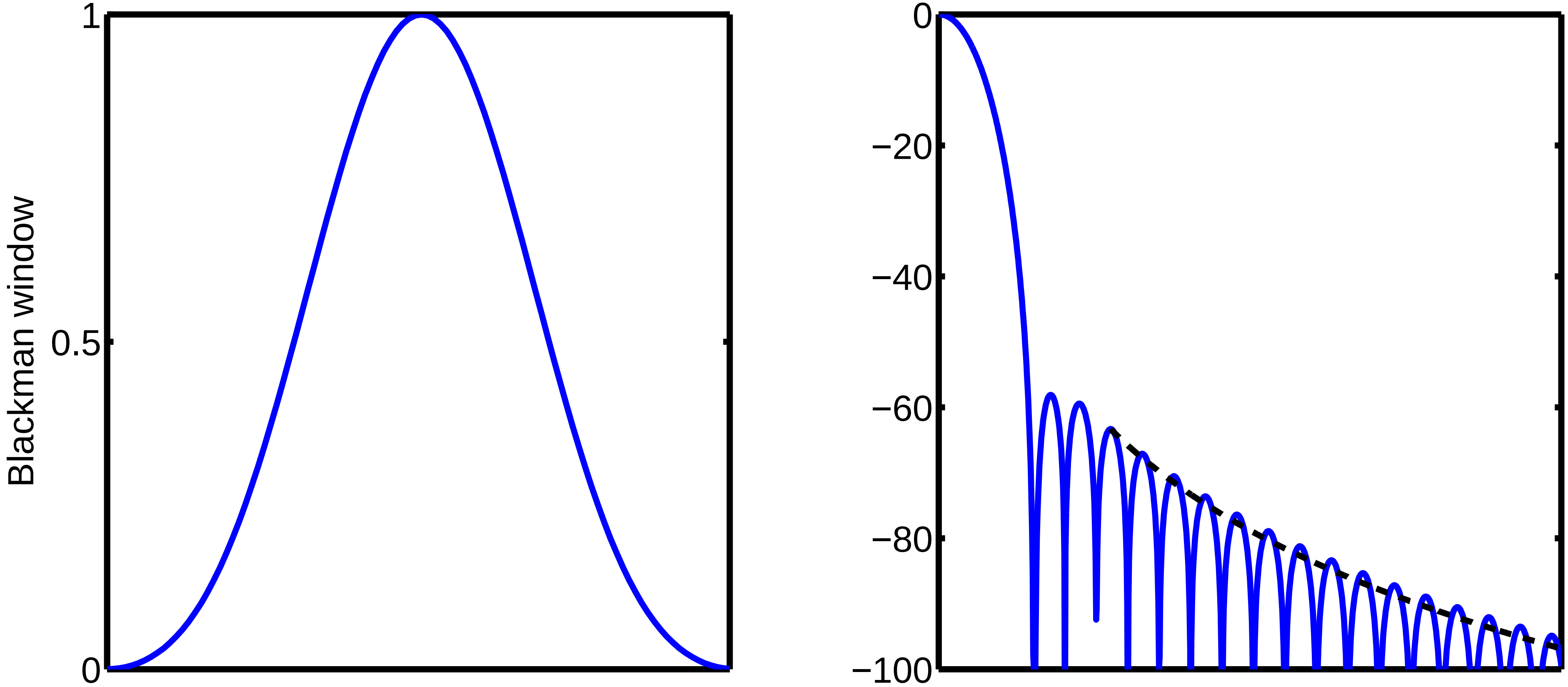
Discontinuous ($w_N(0) = 0.54 - 0.46 = 0.08$)




Decay $F(s) \sim \frac{K}{1+|s|}$

# Blackman Window

Discontinuity in the 2nd derivative




Decay $F(s) \sim \dfrac{K}{1+|s|^3}$

# Regularity and decay and duality

Duality implies that $\mathcal{F}\{F(t)\} = f(-s)$, so the above regularity properties work in reverse as well, e.g. if there exists a constant $K$, and $\varepsilon > 0$ such that

$$|f(t)| \leq \frac{K}{1 + |t|^{p+1+\varepsilon}}$$

Then $F$ has at least $p$ continuous derivatives.

Hence, if $f(t)$ has compact support then $F \in C^\infty$.

# Compactness

**Theorem:** If $f \neq 0$ has compact support then $F(s)$ can't be 0 on a whole interval. Similarly, if $F \neq 0$ has compact support then $f(t)$ can't be zero on a whole interval.

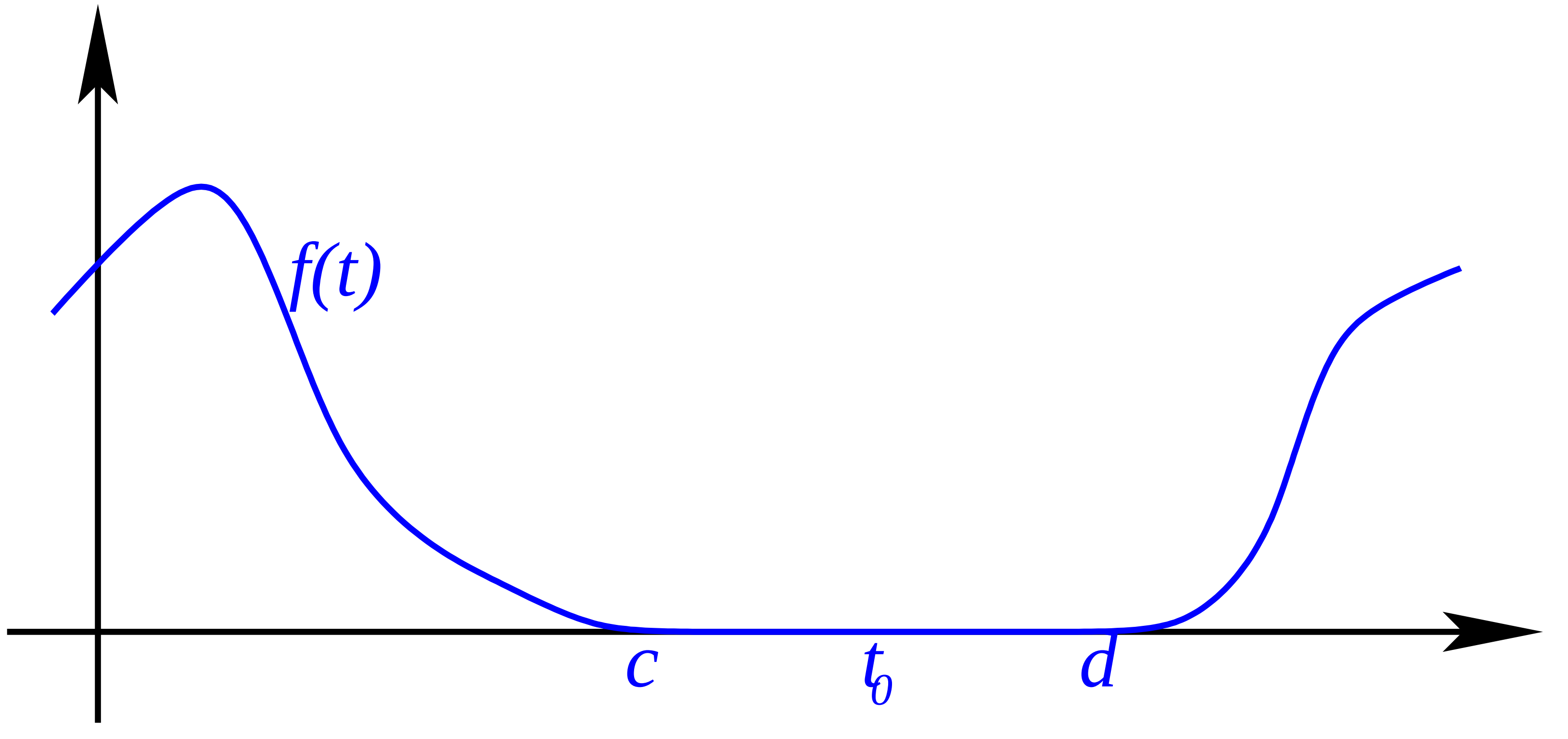
**Proof:** Assume $F(s) \neq 0$ has compact support in the interval $[-b, b]$. From the definition of the IFT

$$f(t) = \int_{-b}^{b} F(s) e^{i2\pi st} \, ds$$

If non-trivial function $f(t) = 0$ for $t \in [c, d]$, then $f^{(n)}(t_0) = 0$ inside the interval $(c, d)$, and so by differentiating $n$ times under the integral at $t_0$,

$$0 = f^{(n)}(t_0) = \int_{-b}^{b} F(s)(i2\pi s)^n e^{i2\pi s t_0} \, ds$$

# Compactness




- The function is flat (constant) at $t_0$
- The derivatives of $f$ at $t_0$ must all be 0

# Compactness

**Proof:** We can write the IFT as

$$f(t) = \int_{-b}^{b} F(s) e^{i2\pi s(t-t_0)} e^{i2\pi s t_0}\, ds$$

And we can expand the exponential $\exp(i2\pi s(t-t_0))$ in a power series about $t_0$ to get

$$f(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \int_{-b}^{b} F(s)(i2\pi s)^n e^{i2\pi s t_0}\, ds$$

However note that we have already shown that each of the integrals in the above sum are zero, and so the sum is zero leading to $f(t) = 0$, which contradicts the assumption that $f(t)$ is nontrivial (i.e. $f(t) \neq 0$ for some $t$).

# Meaning

Given a

- more compact
- irregular
- sharper

function in the time (frequency) domain we get a

- less compact
- smoother
- wider

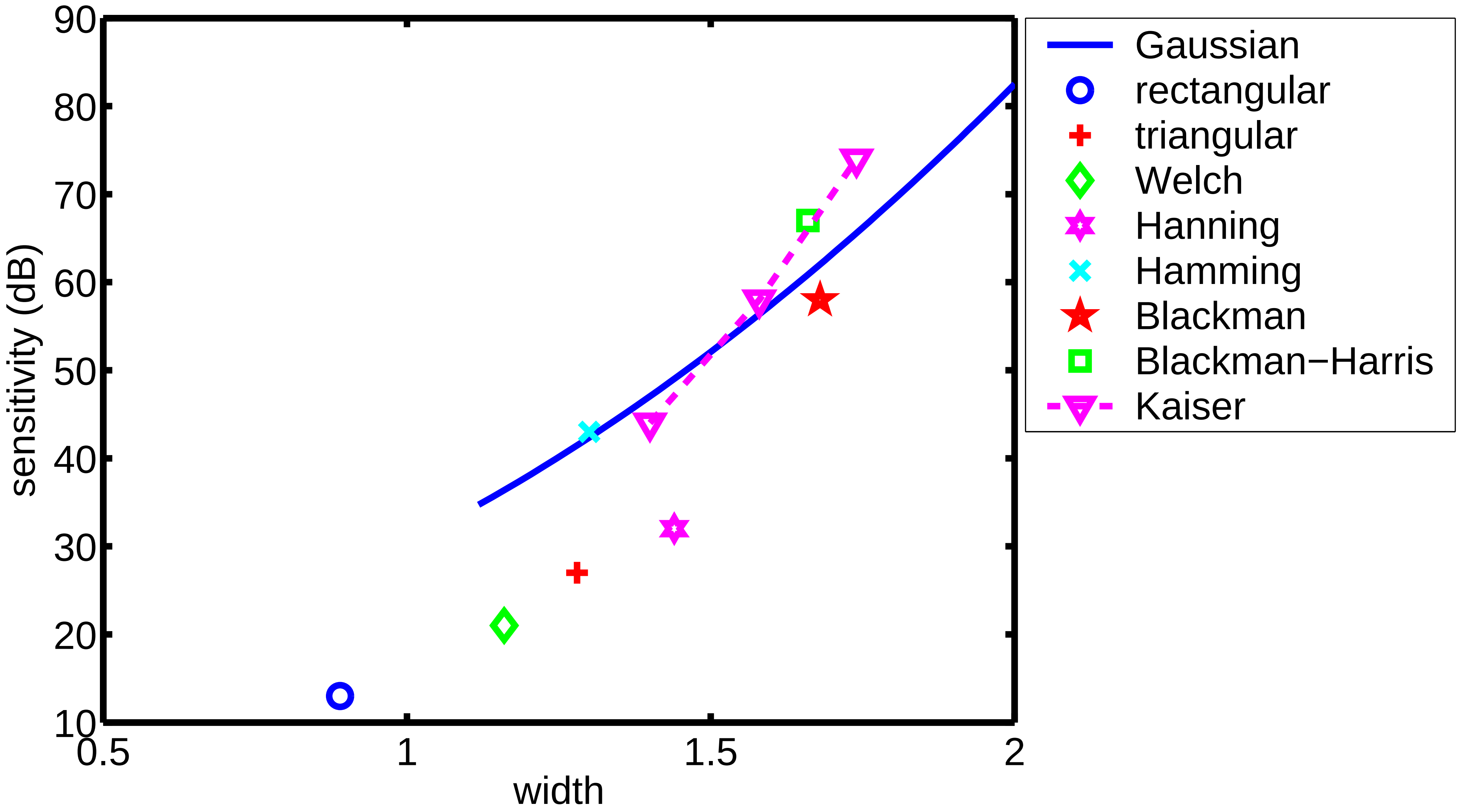
function in the frequency (time) domain.

# Windows summary

| Name | max side lobe | width | roll off polynomial |
|---|---|---|---|
| rectangular | -13 dB | 0.89 | $K/(1+\|s\|)$ |
| triangular (Bartlett) | -27 dB | 1.28 | $K/(1+\|s\|^2)$ |
| Welch (Riesz) | -21 dB | 1.16 | $K/(1+\|s\|^2)$ |
| Hanning | -32 dB | 1.44 | $K/(1+\|s\|^3)$ |
| Hamming | -43 dB | 1.30 | $K/(1+\|s\|)$ |
| Blackman | -58 dB | 1.68 | $K/(1+\|s\|^3)$ |
| Blackman-Harris | -67 dB | 1.66 | $K/(1+\|s\|)$ |
| Kaiser (B=6) | -44 dB | 1.40 | $K/(1+\|s\|)$ |
| Kaiser (B=8) | -58 dB | 1.58 | $K/(1+\|s\|)$ |
| Kaiser (B=10) | -74 dB | 1.74 | $K/(1+\|s\|)$ |
| Gaussian ($\alpha$=4.5) | -56 dB | 1.55 | $K/(1+\|s\|)$ |

Where available results from "On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform", F.J.Harris, Proc. of the IEEE, Vol.66, No.1, Jan. 1978, pp.51-83.

# Sensitivity vs resolution



# Transient Signal Analysis

We have previously seen that the Fourier transform is not appropriate for analyzing transient signals, but what should we do then? The first step is to look at the Short-Time or Windowed Fourier Transform.

# Transient signals

- all signals are transient
  - they have a start and stop at least
- sometimes this doesn't matter
- often it does
  - conversation is full of transients
  - music
  - images
- the Fourier transform
  - the Fourier transform is nice because it diagonalises **time-invariant** linear systems
  - doesn't localize in time at all
- we need something more for transient signals

# EM Frequency Band Allocations

| Frequency Band | Designation | Typical use |
|---|---|---|
| 3-30 kHz | Very Low Frequency (VLF) | Long-range navigation |
| 30-300 kHz | Low Frequency (LF) | Marine Communications |
| 300-3000 kHz | Medium Frequency (MF) | AM radio |
| 3-30 MHz | High Frequency (HF) | Jindalee, Amateur Radio |
| 30-300 MHz | Very High Frequency (VHF) | FM radio, VHF TV |
| 300-3000 MHz | Ultra High Frequency (UHF) | UHF TV, radar |
| 3-30 GHz | SuperHigh Frequency (SHF) | Satellite Comms |

From p. 308 of Philips, Parr and Riskin.

- Audible sound frequencies: $\sim 20 - 20,000 Hz$
- AM radio frequencies: $535 - 1615 kHz$
- FM radio frequencies: $88 - 108 MHz$
- how should we carry sound on radio?

# Amplitude Modulation (AM)

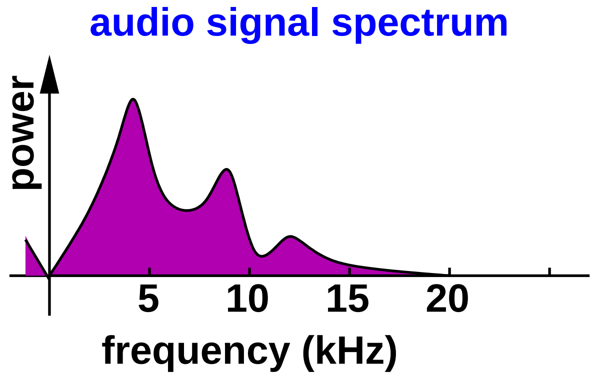
AM Radio

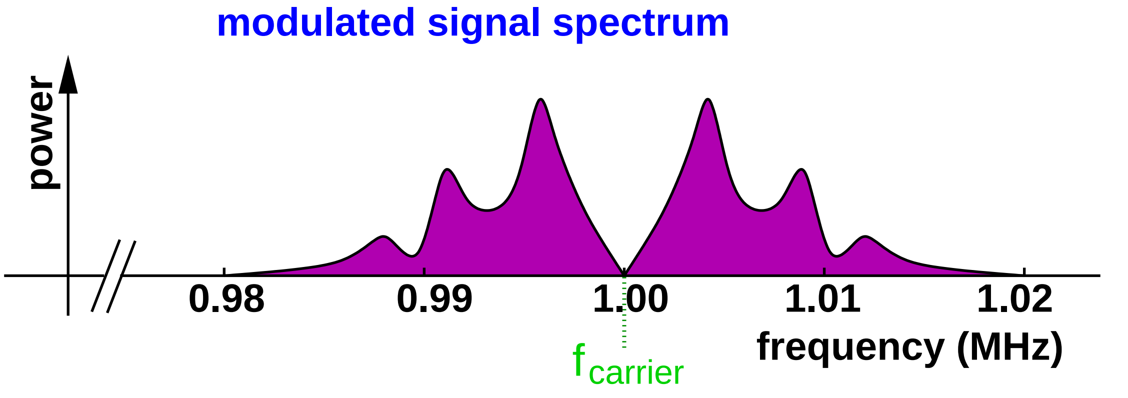
- modulation of signal $x(t)$ with a cosine function

$$y(t) = \cos\left(2\pi f_{\text{carrier}} t\right)\left[1 + x(t)\right]$$

- $f_{\text{carrier}} \in [535 - 1615]kHz$
- modulation property of FT

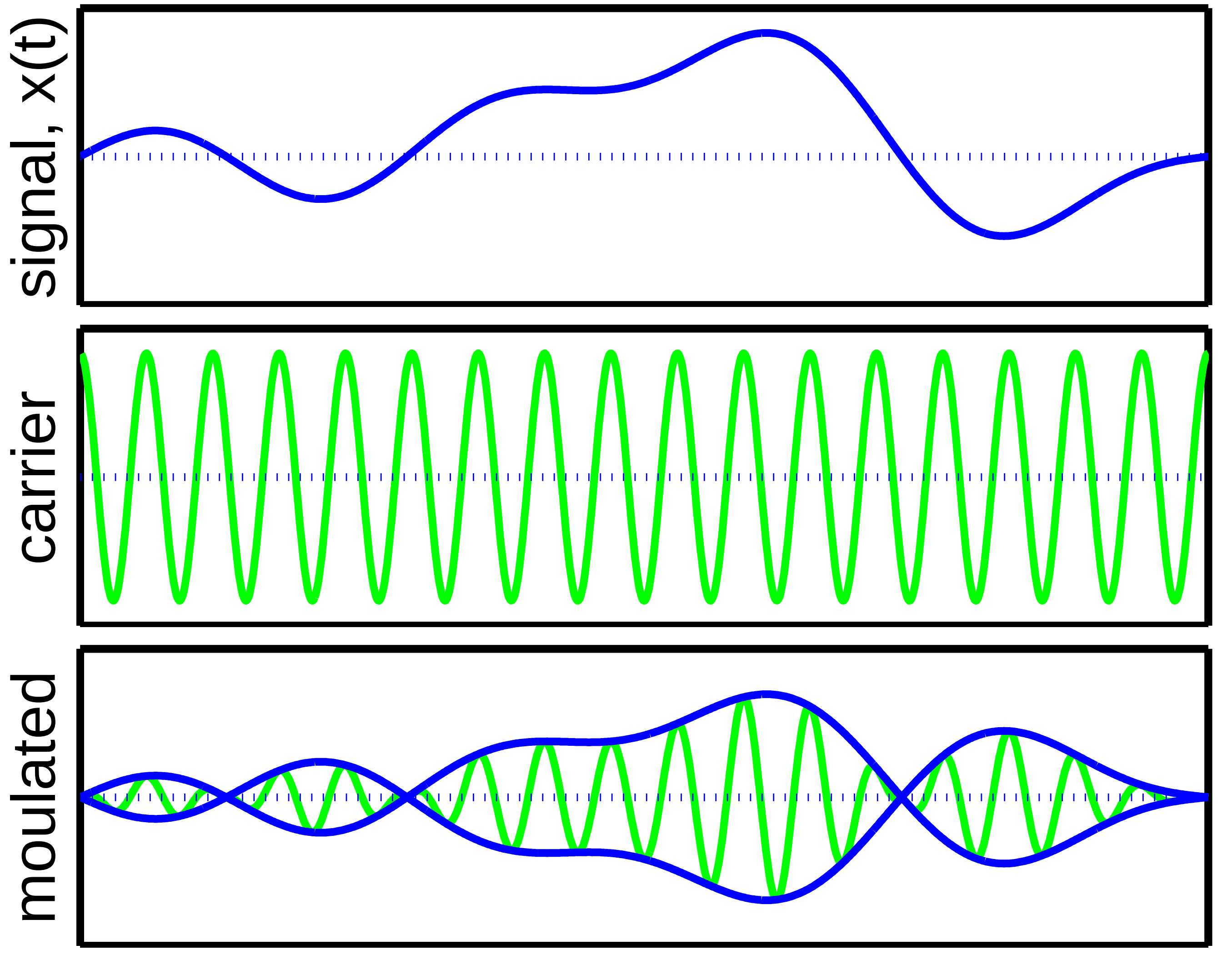
$$\mathcal{F}\left\{ f(t)\cos(2\pi s_0 t)\right\} = \frac{1}{2}F(s-s_0) + \frac{1}{2}F(s+s_0)$$



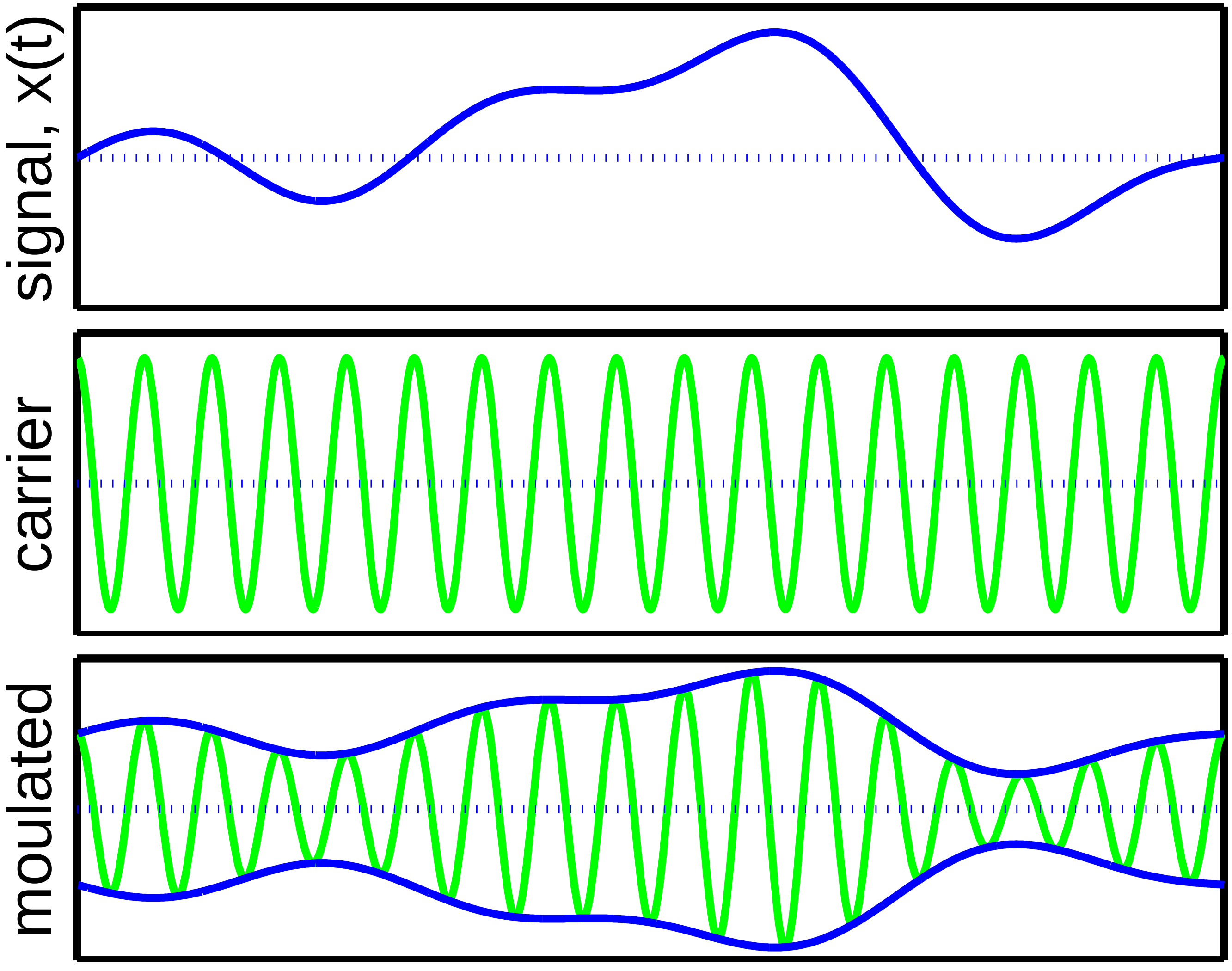
# Amplitude Modulation (AM)

$$y(t) = \cos\left(2\pi f_{\text{carrier}} t\right) \left|x(t)\right|$$



# Amplitude Modulation (AM)

$$y(t) = \cos\left(2\pi f_{\text{carrier}} t\right)\left[1 + x(t)\right]$$



# Frequency Modulation (FM)

FM Radio (88-108 MHz in the US)

- modulate the frequency of the signal
- given a signal $x(t)$ to be transmitted

$$y(t) = \cos(2\pi\phi(t))$$

  where $\phi(t)$ is now a (non-linear) function of time, depending on the signal $x(t)$ to be transmitted.

- **instantaneous frequency** is the rate of change of phase, e.g.

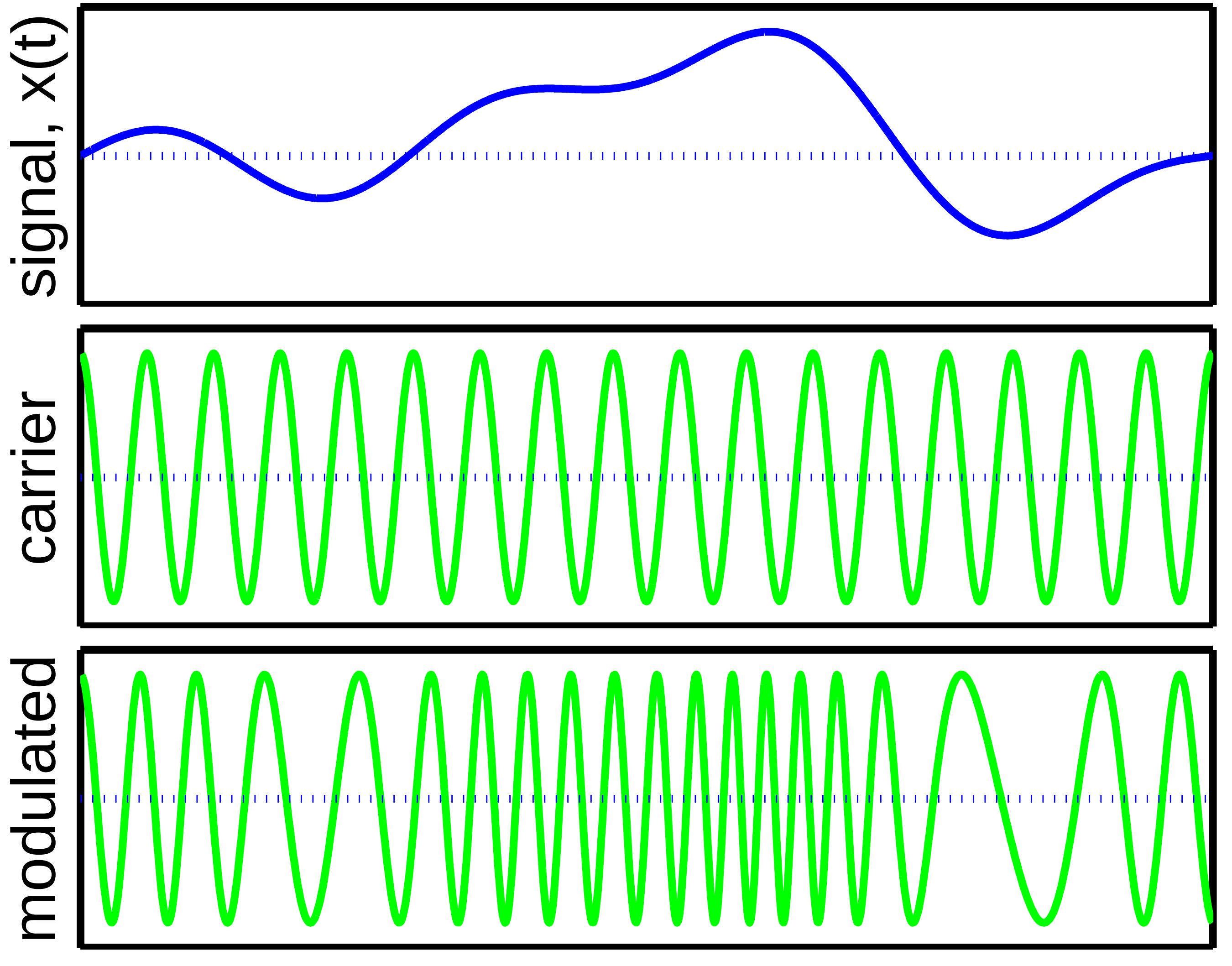
$$f(t) = \frac{d}{dt}\phi(t)$$

- so take

$$\phi(t) = \int_0^t f_{\text{carrier}} + x(t)\,dt$$

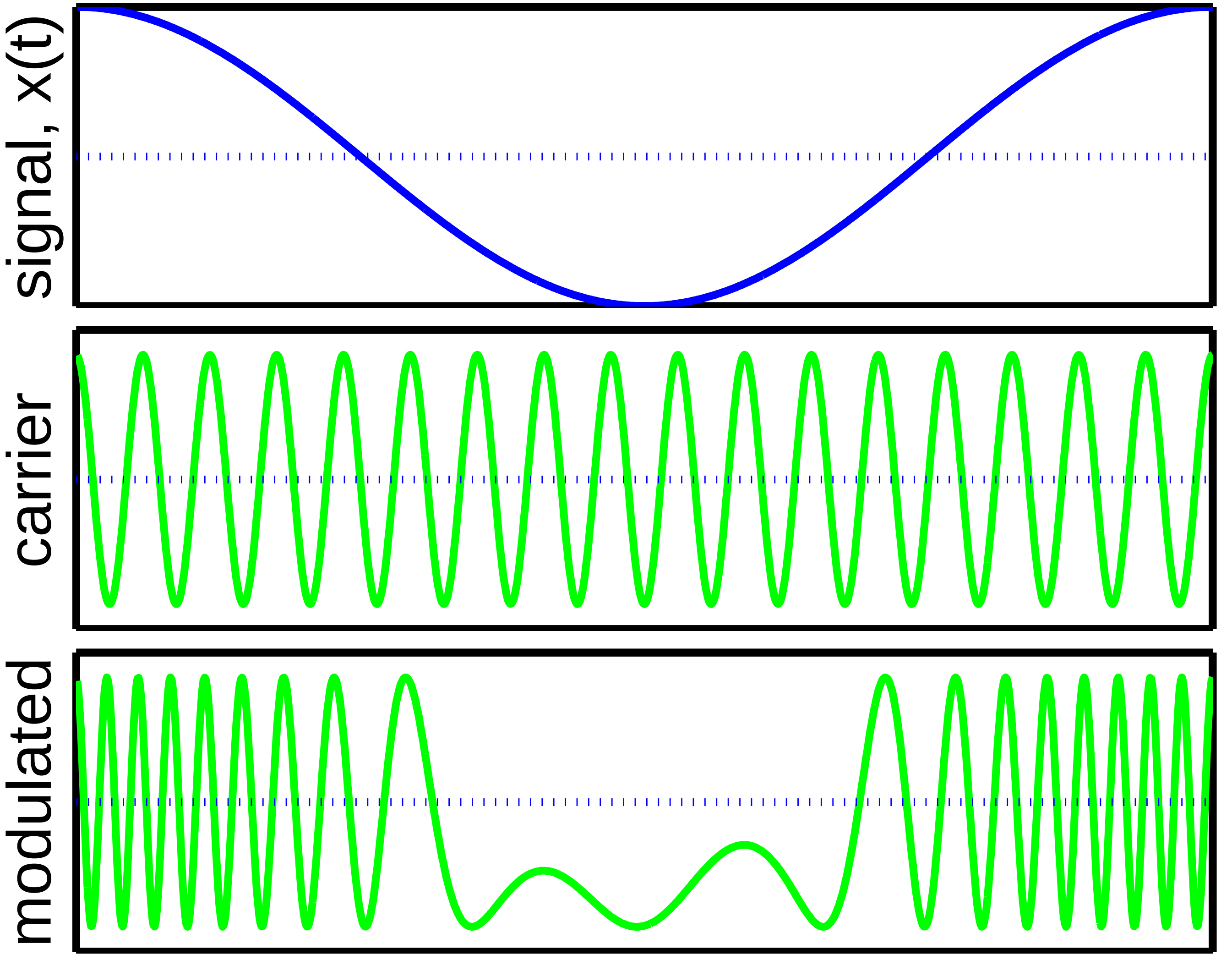
# Frequency Modulation (FM)

$y(t) = \cos(2\pi\phi(t))$ **where** $\phi(t) = f_{\text{carrier}} t + \beta \int_0^t x(t)\, dt$



# Frequency Modulation (FM)

The FM signal is transient (even if input isn't)



# Envelope and phase

We can change two things

- Envelope (amplitude modulation)
- Phase (frequency modulation)

Result is a signal

$$y(t) = A(t)\cos\left(2\pi\phi(t)\right)$$

In transient analysis of this signal we would like to be able to determine $A(t)$ and $\phi(t)$.

- note that a real signal (e.g. music) would consist of a superposition of a number of such terms, e.g.
  - plucked string has a number of harmonics
  - each decays at different rates

# A Chirp

Both Amplitude and Frequency Modulation can happen at once, a simple example being a chirp. Examples:

- a linear chirp

$$y(t) = A(t)\cos\left[2\pi(bt^2 + ct)\right]$$

Instantaneous frequency

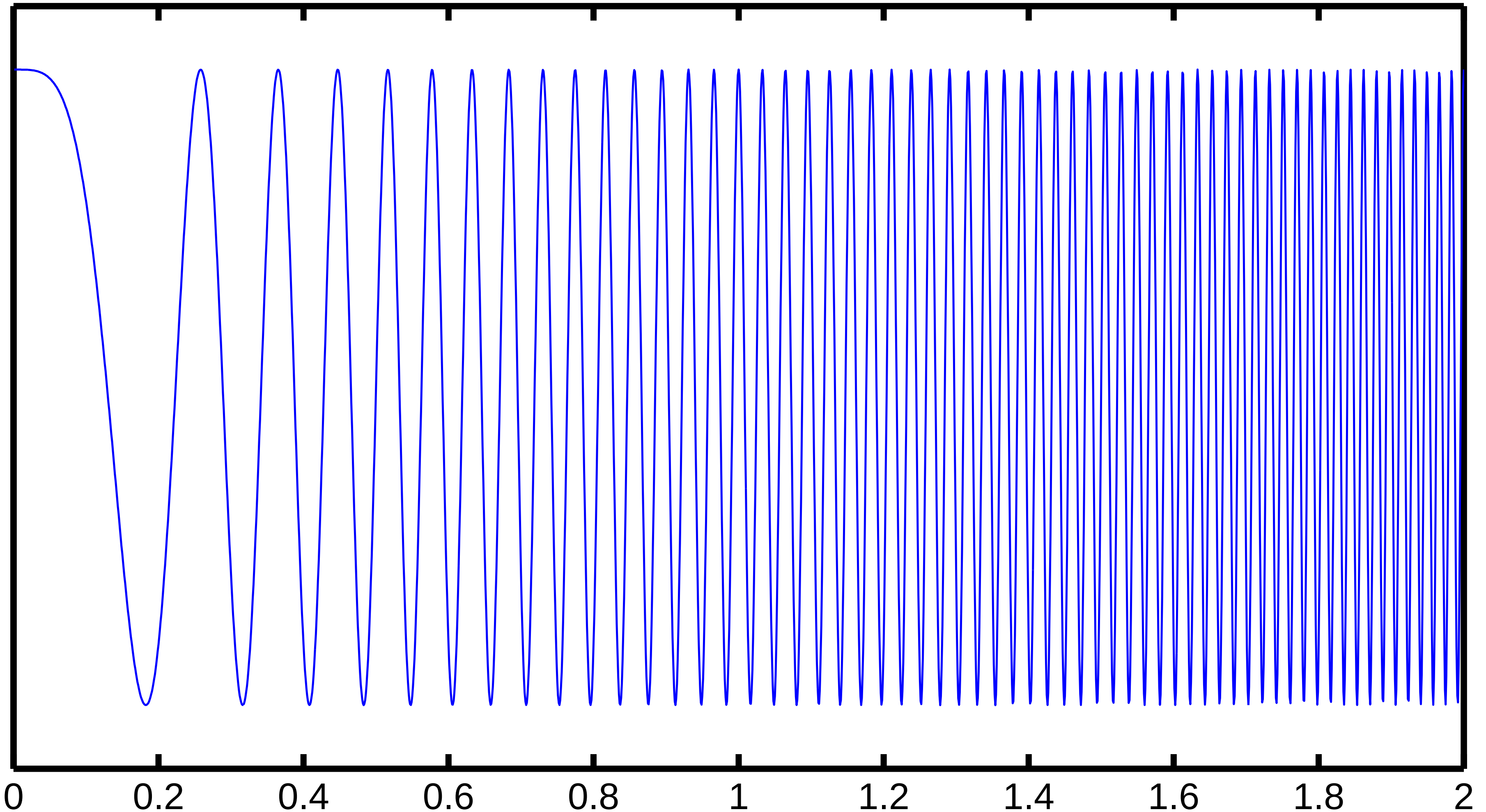
$$f(t) = \frac{d}{dt}\left[bt^2 + ct\right] = 2bt + c$$

- a hyperbolic chirp

$$y(t) = \cos\left(\frac{2\pi\alpha}{\beta - t}\right) \quad \text{and} \quad f(t) = \frac{\alpha}{(\beta - t)^2}$$

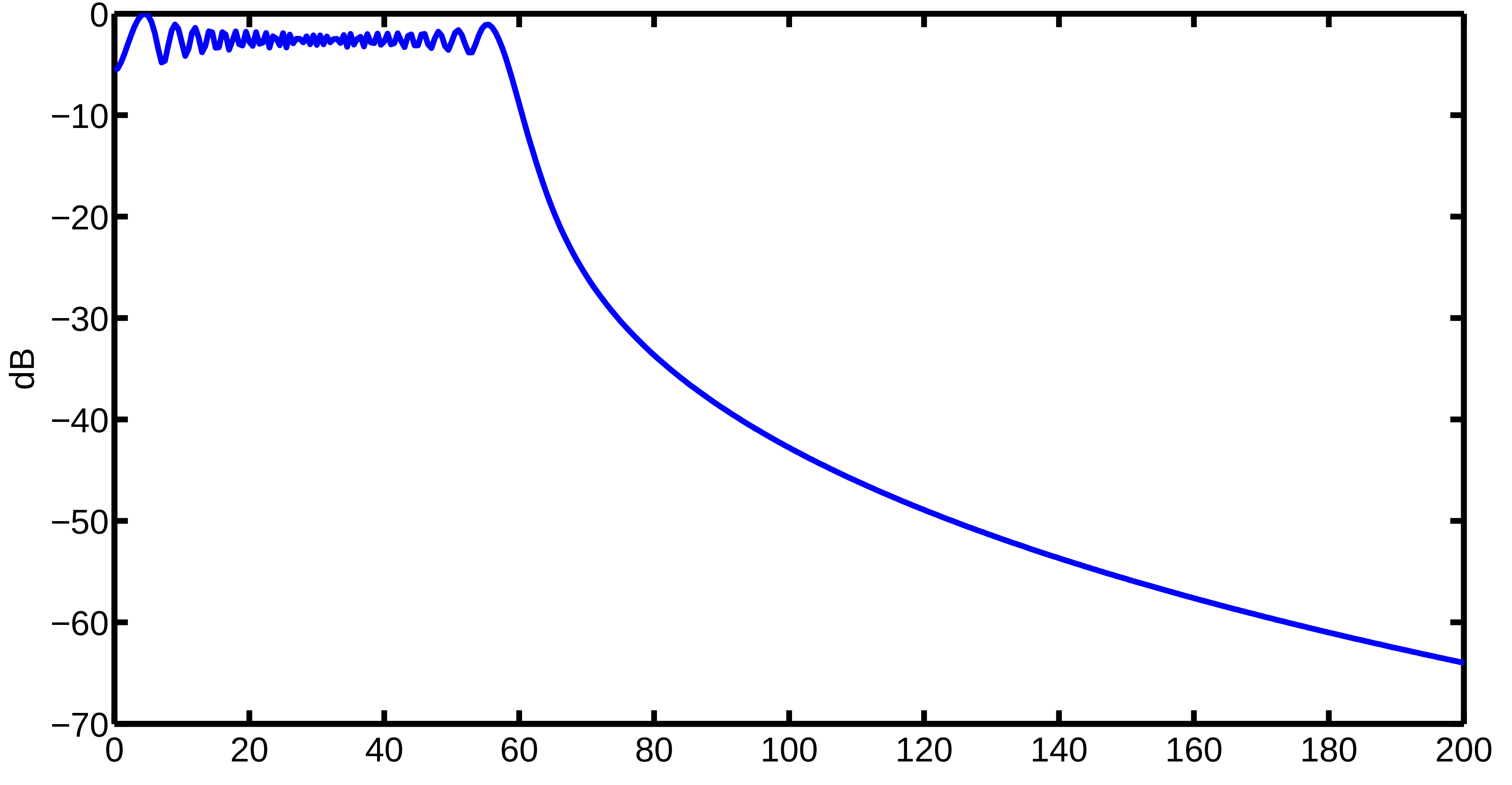
# An Example Chirp

A Chirp $y(t) = \cos(2\pi 15 t^2)$, so instantaneous freq. $f(t) = 30t$



# A Chirp

DFT of a chirp



# Instantaneous frequency

Just to reiterate, given a signal

$$y(t) = A(t)\cos\left[2\pi\phi(t)\right]$$

The **Instantaneous frequency** is

$$f(t) = \frac{d\phi}{dt}$$

# Windows, windows everywhere

We can use windowing functions in other ways

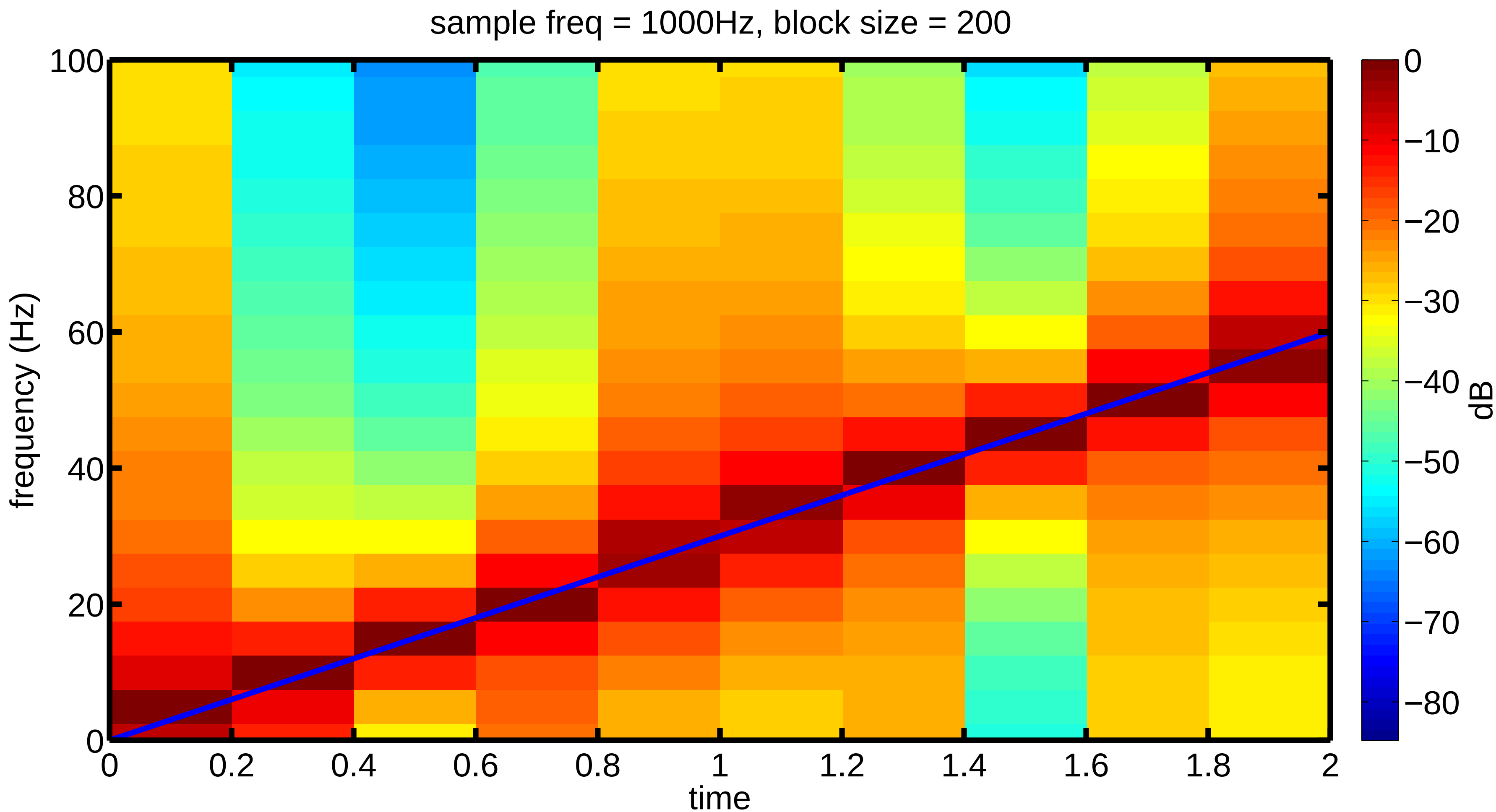
- analysis of transient signals
- use windows to select a chunk of data
- move the window along
- so perform FT of a series of functions

$$g(\tau;t) = f(\tau)w^*(\tau - t)$$

- we get the Short-Time Fourier Transform (STFT)

# STFT of a Chirp

STFT of the chirp from slide ??.



# Short Time Fourier Transform

- The Fourier transform goes from time- to frequency-domain
  - lose all time dependence
- but, e.g. music does not have same frequency over long time periods
- want to frequencies over shorter time periods
- get STFT by applying time-shifted window function $w(\tau - t)$

$$STFT\{f;t,s\} = \int_{-\infty}^{\infty} f(\tau) w^*(\tau - t) e^{-i2\pi s\tau} d\tau$$

Magnitude$^2$ of the STFT results in the **spectrogram**.

$$\text{spectrogram}(f;t,s) = |STFT\{f;t,s\}|^2$$

# Discrete time STFT

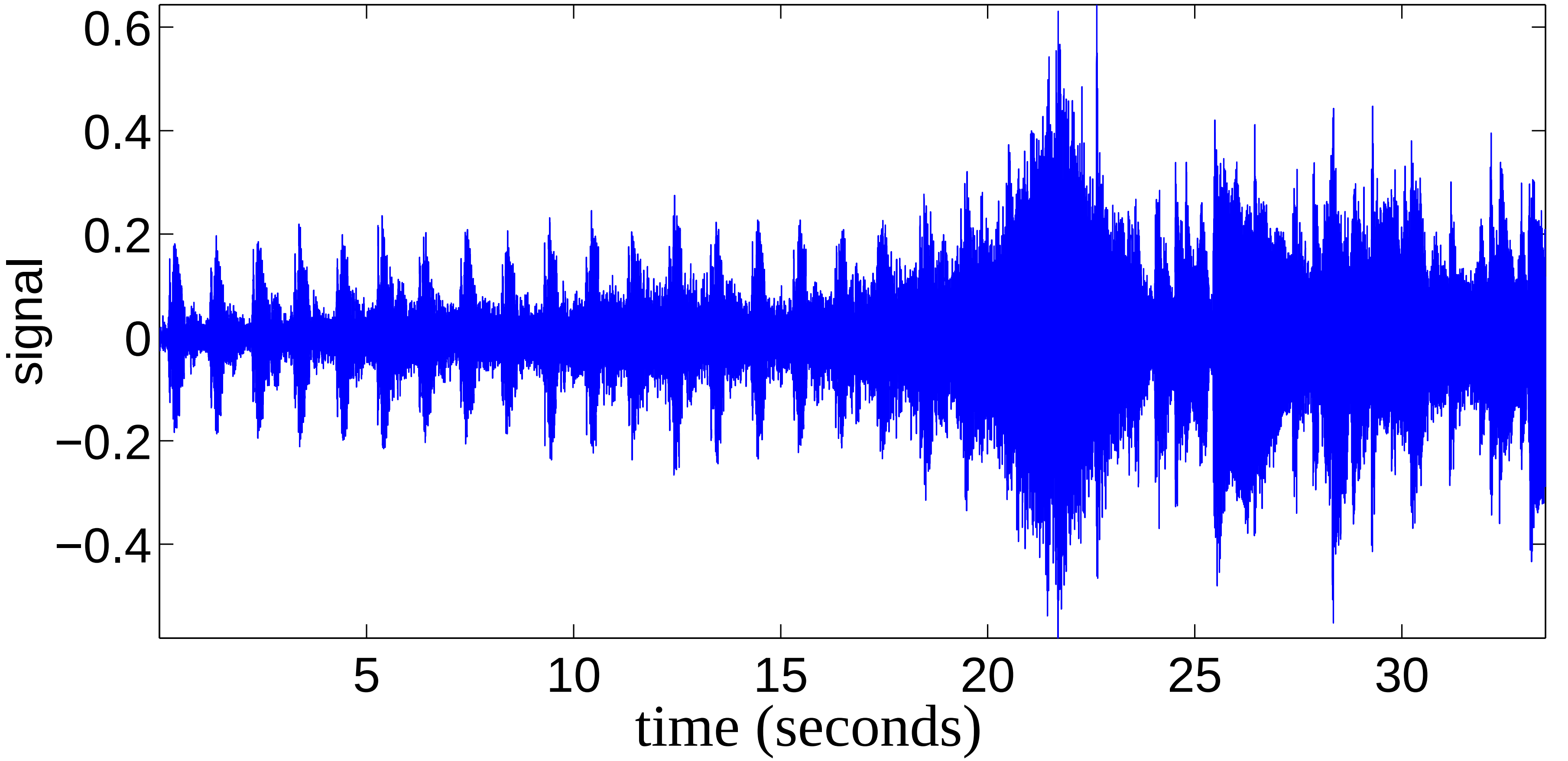
- apply a standard window (e.g. Hamming)
- DFT a block of data of size $M$ from $n$ to $n+M-1$.
- do so for all $n$.

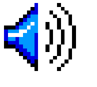
$$DSTFT\{x;n,k\} = X(n;k) = \sum_{m=n}^{n+M-1} f(m)w^*(m-n)e^{-i2\pi ikm/N}$$

- In $X(n;k)$ the $n$ indexes time (in the trans. domain)
- In $X(n;k)$ the $k$ indexes frequency (as in the DFT)
- often it is only performed on non-overlapping blocks
  - only calculate $X(n;k)$ at time points $n = 0, M, 2M, \ldots$

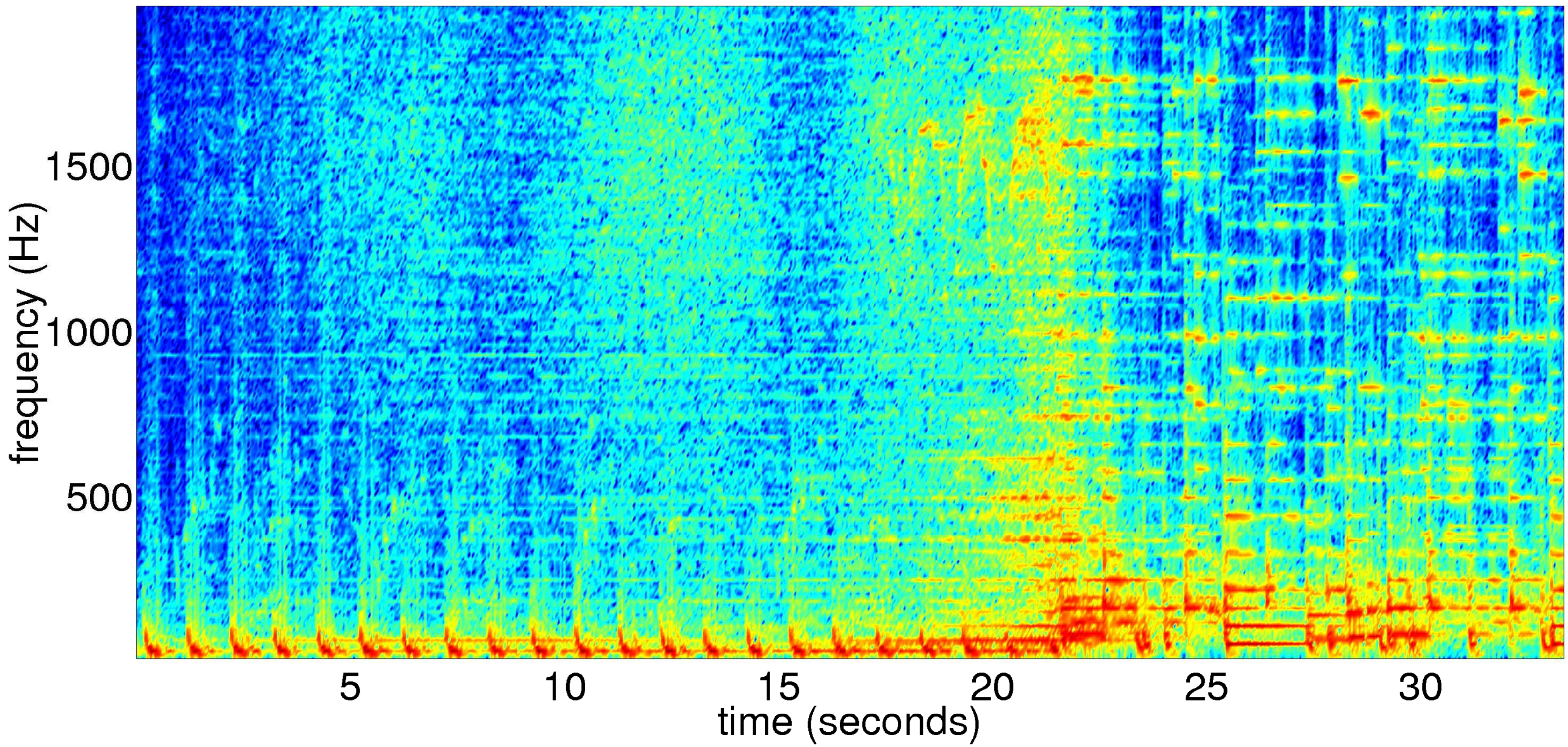
# Example spectrogram

Most sounds aren't continuous, they are **transient**




Dark side of the moon: Breathe clip 

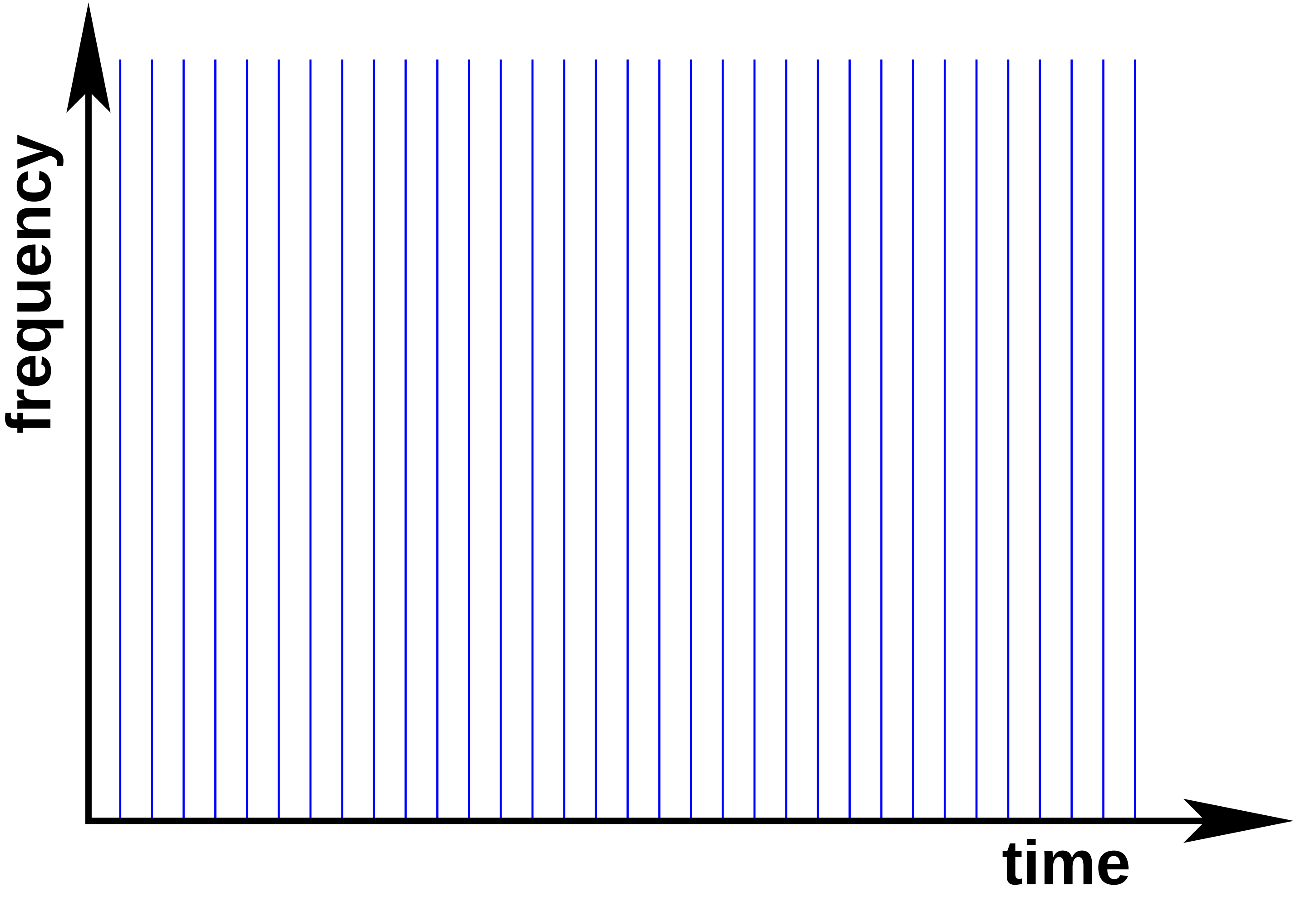
# Example spectrogram




Dark side of the moon: Breathe clip

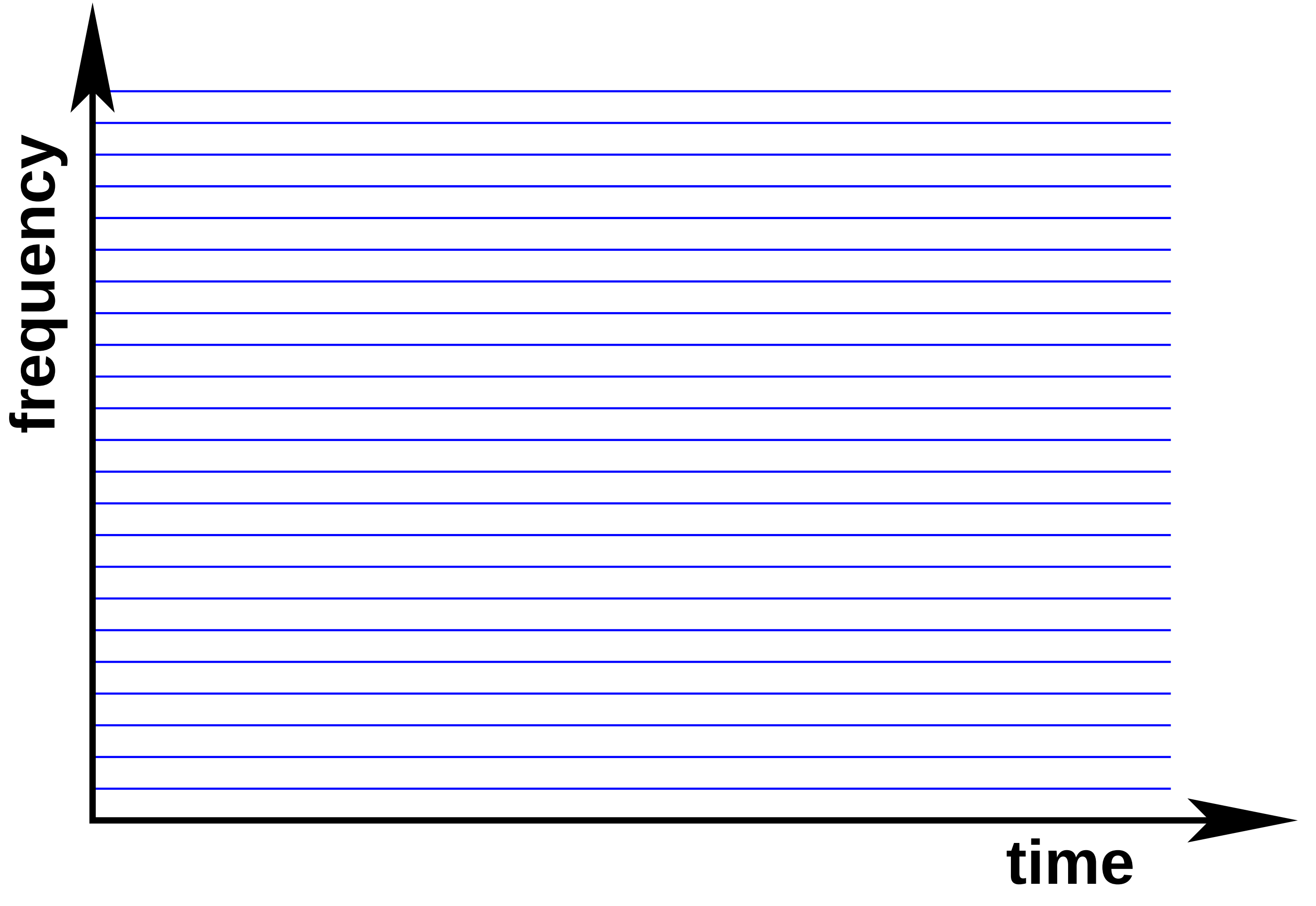
# Cutting up the time-frequency space

Time domain



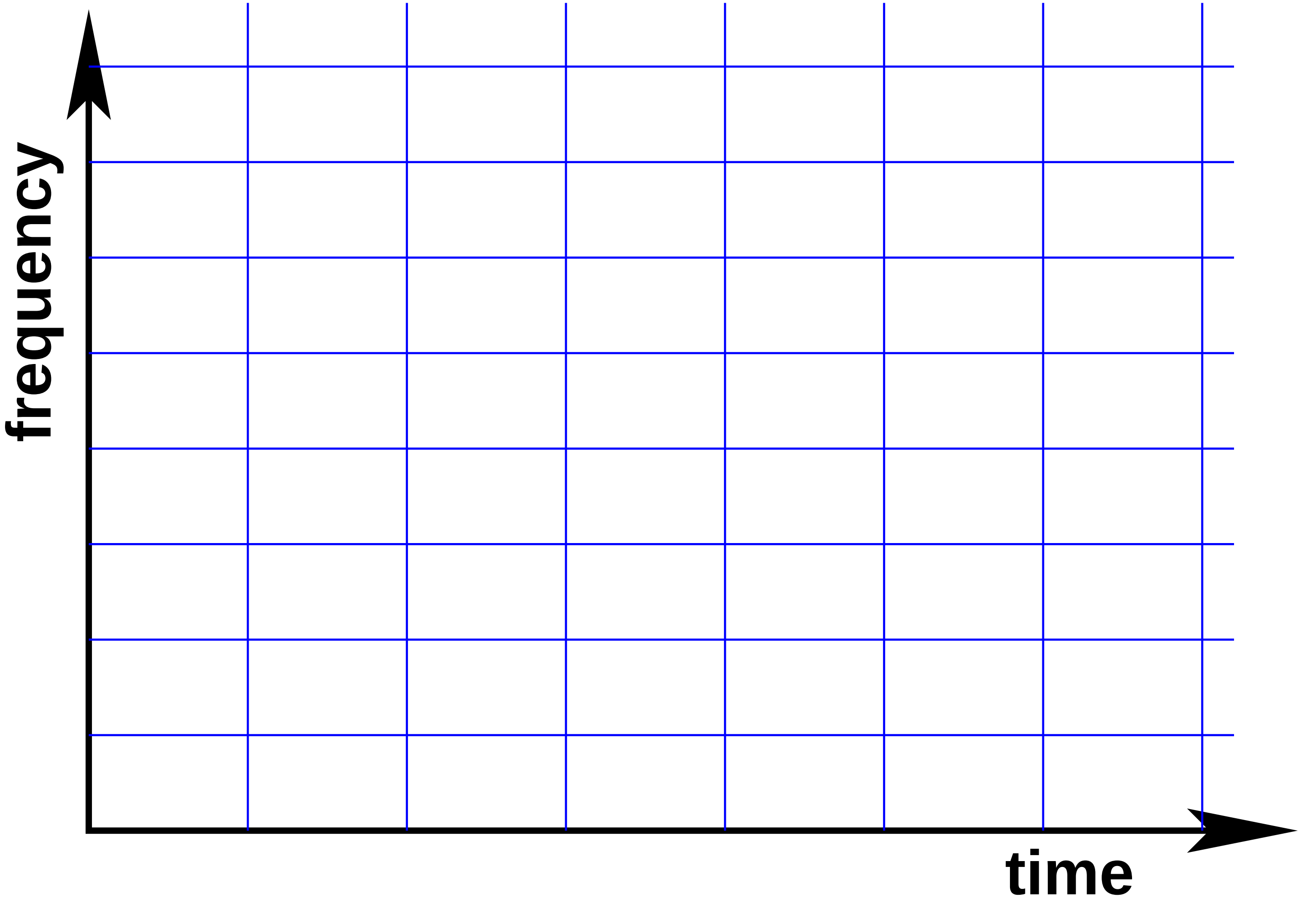
# Cutting up the time-frequency space

Frequency domain (Fourier transform)



# Cutting up the time-frequency space

STFT



# Uncertainty

Fundamental limitation on STFT is uncertainty

- if we make the window short, we get good time resolution, but poor frequency resolution
- if we make the window long, we get poor time resolution, but good frequency resolution
- we can't do better in both
- there is an uncertainty bound between time and frequency

# Resolution

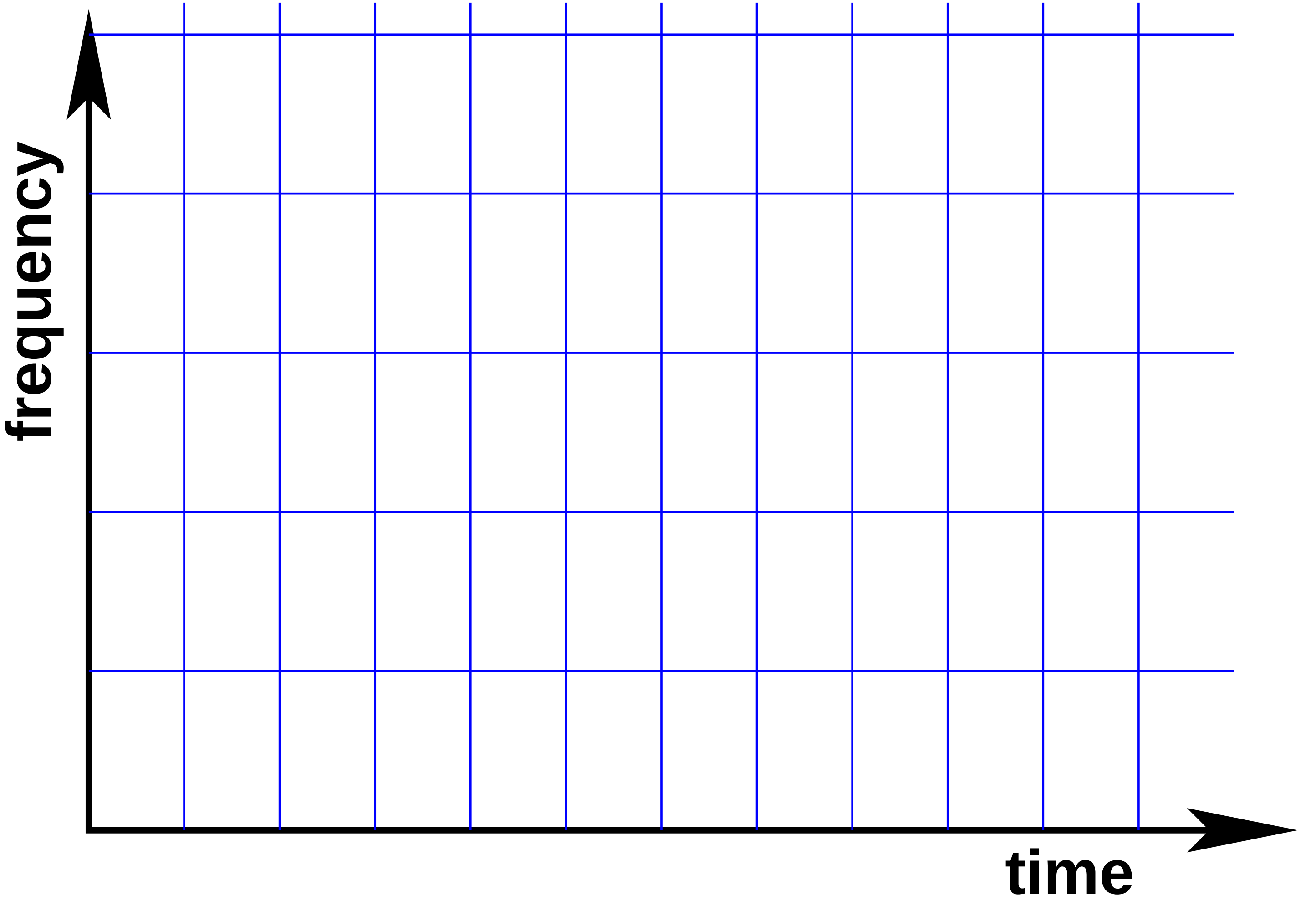
Given rectangular window of width $M$ samples, and sampling intervals $t_s$

- time resolution is just $Mt_s$
  - signals have to be in different boxes to be resolved
- frequency resolution is $1/Mt_s$
  - standard frequency resolution for a series with sample rate $f_s = 1/t_s$ and $M$ samples.

Notice that the product of the two resolutions is a constant!

# Cutting up the time-frequency space

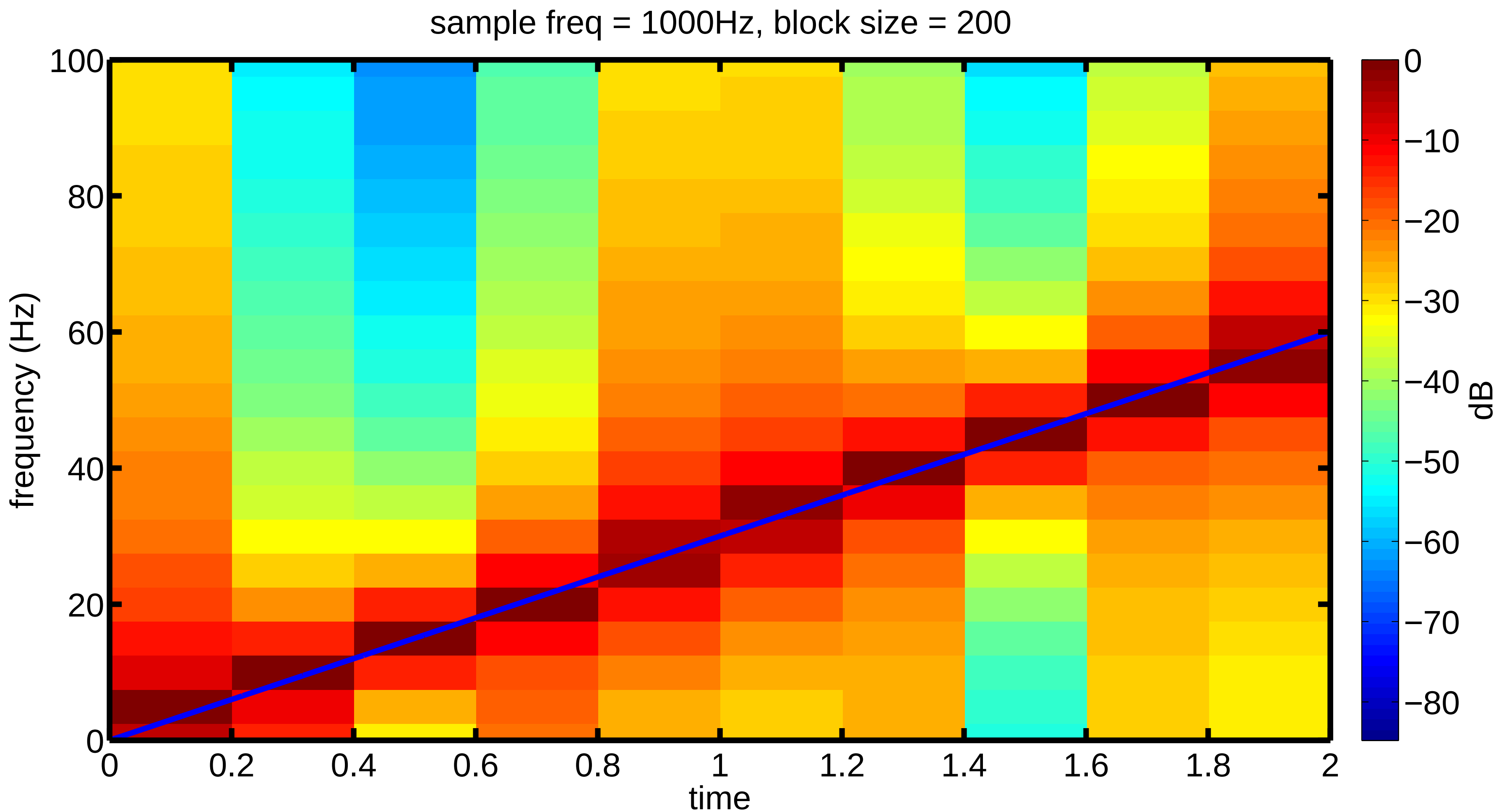
STFT (different width window)




Areas of boxes don't get smaller!

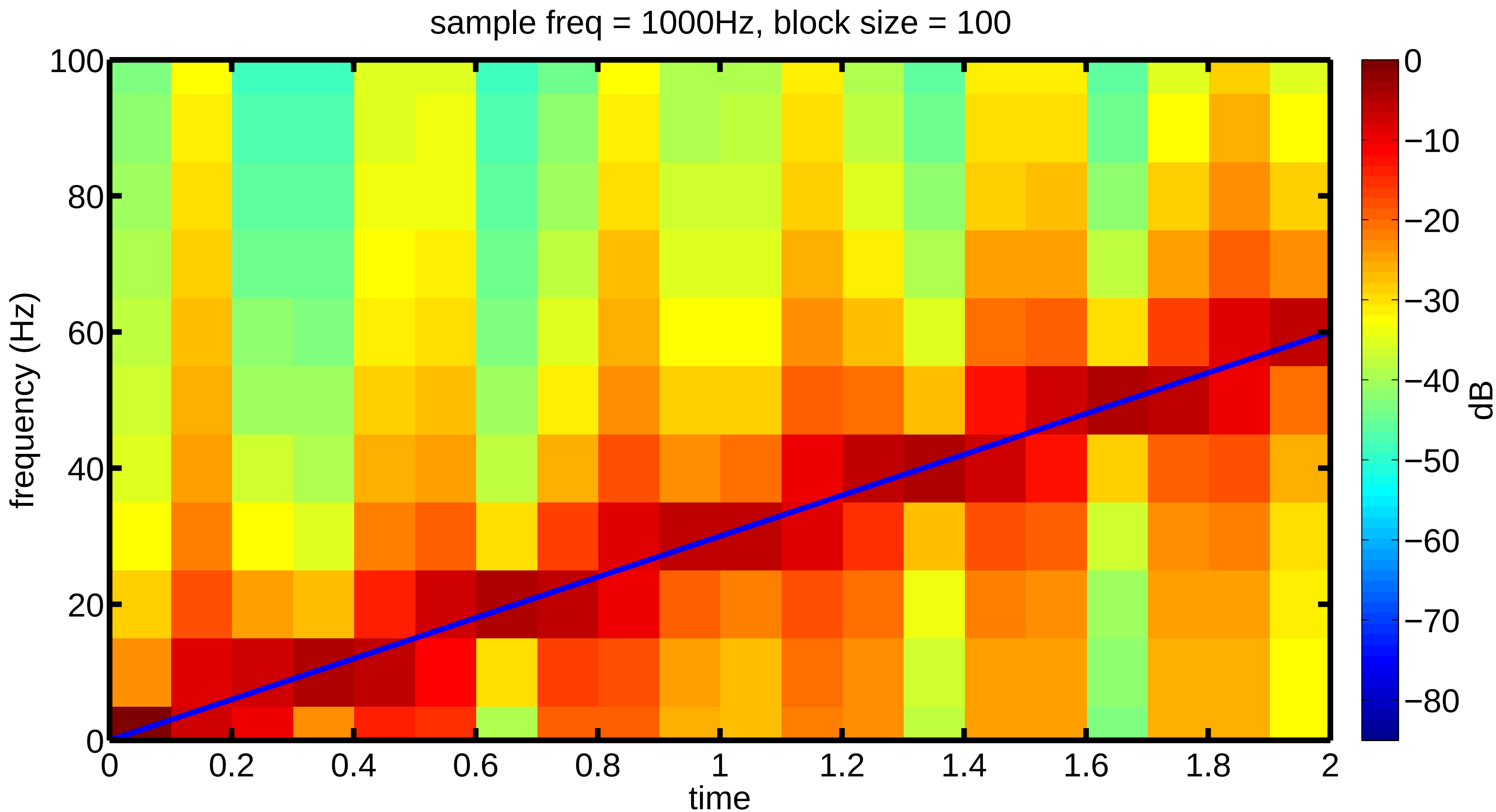
# Spectrogram of a Chirp

## Spectrogram of a chirp



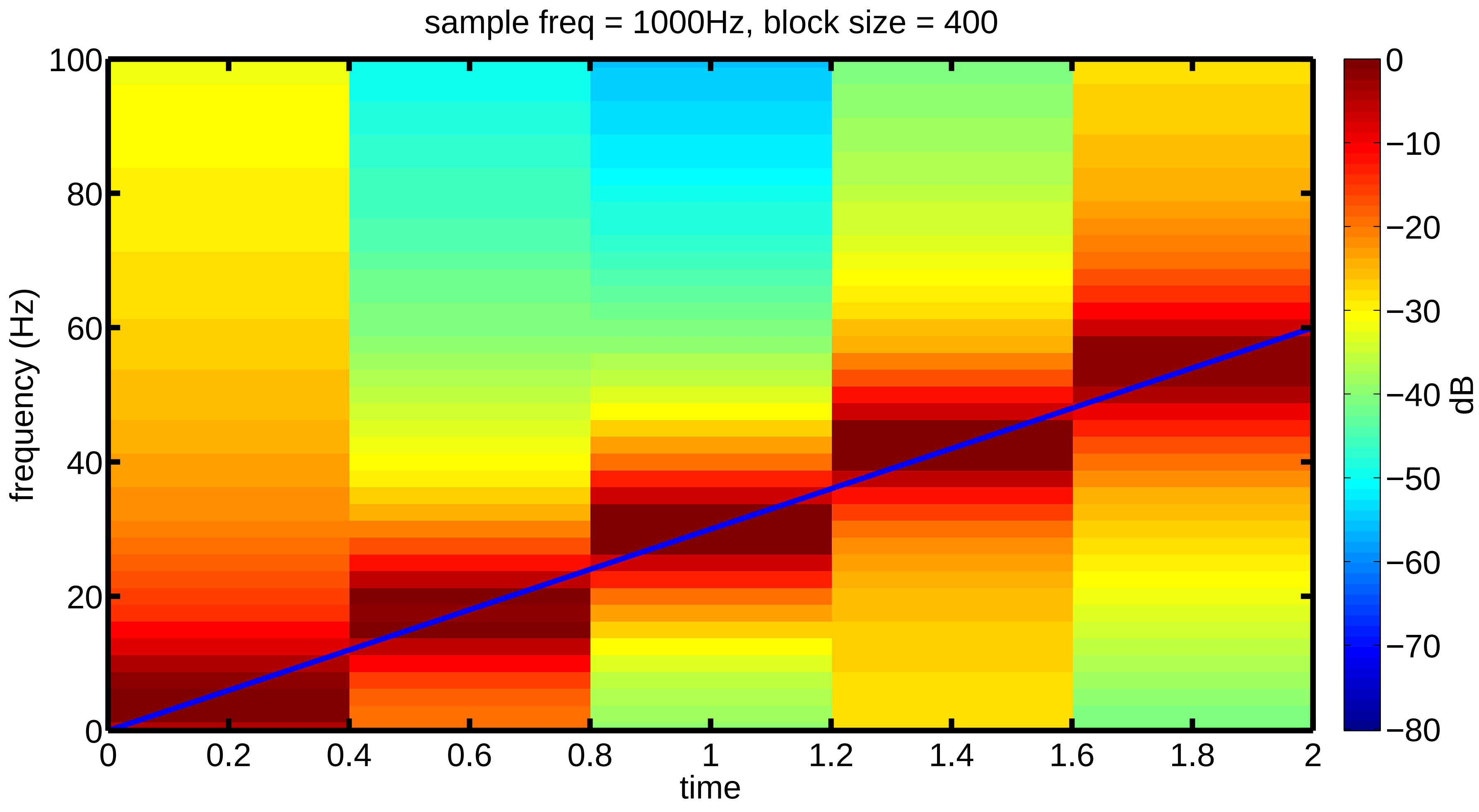
# Spectrogram of a Chirp

## Spectrogram of a chirp



# Spectrogram of a Chirp

Spectrogram of a chirp



# Windows

- STFT with a window: sometimes called the Windowed FT (WFT)
- What window should we use?
- Choice involves tradeoffs
  - length of window (and hence computational cost) (e.g. does it have compact support)
  - size of uncertainty (Gabor function has minimal uncertainty region)
  - regularity of window, and roll off in Fourier domain
  - windows side-lobes vs its width in Fourier domain
  - can scale window and tradeoff frequency for time resolution

# Limitations of the STFT

- computational cost $O(nm \log m)$
- time/frequency resolution tradeoff
  - small $m$ better time, worse frequency resolution
  - large $m$ better frequency, worse time resolution
- time/frequency resolution tradeoff is fixed
  - higher frequencies can change faster than lower frequencies
  - would be nice to have appropriate resolution for each frequency

# The answer: wavelets

- next lecture