Transform Methods & Signal Processing

lecture 03

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Discrete signals

In theory there is no difference between theory and practice. In practice there is.

Yogi Berra

Discrete signals

Real signals (these days) are discrete

- Moore's law (speed of digital hardware increases by a factor of two every 18 months, or the number of transistors on a chip doubles, or the cost halves).

  "Cramming more components into integrated circuits", Gordon E. Moore, Electronics, Vol. 38, No. 8, April, 1965.

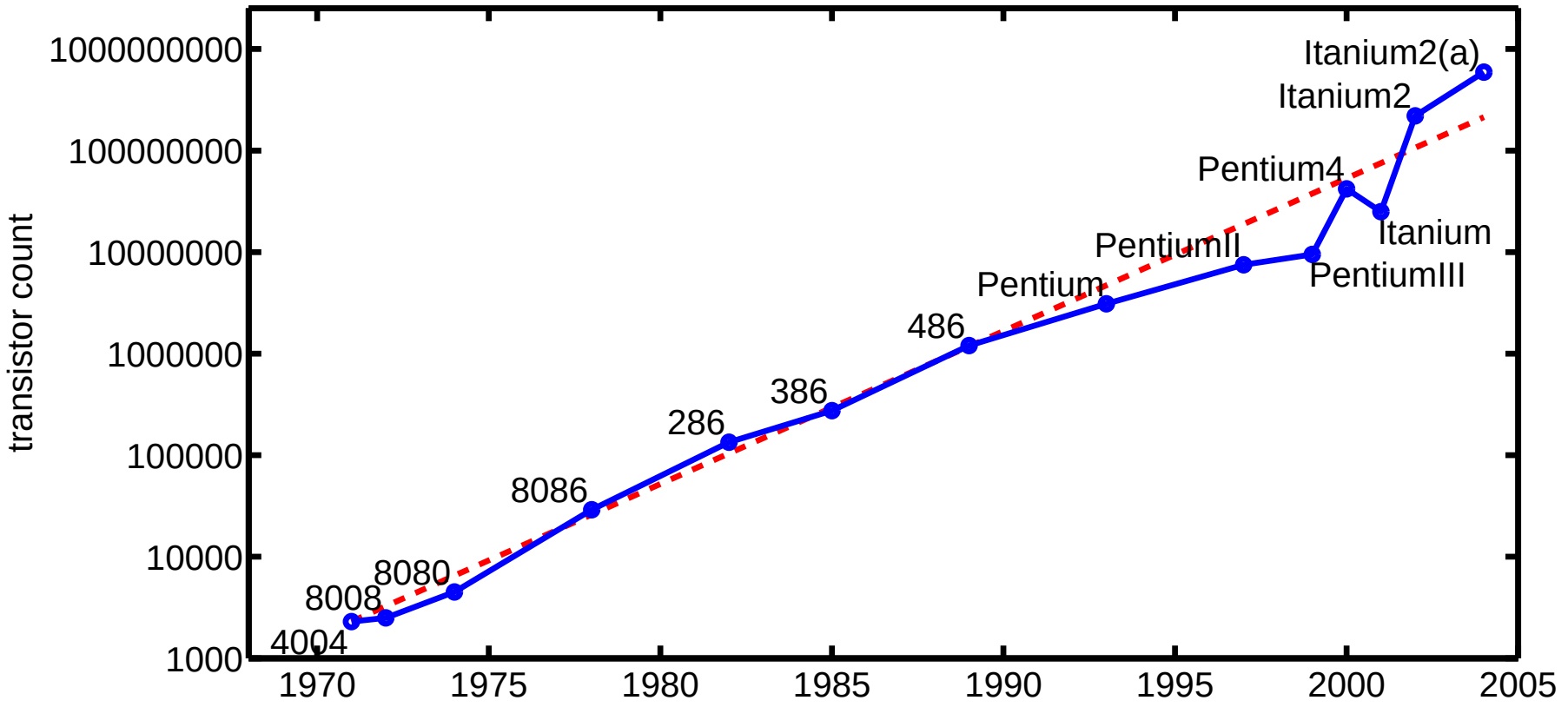
- Easier/cheaper to use standard DSP solution.

  e.g. CD players — we can get nominally better results from a LP record, and a really good player, but CD's cost orders of magnitude less for **almost** indistinguishable results.

- If it isn't cheap enough today, it will be in a year.

Moore's Law

Moore's law: the speed of digital hardware increases by a factor of two every 18 months, or the number of transistors on a chip doubles, or the cost halves.



Actually looks more like a factor of 2 every 2 years.

Gates's law

Gates's Law: The speed of software halves every 18 months.

Gates's law does not apply to DSPs (they use small embedded OSes).

Parkinson's Law of Data: Data expands to fill the space available for storage

Parkinson's law of data **does** typically apply. As chips get faster, we sample at higher resolution, and faster sampling rates...

Real signals

In theory there is no difference between
theory and practice. In practice there is.

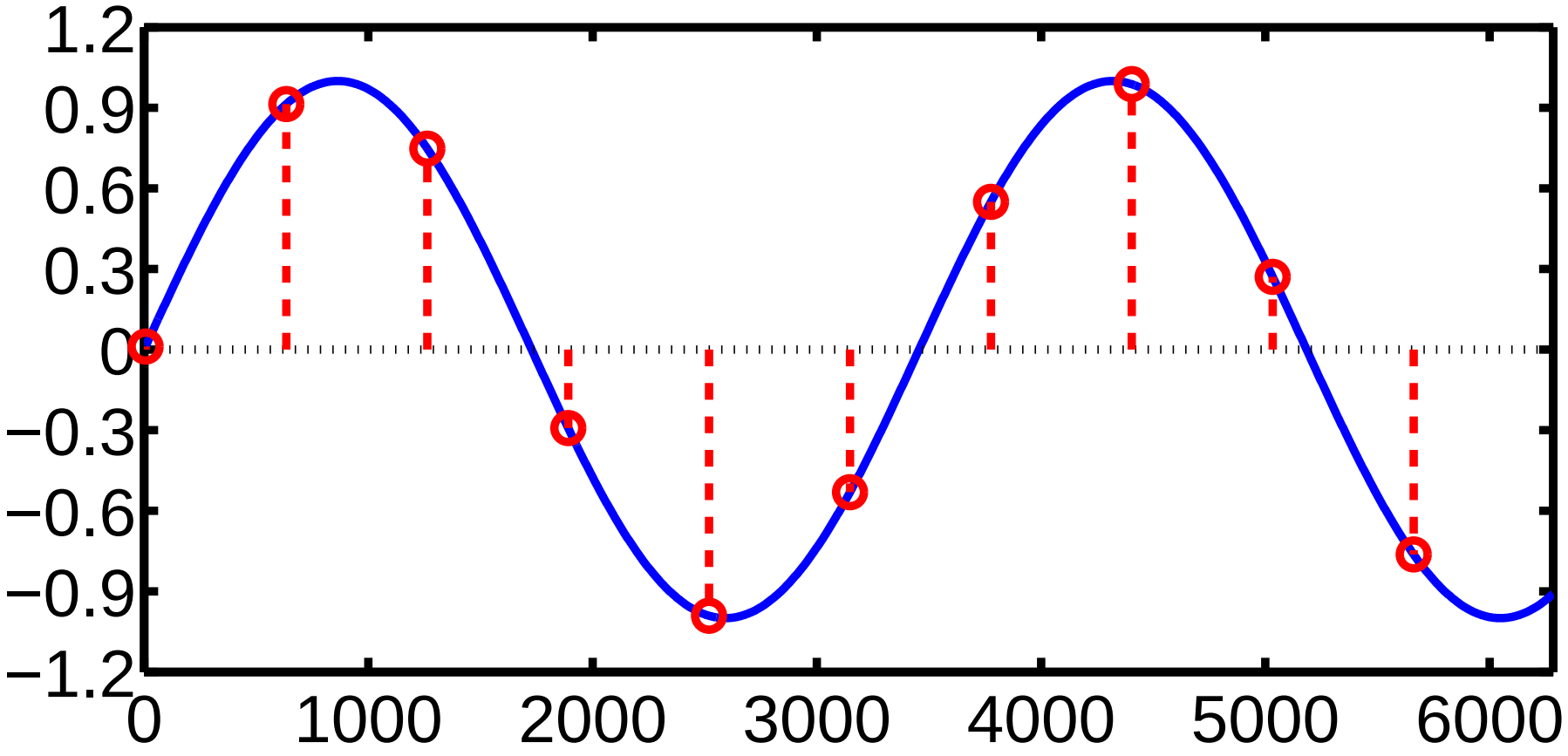
Yogi Berra

Real data is

- finite (integrals convergence much easier)
- discrete time
- discrete valued

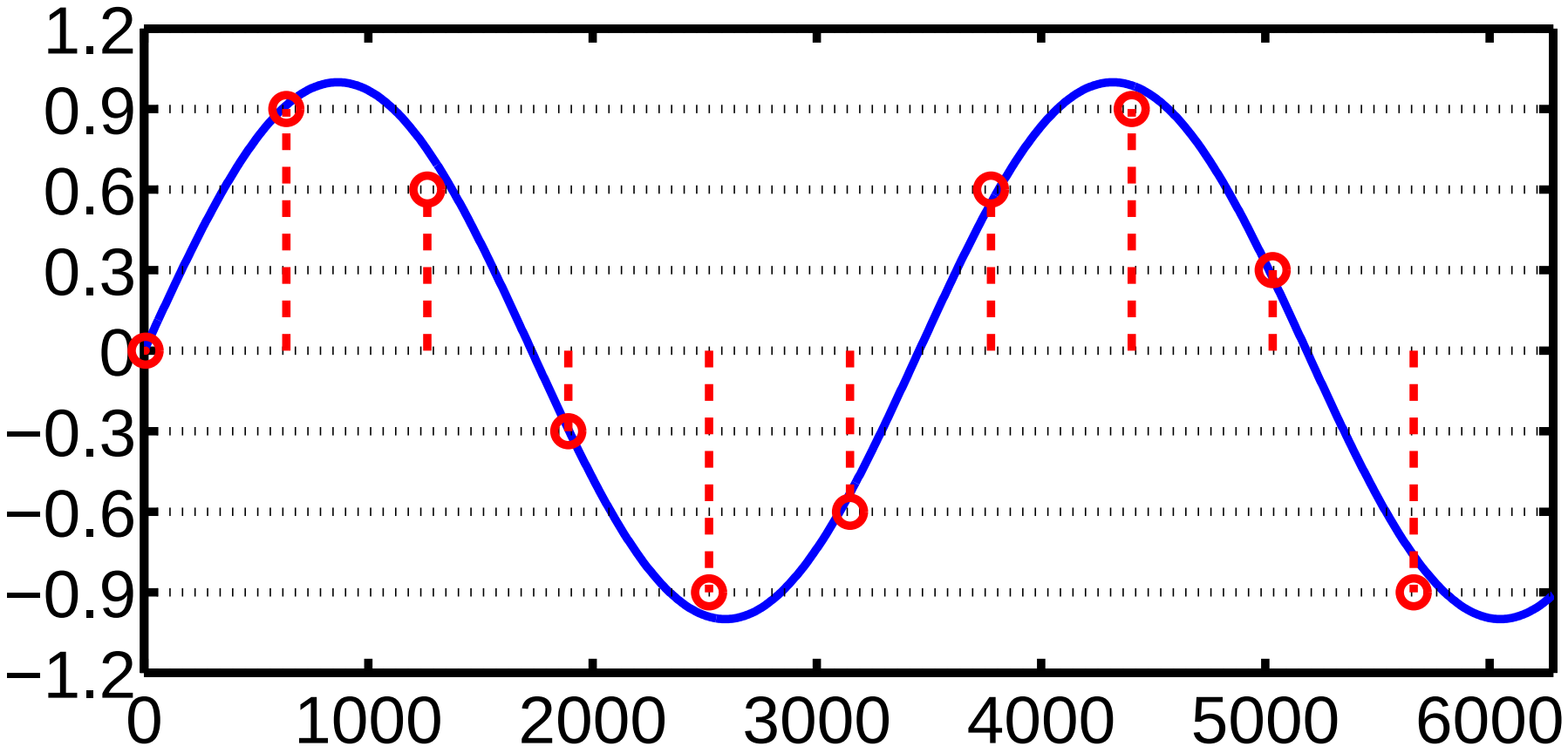
Discrete time

- Real signals are **discrete-time**
- We can **sample** a continuous function to get a discrete approximation



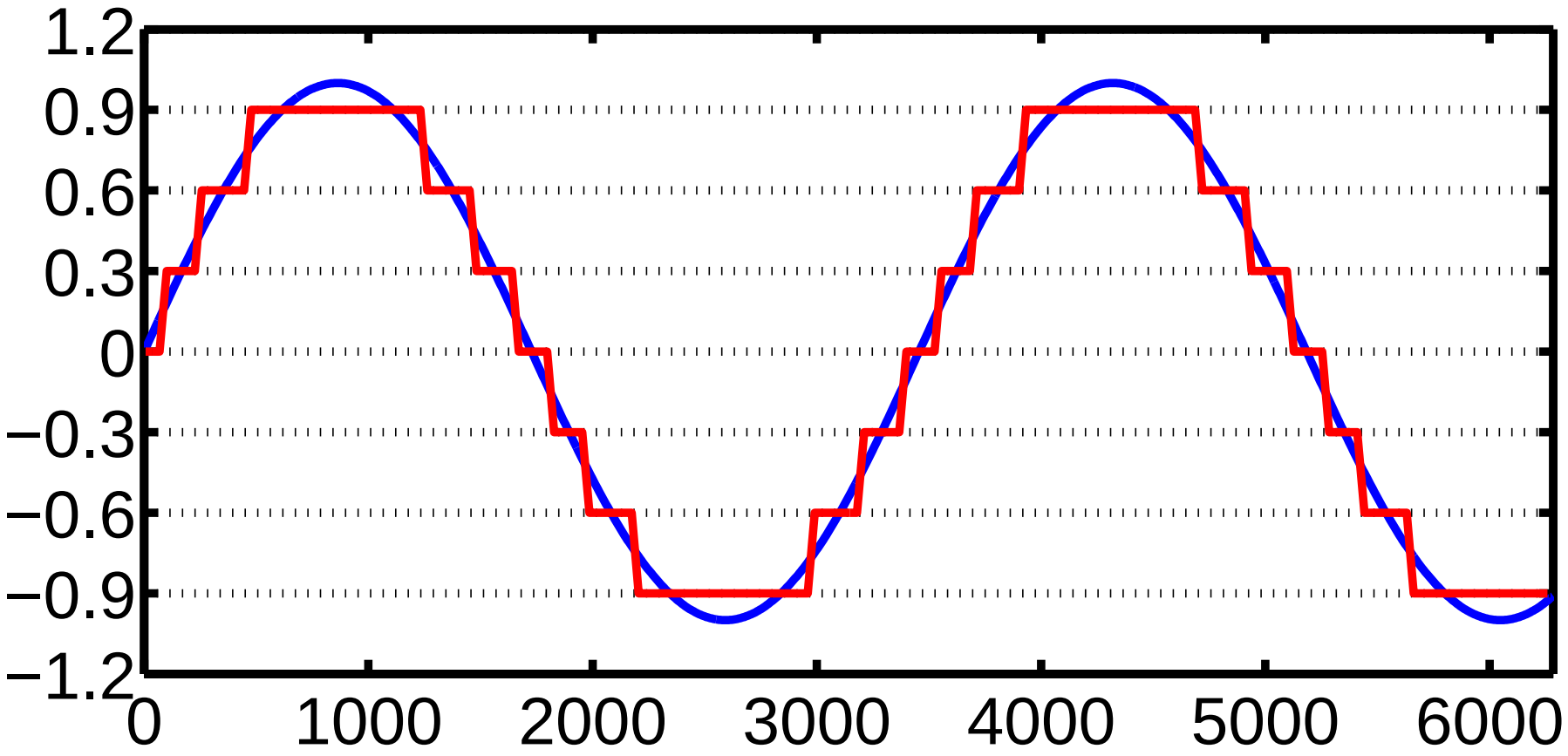
Quantization: discrete-value

- Real signals are **discrete-valued**
- Analogue to Digital conversion: sample in time, and quantise



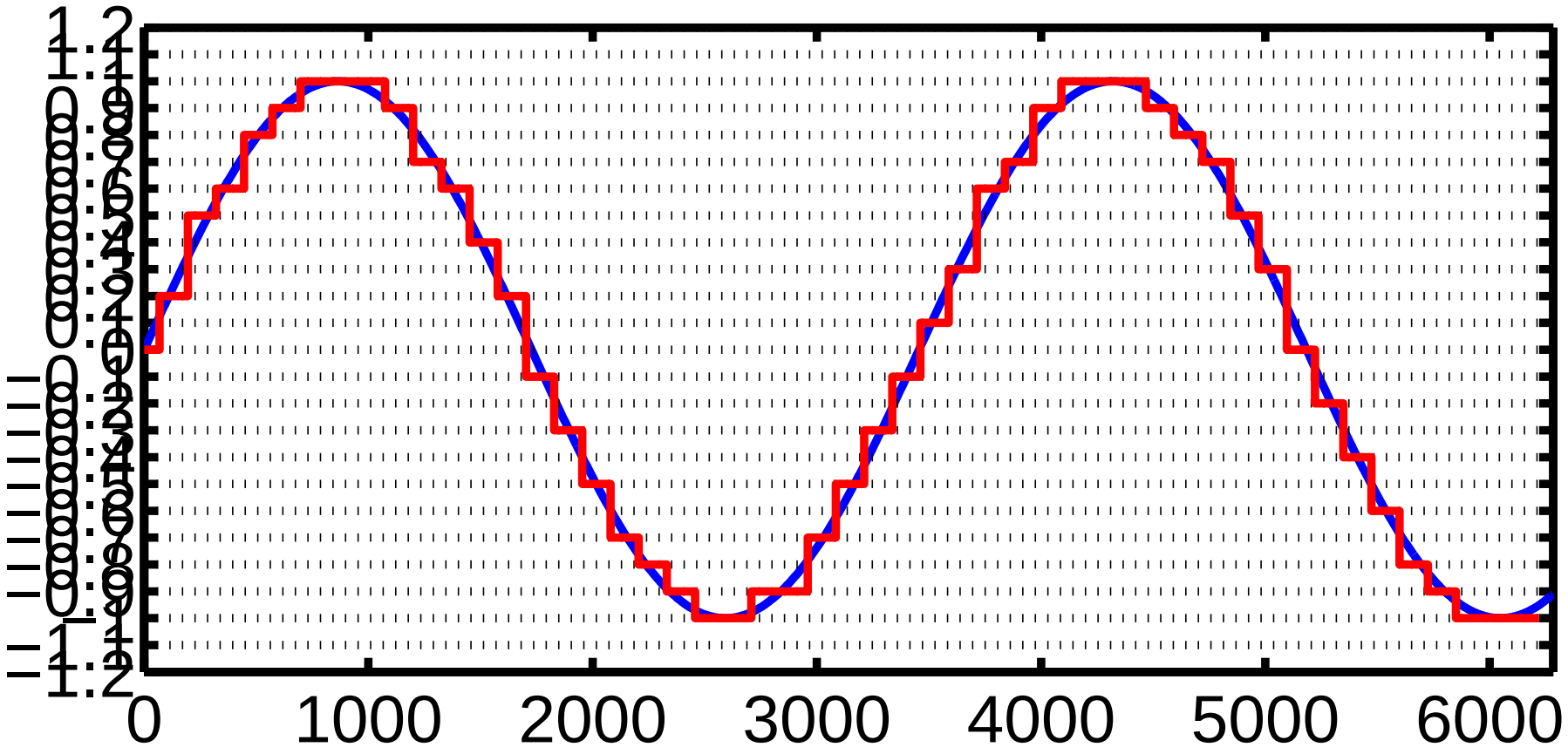
Approximation

- Faster sampling => better approximation
- More details later



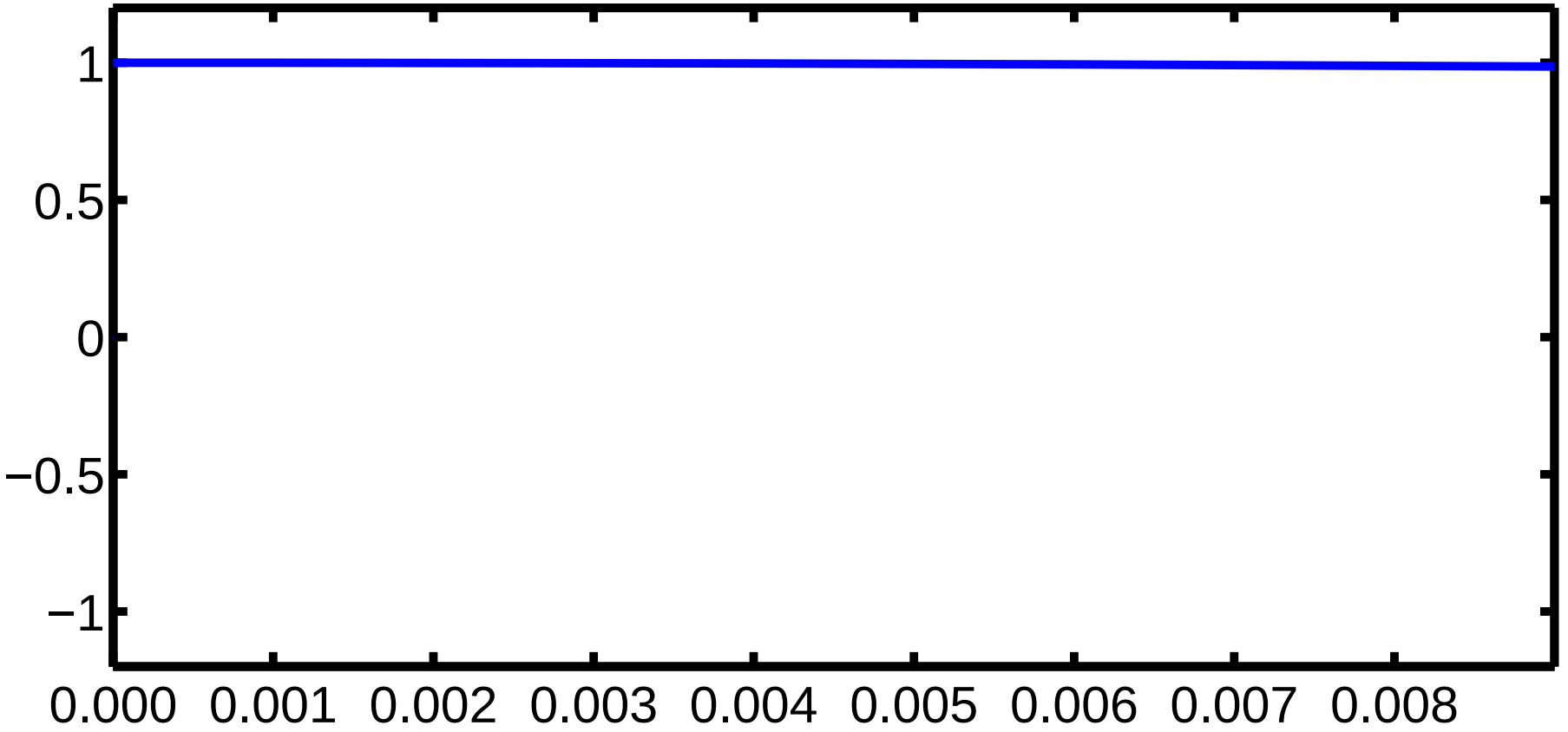
Approximation

- Finer quantization => better approximation
- More details later



Approximation

- Longer data sets => better approximation
- More details later



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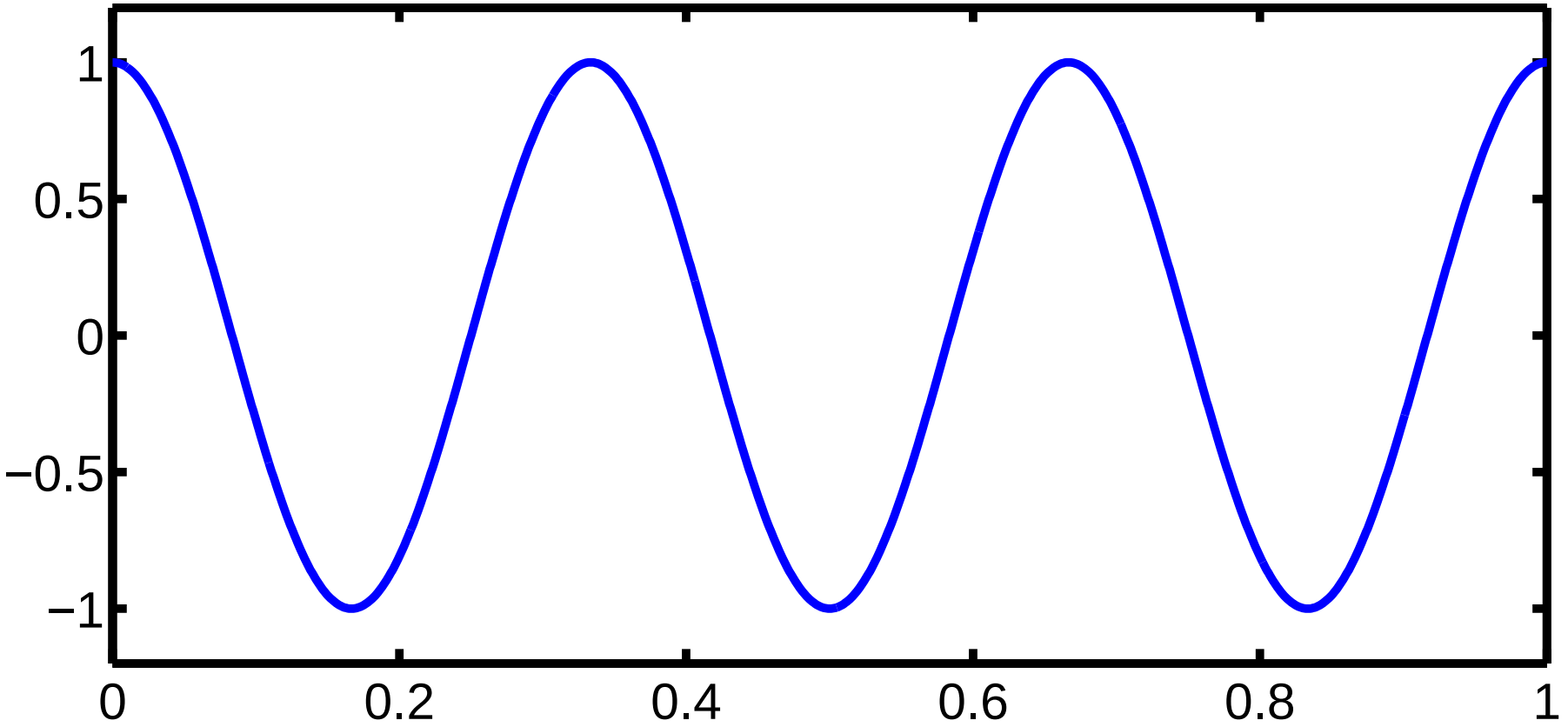


Sampling

Sampling produces a new time series, with discrete index, e.g.

$$x(n) = f(nt_s)$$

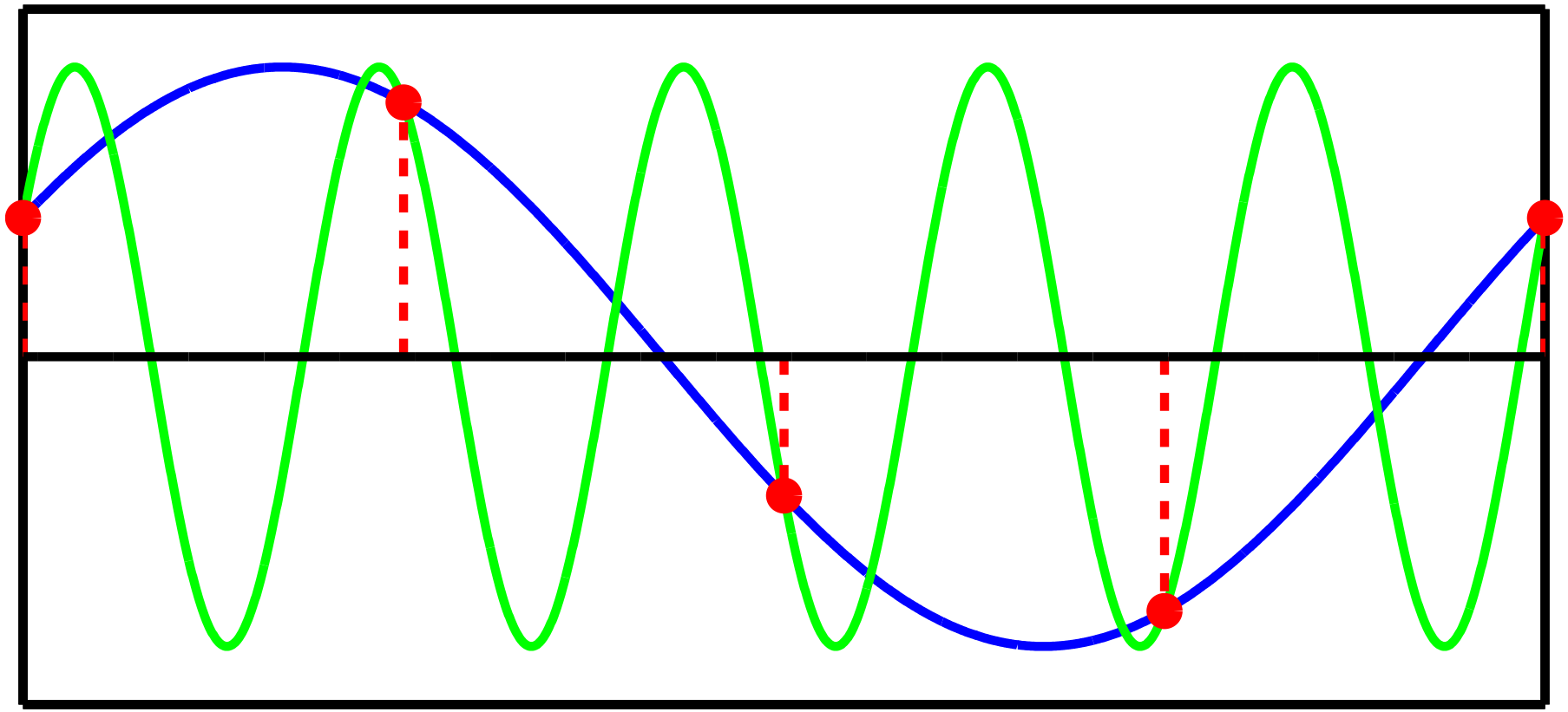
where t_s is the **sampling interval**

The **sampling frequency** is $f_s = 1/t_s$.

e.g. sampling frequency for CDs is 44.1 kHz

Aliasing

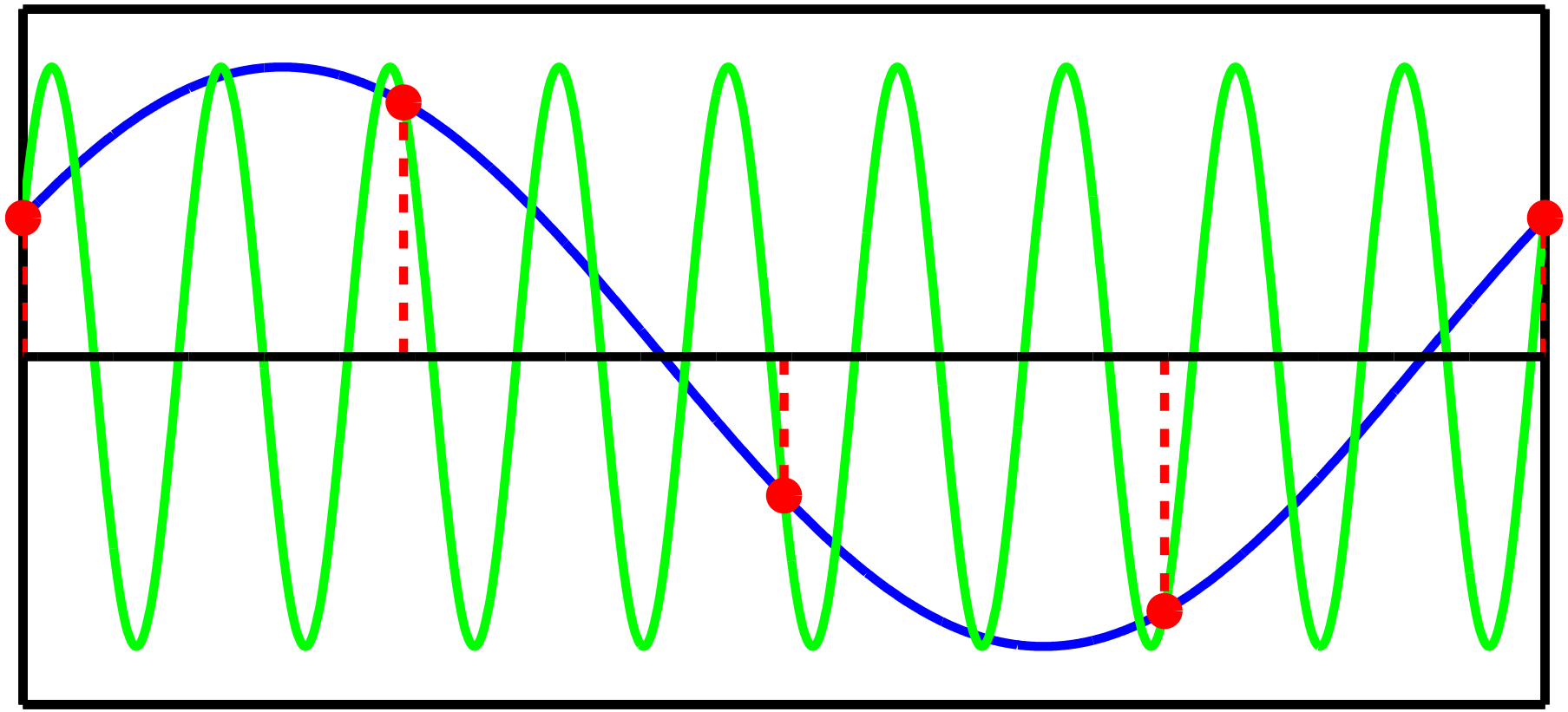
A critical issue for sampling is **aliasing**



Samples might be caused by different underlying signals.

Aliasing

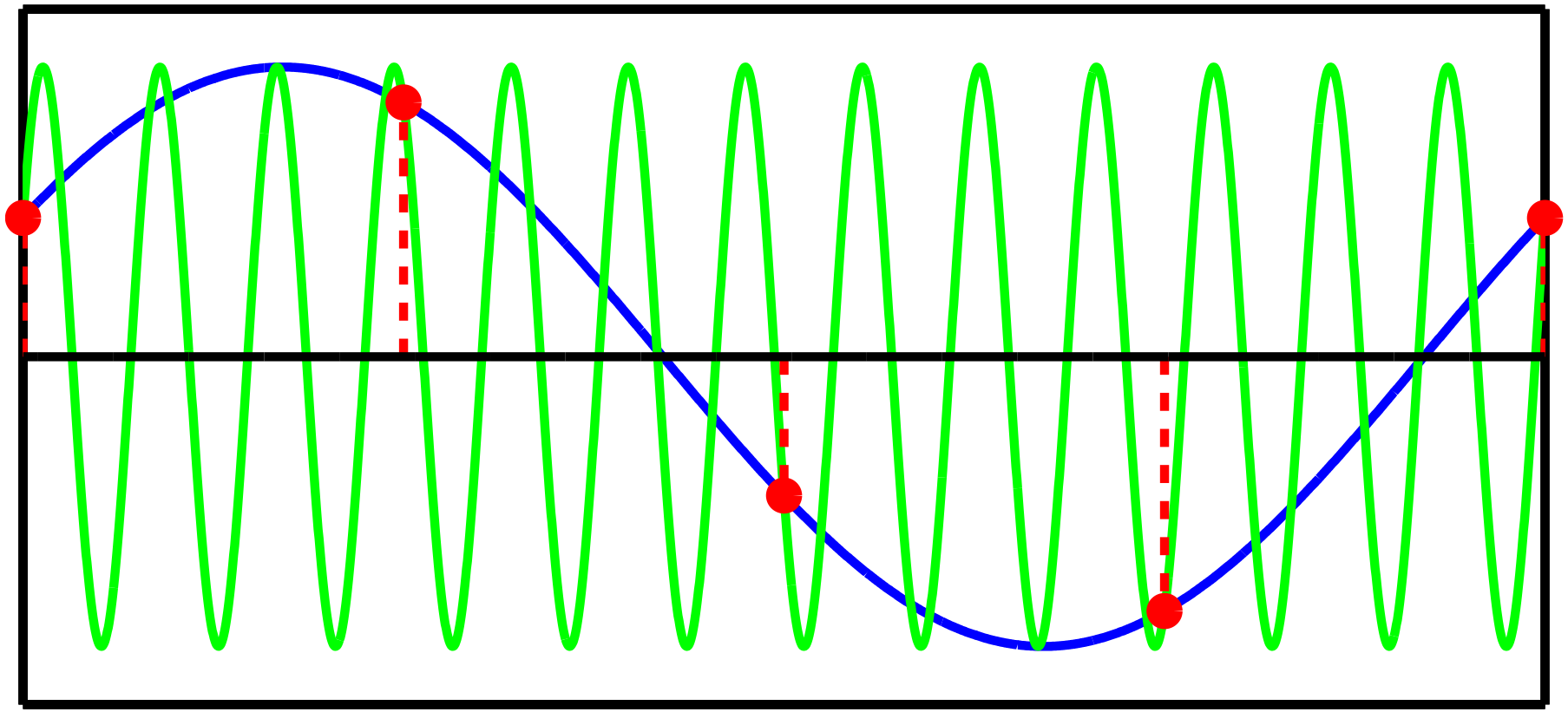
A critical issue for sampling is **aliasing**



More than one ambiguity.

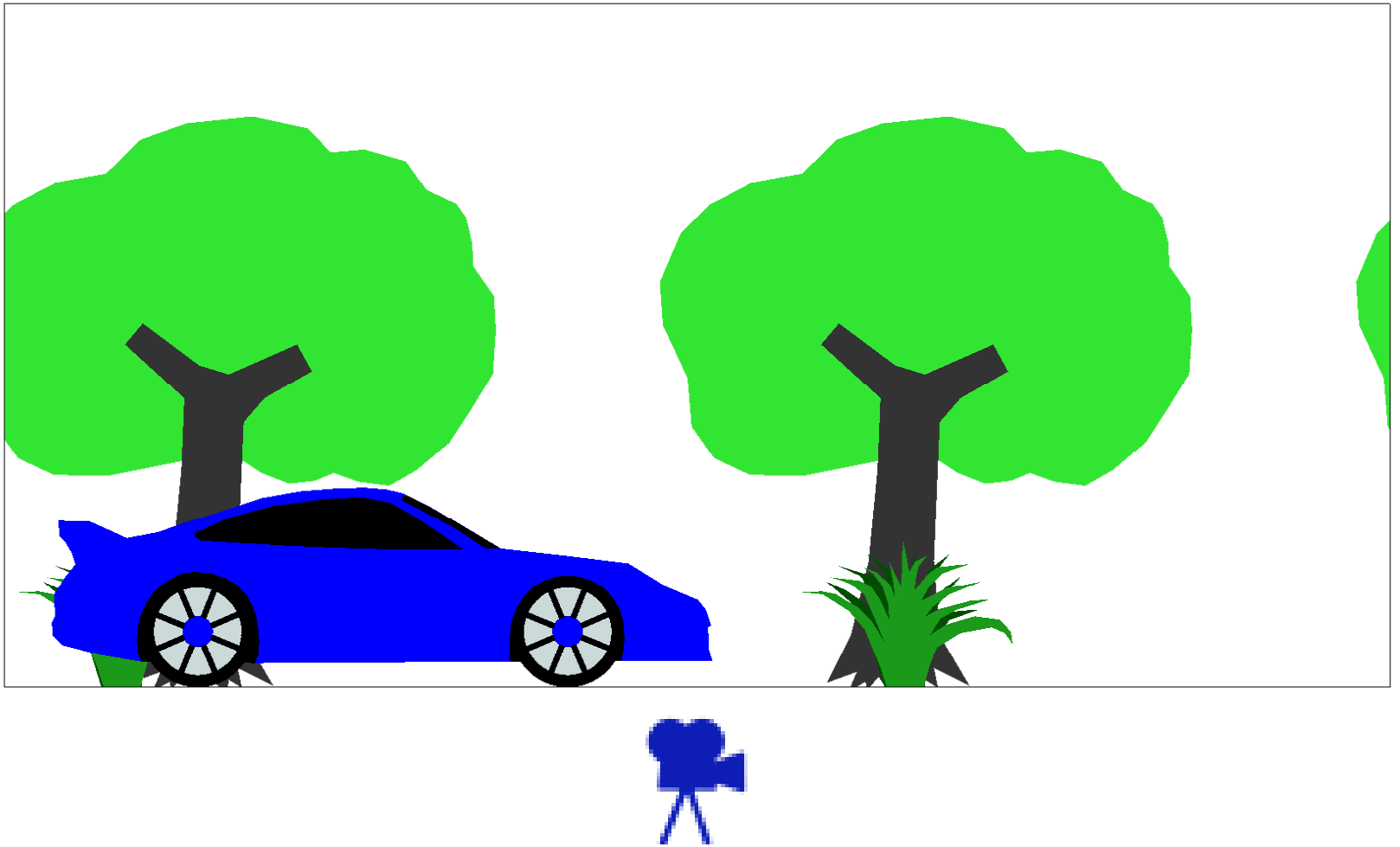
Aliasing

A critical issue for sampling is **aliasing**



An ∞ number of possibilities...

Another example of aliasing



Aliasing: time domain view

- Signal with frequency f_0, given by $f(t) = \sin(2\pi f_0 t)$
- Sampling interval t_s, and sampling frequency $f_s = 1/t_s$.
- Sampled signal is $x(n) = f(nt_s) = \sin(2\pi f_0 n t_s)$
- We can always add $2\pi m$ (where m is an integer) to a sin function without impact, e.g.

$$\begin{aligned} x(n) &= \sin(2\pi f_0 n t_s) \\ &= \sin(2\pi f_0 n t_s + 2\pi m) \\ &= \sin\left(2\pi\left[f_0 + \frac{m}{nt_s}\right] nt_s\right) \\ &= \sin(2\pi[f_0 + f_s k]\, nt_s) \text{ where } m = kn. \end{aligned}$$

So there is an ambiguity in $x(n)$ about frequencies $f_0 + f_s k$ for integer k.

Aliasing: frequency domain view

Consider a **delta train** or **Dirac comb** defined by

$$d(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$$

We can consider sampling of a function $f(t)$ to be equivalent to taking the product with a delta train, e.g.

$$x(t) = d(t/t_s)\, f(t)$$

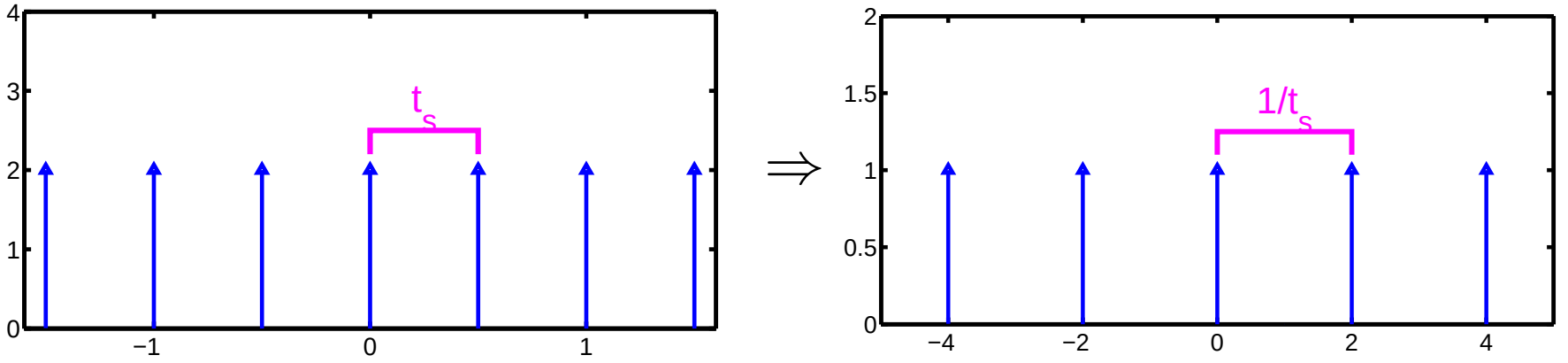
From the convolution, and the duality theorems, we can see that the FT of $x(t)$ will be the convolution of the FTs of $d(t)$ and $f(t)$.

The FT of the delta train is $\mathcal{F}\{d(t/t_s)\} = |t_s| d(t_s s)$

Delta train

A train of delta functions $d(t/t_s) = \sum_{n=-\infty}^{\infty} \delta(t/t_s - n)$ has Fourier transform which is also a delta train, e.g.

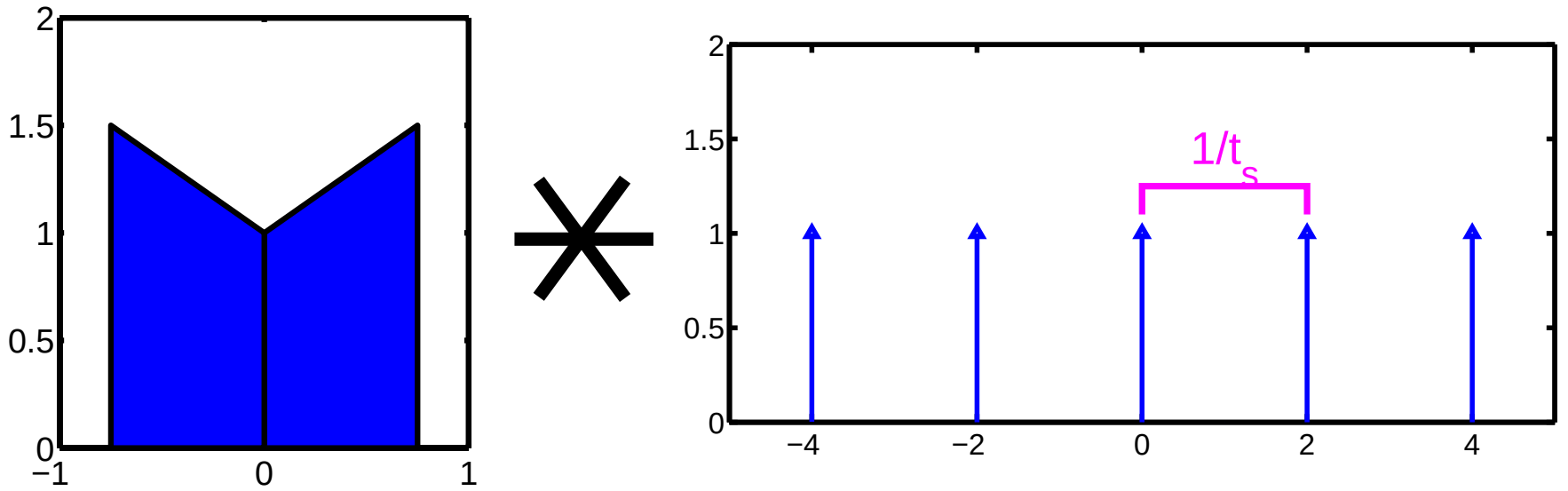
$$\mathcal{F}\{d(t/t_s)\} = |t_s| d(t_s s)$$

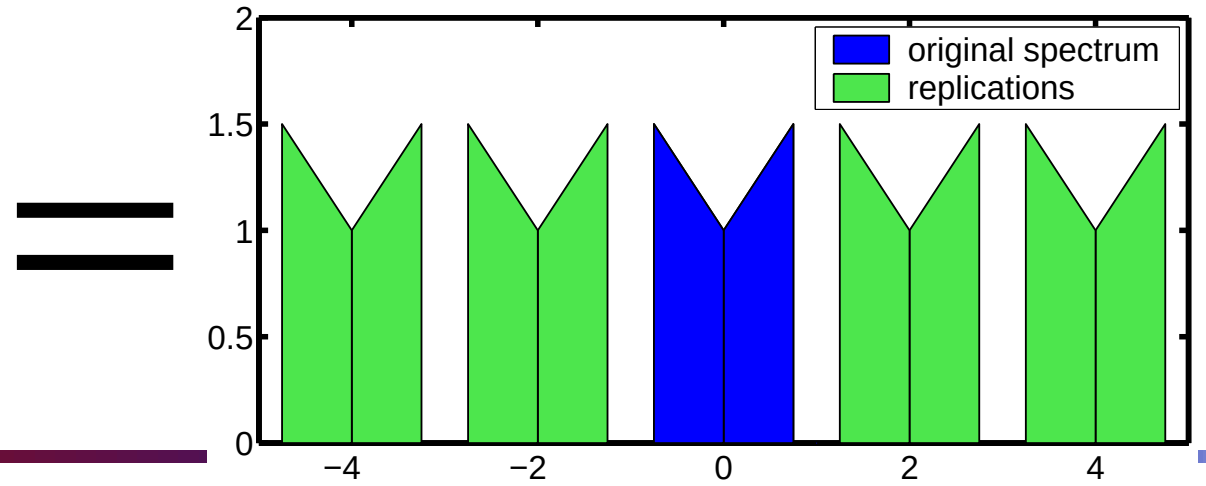


Aliasing: frequency representation

$$x(t) = d(t/t_s)\, f(t) \Rightarrow X(s) = d(t_s s) * F(s)$$

Convolution of a delta train with a function looks like:





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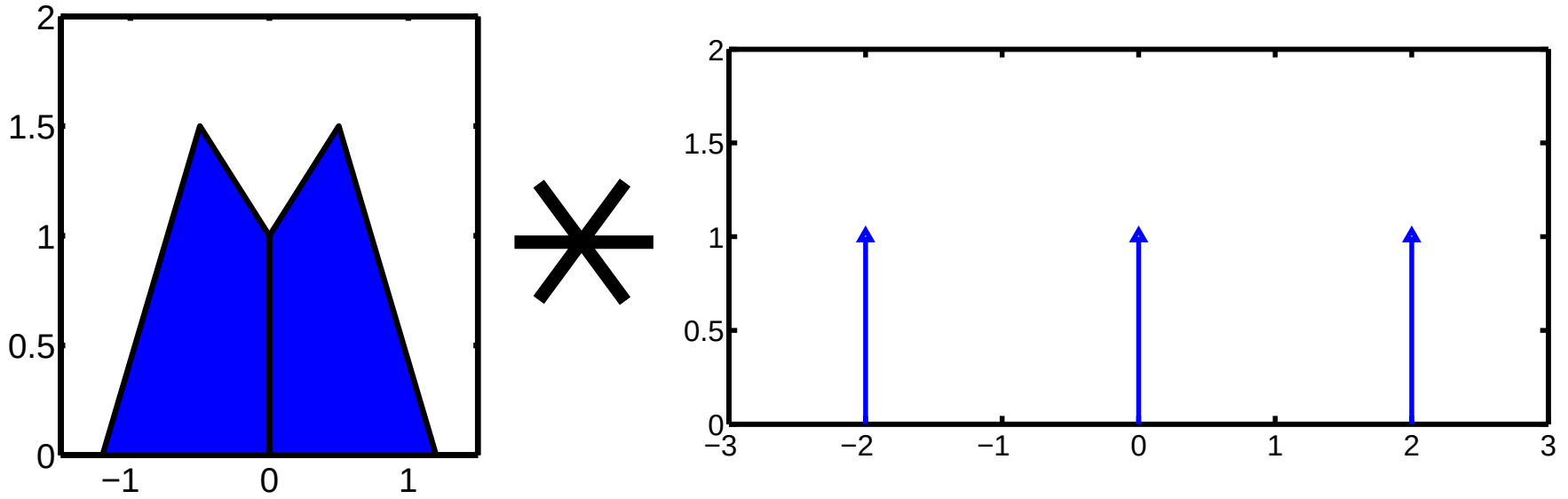

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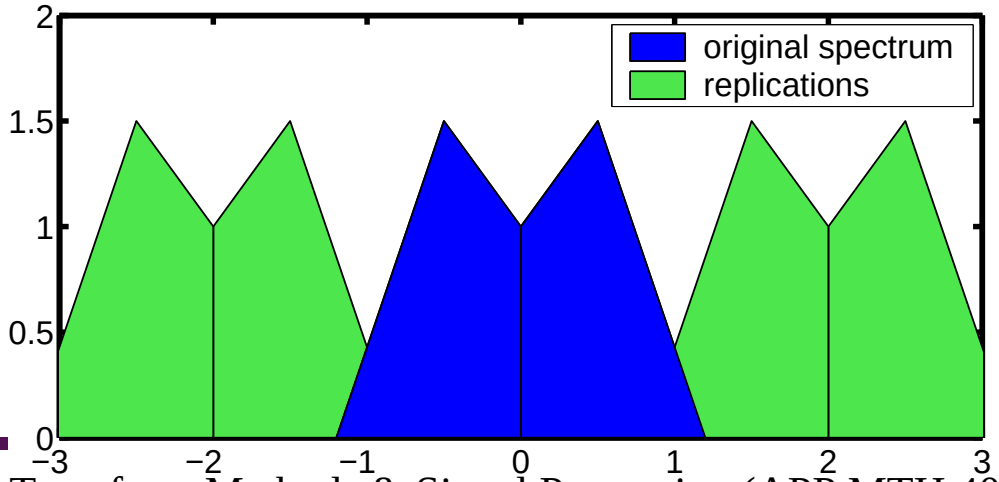


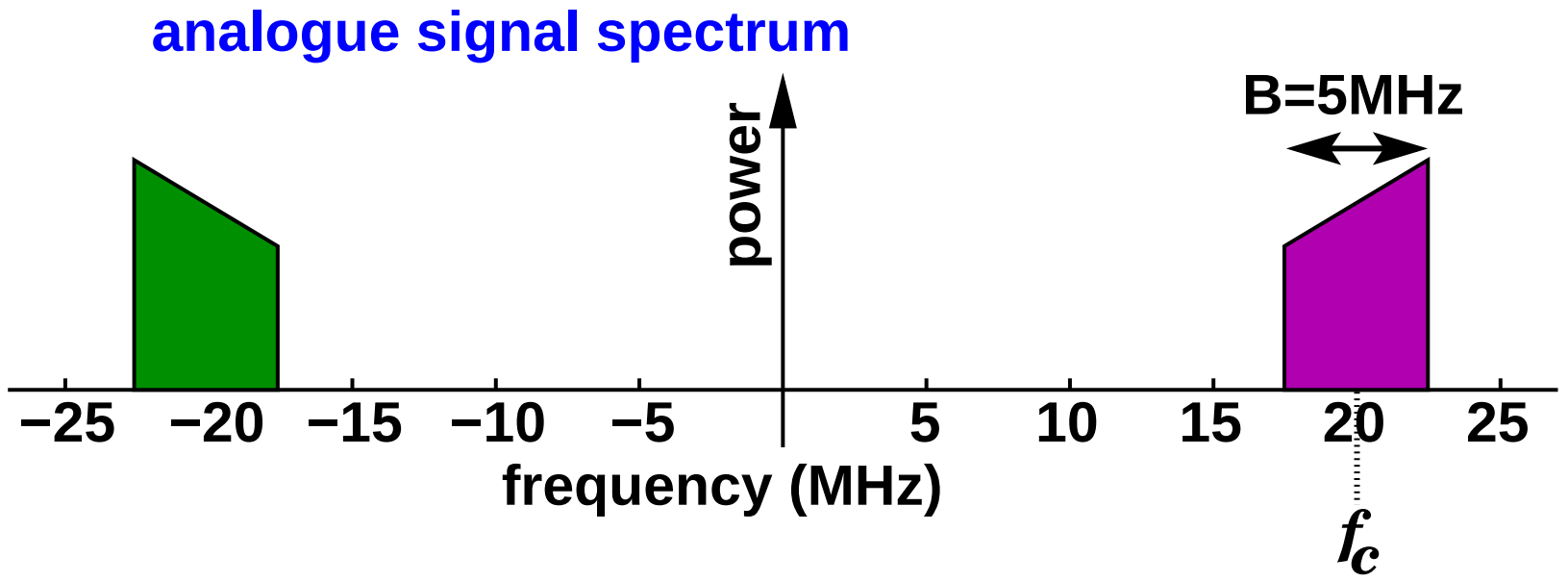
Nyquist sampling theorem

Assume the spectrum of the signal is zero above a critical frequency f_c. We call this the bandwidth of the signal.
For sampling frequencies $f_s > 2f_c$, the spectra above won't overlap. If $f_s < 2f_c$ aliasing becomes a problem.

- the critical sampling rate referred to by, e.g. the Nyquist rate, or Shannon (1949) or Whittaker (1935) sampling theorem.
- the sampling frequency must be greater than twice the highest frequency present in the signal
- need to bandlimit the input signal before sampling
- bandwidth does not need to be centered on zero Hz.

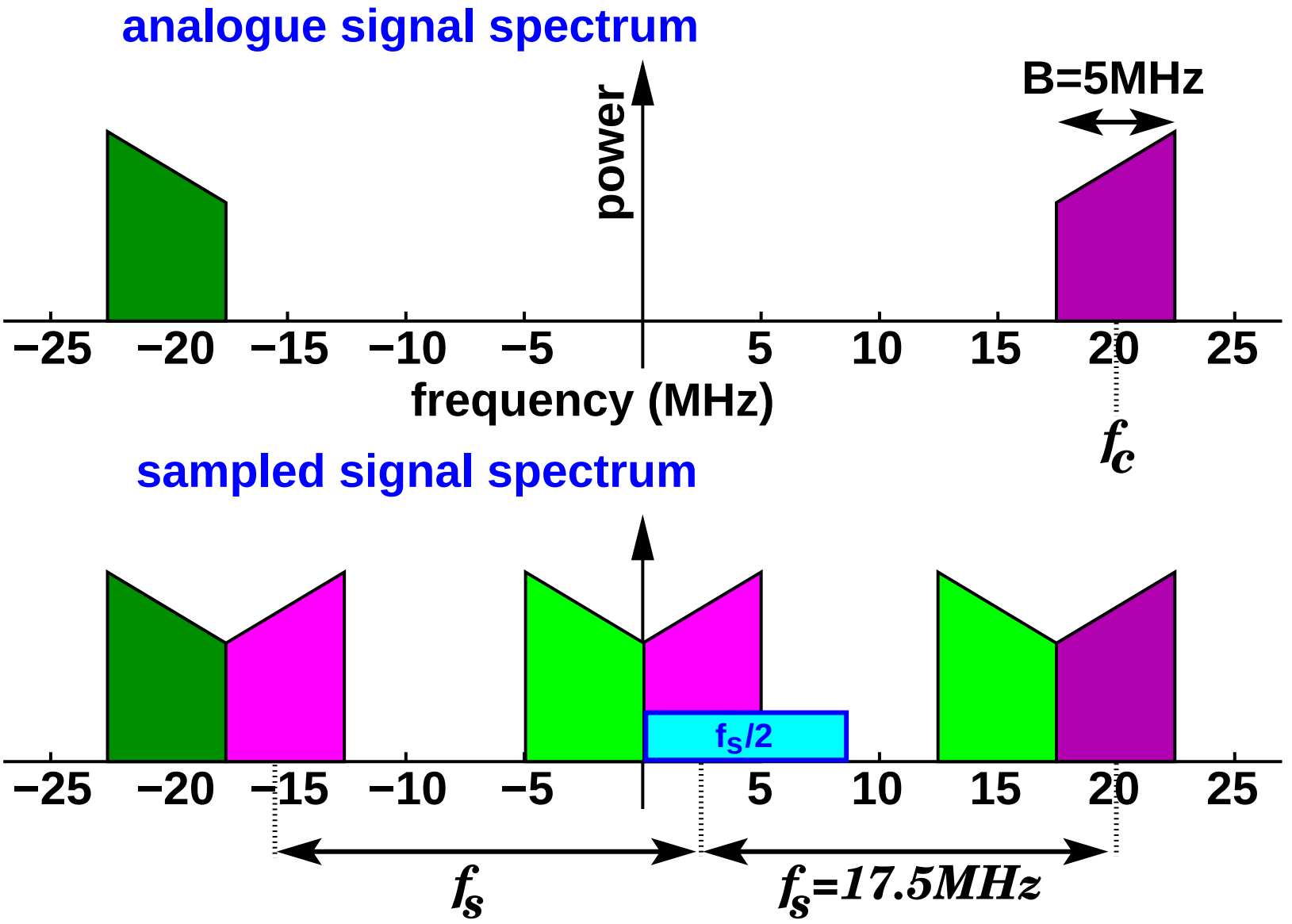
Example

Analogue signal with central frequency 20 MHz, and 5 MHz bandwidth.

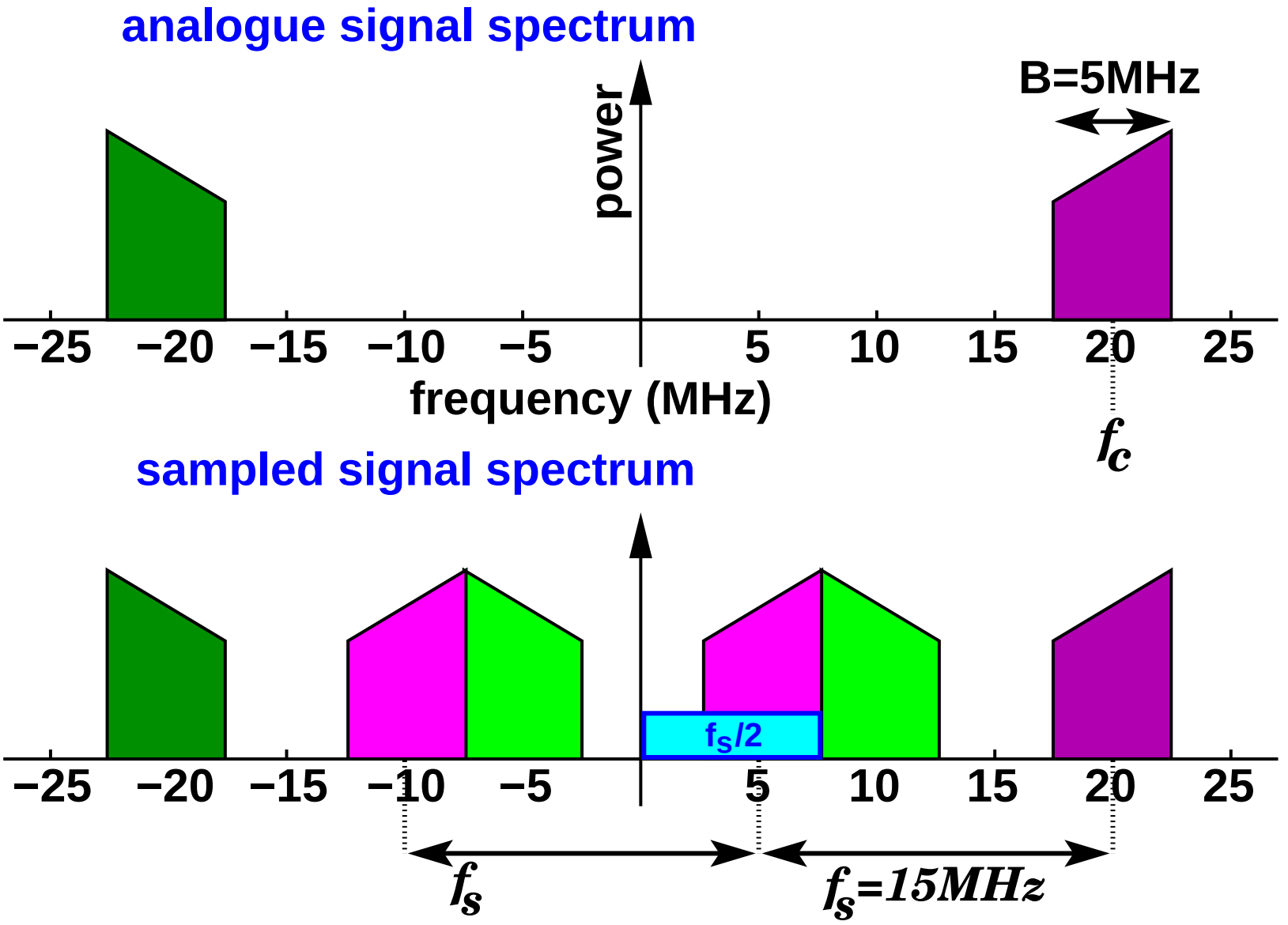


To include entire spectrum, we need to sample at $2 \times 22.5 = 45$ MHz.

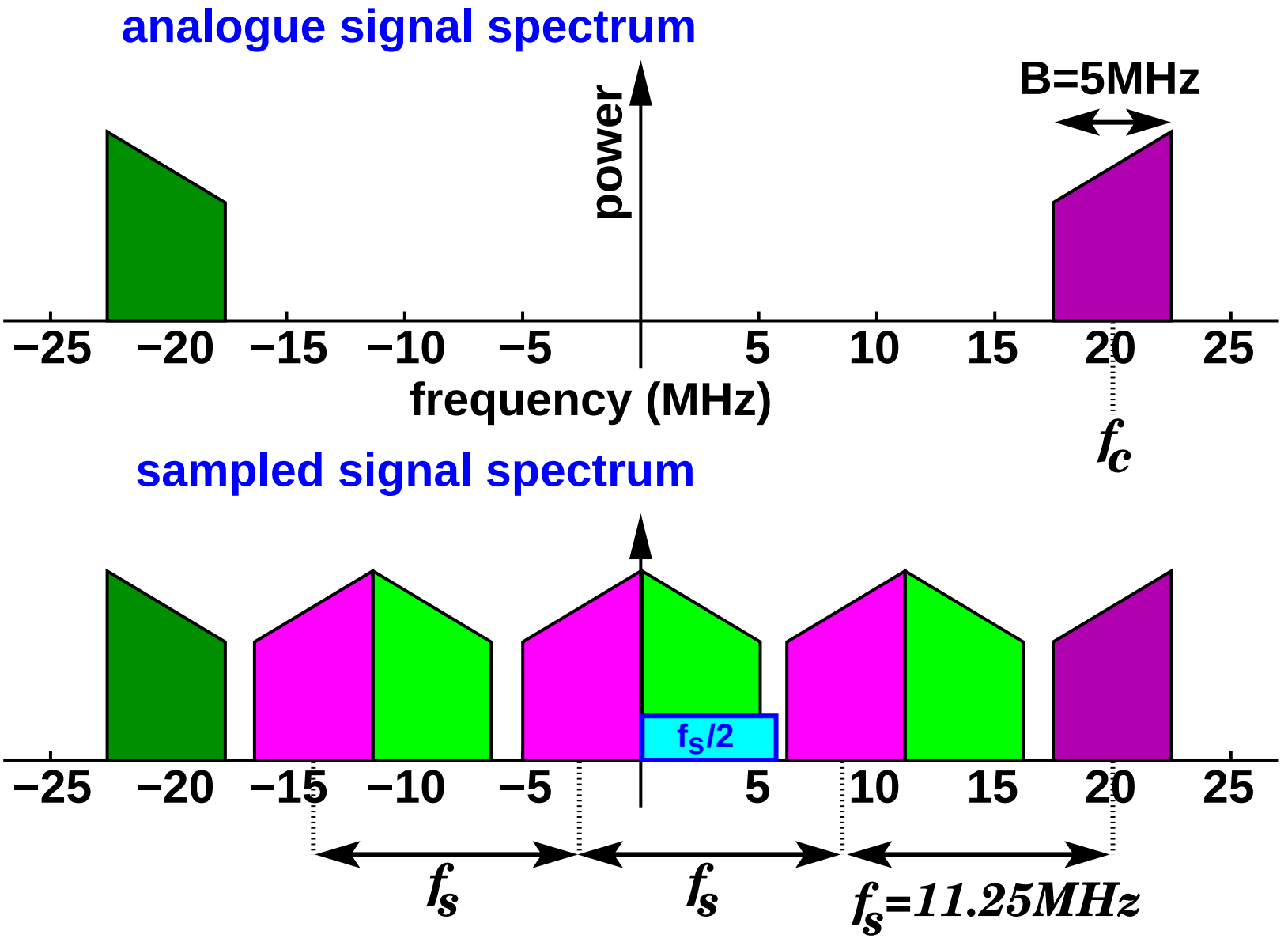
Example: Sample at 17.5 MHz



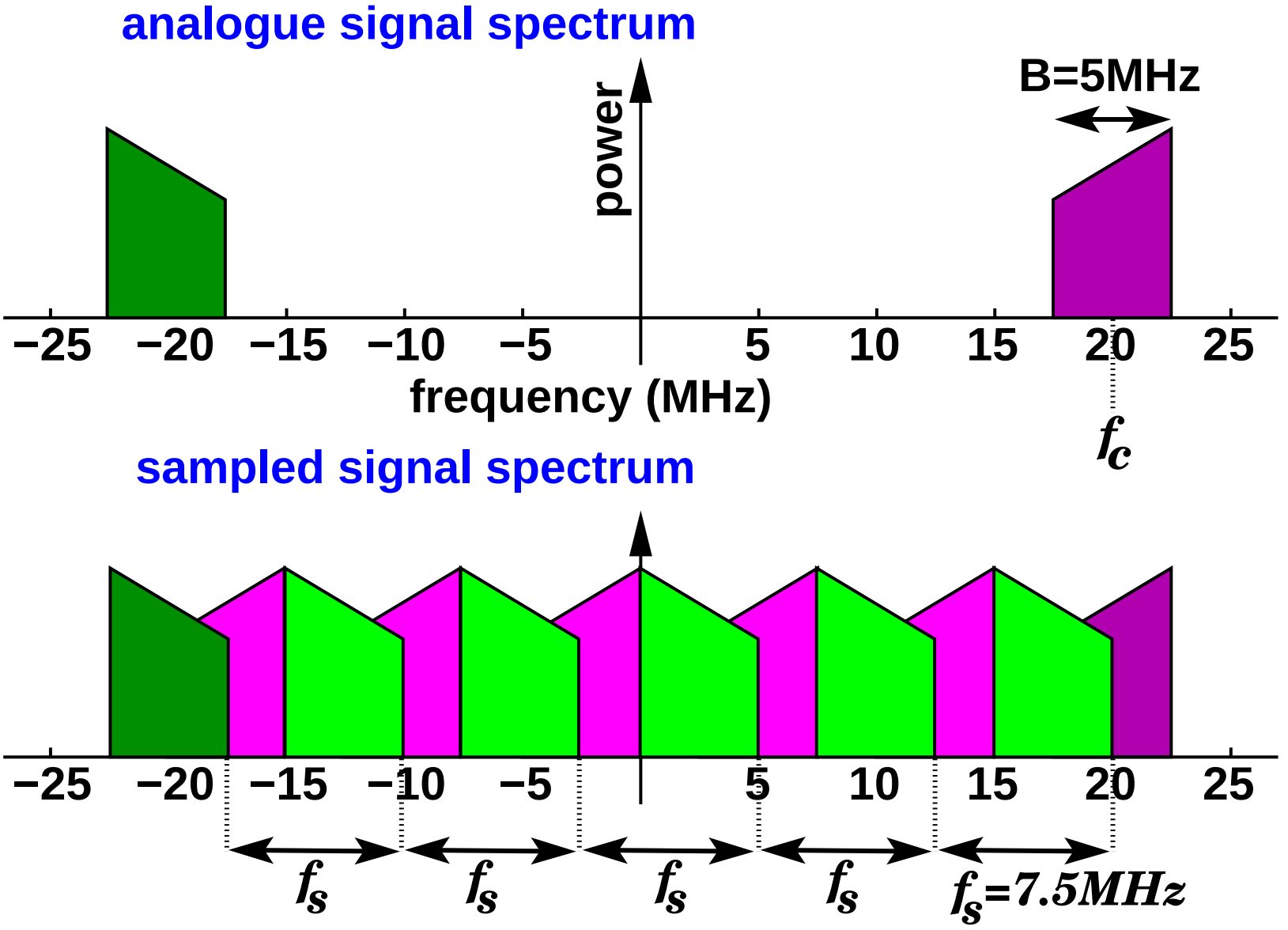
Example: Sample at 15 MHz



Example: Sample at 11.25 MHz

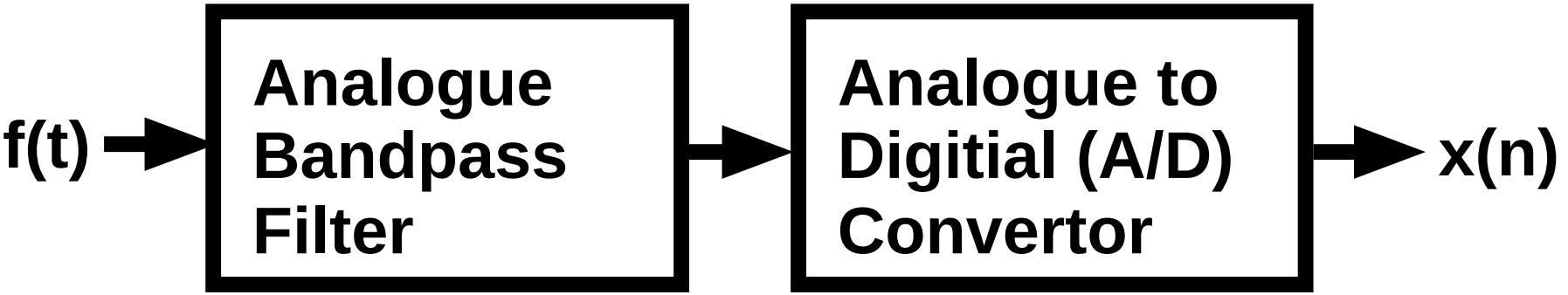


Example: Sample at 7.5 MHz



Bandlimiting

When sampling from real signals, one **must** bandlimit the input!



Shouldn't push the boundaries with sampling, and filters

- analogue filter might not be ideal
- sample clock generation instabilities
- imperfections in A/D quantization.

Hence, include guard bands around bandwidth of interest.

Some more sampling theory

Shannon Sampling Theorem:

"If a function $f(t)$ contains no frequencies higher than W cycles per second, it is completely determined by giving its ordinates at a series of points spaced $(1/2W)$ seconds apart."

- so we can reconstruct $f(t)$ from its samples
  - if the signal is bandlimited
  - samples spaced $(1/2W)$
  - Hence Nyquist result

Shannon theorem

Proof sketch: Assume function is bandlimited so $F(s) = 0$ for $|s| > W$, then the IFT is

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{i2\pi st}\, ds = \int_{-W}^{W} F(s) e^{i2\pi st}\, ds$$

If instead, we make, F periodic, with period $2W$ then we can find a Fourier series for it, e.g.

$$F(s) = \sum_{n=-\infty}^{\infty} A_n e^{i\pi ns/W}$$

where,

$$A_n = \frac{1}{2W} \int_{-W}^{W} F(s) e^{-i\pi ns/W}\, ds = \frac{1}{2W} f\left(\frac{n}{2W}\right)$$

Shannon theorem

Proof sketch:
We can represent $F(s)$ perfectly with the Fourier series coefficients A_n, but these are just proportional to the function sampled at uniform intervals, e.g. $A_n \propto f\left(\frac{n}{2W}\right)$.

Hence, the samples completely define the FT F, and hence the function f. □

Shannon interpolation

Reconstruction of original signal from IFT

$$
\begin{aligned}
f(t) &= \int_{-W}^{W} F(s) e^{-i2\pi st}\, ds \\
&= \int_{-W}^{W} \sum_{n=-\infty}^{\infty} A_n e^{i\pi ns/W} e^{i2\pi st}\, ds \\
&= \sum_{n=-\infty}^{\infty} A_n \int_{-\infty}^{\infty} r(s/2W) e^{i2\pi s(-t+n/2W)}\, ds \\
&= \sum_{n=-\infty}^{\infty} 2WA_n \int_{-\infty}^{\infty} r(-s) e^{i2\pi s(2Wt-n)}\, ds \\
&= \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \operatorname{sinc}(2Wt-n)
\end{aligned}
$$

Shannon interpolation

Assume we sampled at the Nyquist rate, i.e. $f_s = 2W$, or $t_s = 1/2W$, then the sample points would be

$$f\left(\frac{n}{2W}\right)$$

The summation

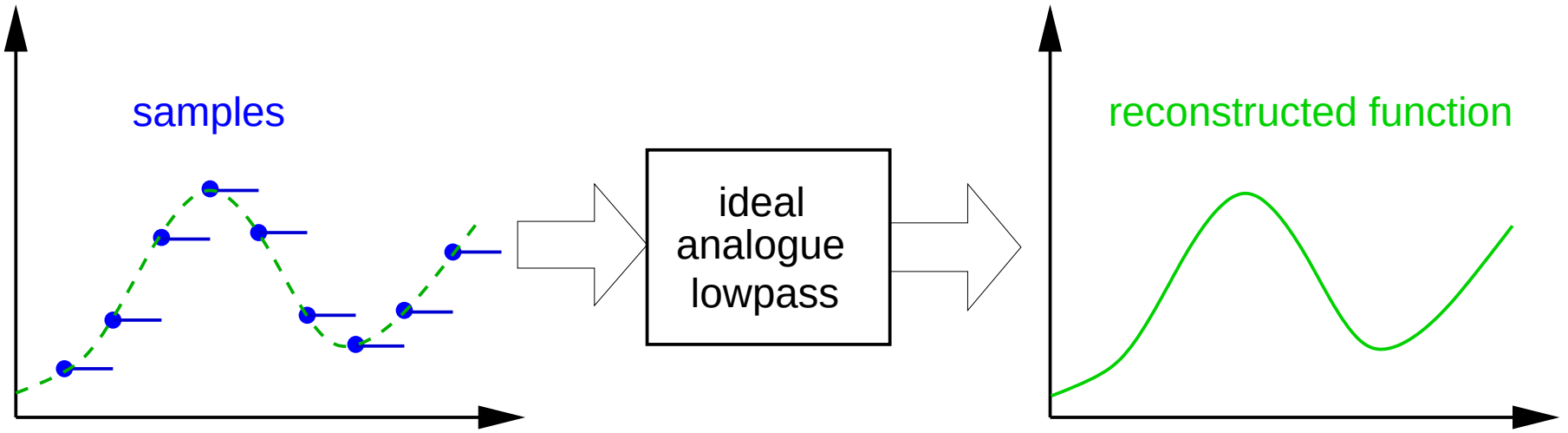
$$f(t) \quad = \quad \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \operatorname{sinc}(2Wt - n)$$

The above formula represents a "convolution" of the sampled signal with a sinc function. We will learn about convolutions later, but note that this convolution acts to (perfectly) filter out high frequencies.

Digital to Analogue converter

Interpretation

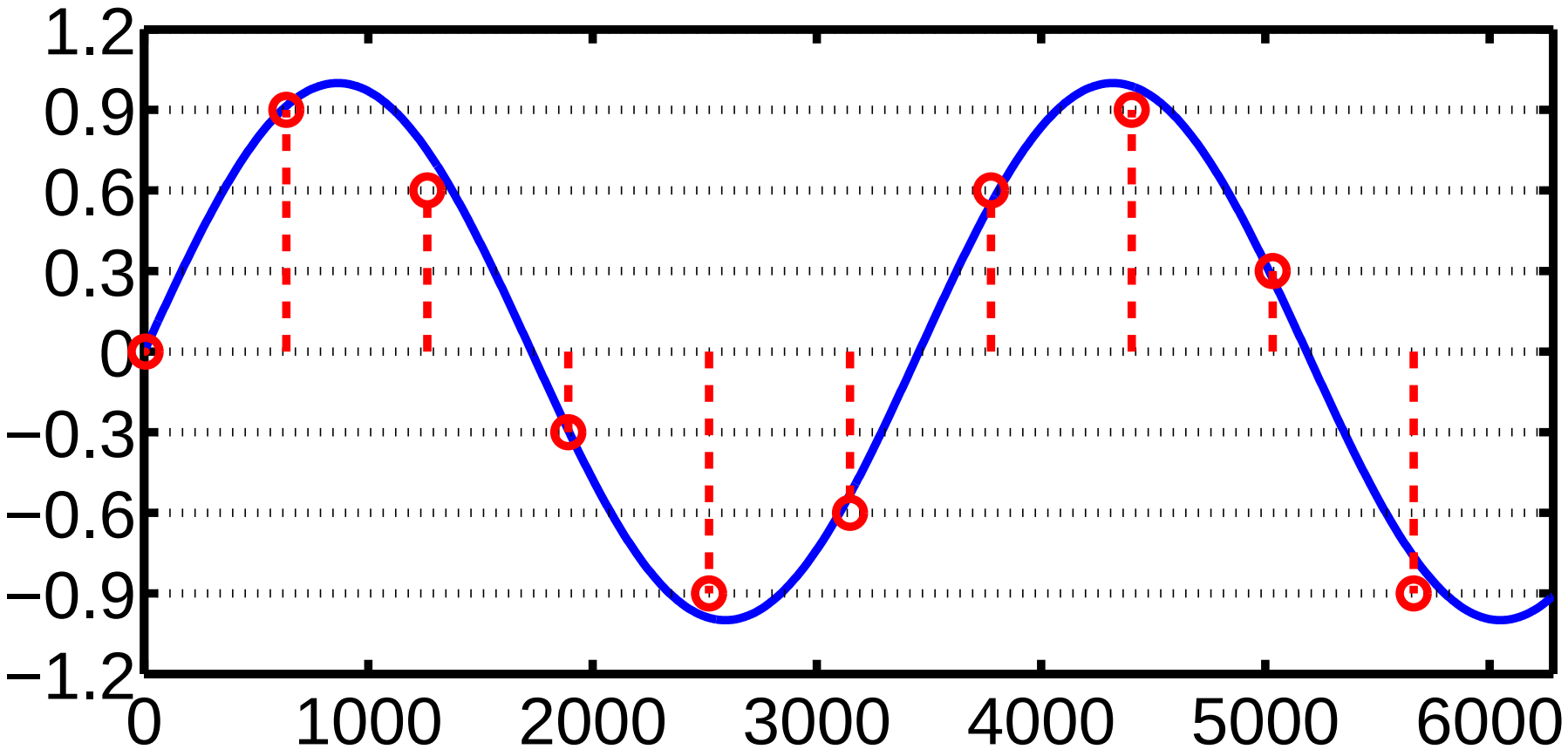
- convolution with sinc
- equivalent to ideal analogue low-pass filter



- this is essentially what a Digital to Analogue converter tries to do
- have to build analogue filter — hard to make it ideal

Quantization: discrete-value

Quantise real number values so they can be represented on a computer (or DSP) in a binary format. This is the essence of "digital" technology.



Dynamic range

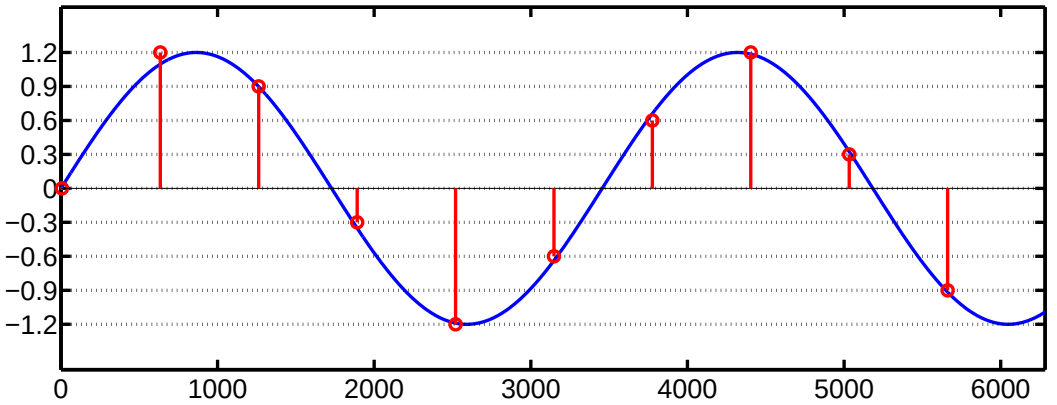
Dynamic range expresses the range of values we can represent in our digital format, e.g.

- assume fixed point representation with b bits.
- largest value representable is $(2^b - 1)\delta$
- smallest value representable is δ
- dynamic range $= 20\log_{10}\frac{(2^b-1)\delta}{\delta} \simeq b20\log_{10}2 = 6.02b$ dB
- 6 dB per bit

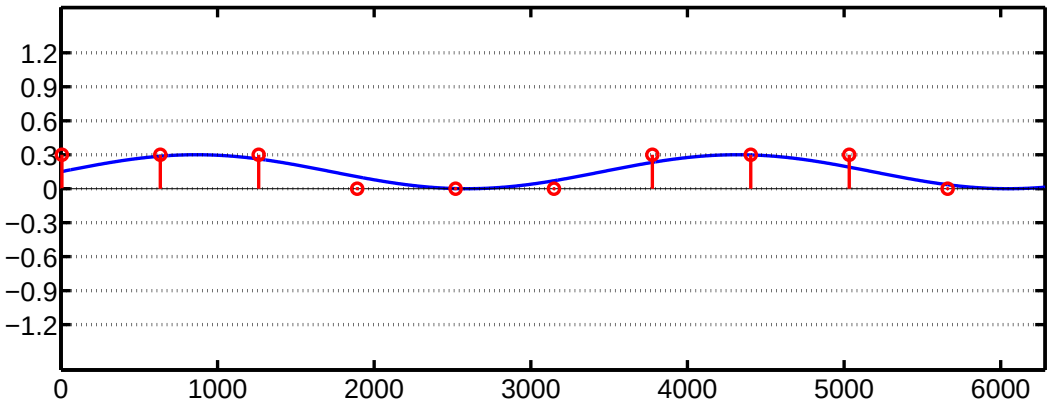
CD's use 16 bit fixed point, so the dynamic range of a CD recorded sound is approximately $16 \times 6 = 96dB$.
Compare to somewhere between 50-70 dB for LPs, depending on the quality of the pressing.

Dynamic range: examples

When the signal just fills the range of possible values, the maximum amplitude of the signal will be $(2^b - 1)\delta$.



The smallest signal (other than zero) that we can represent has maximum amplitude δ.



Dynamic range of the human senses

Human senses aren't really digital, but for purposes of comparison we will consider them here. They are pretty amazing.

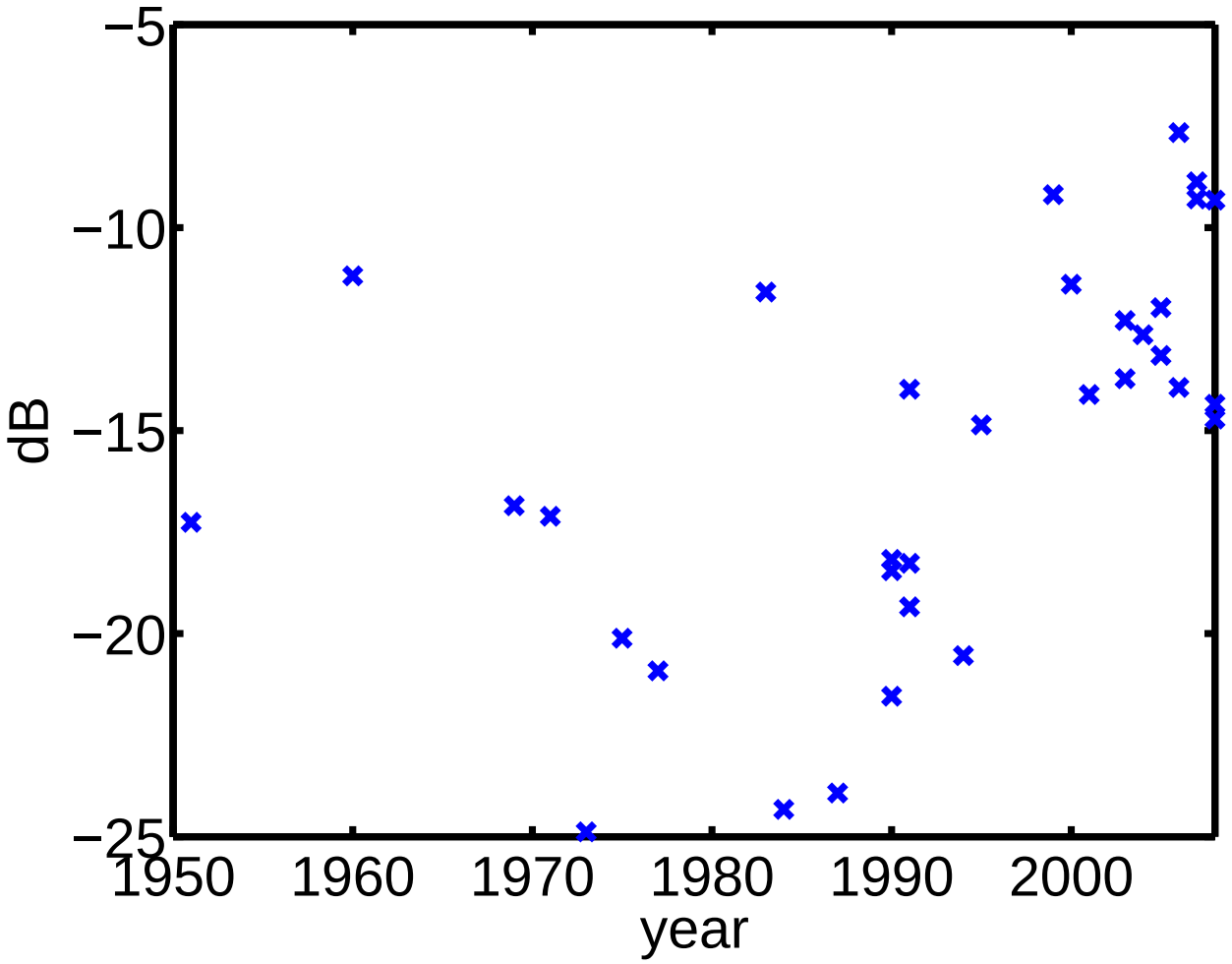
- We have already seen that the human ear has about 130 dB dynamic range.
- The human eye has about 100 dB dynamic range.

Although the dynamic range is very large, its important to note that our senses can't achieve this range simultaneously.

- Loud sounds can mask quieter sounds
- Our eye needs time to adjust to the level of brightness – the range of contrasts is can simultaneously perceive is much smaller.

Death of Dynamic Range

In recent year there is a trend in Pop music to aim for "louder" music at the expense of dynamic range.



Introduced noise

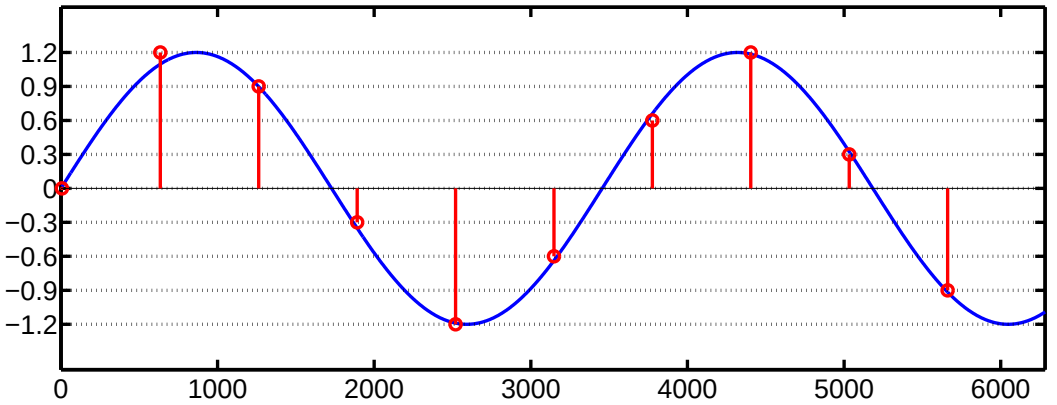
The noise introduced by quantization is of the order of δ the smallest value representable value. We want to compute the SNR (Signal to Noise Ratio).

- assume fixed point representation with b bits.
- noise is of the order of δ.
- SNR depends on loading factor.
  - lightly loaded, then δ is relatively large, and so SNR is small.
  - fully loaded, then SNR is similar to dynamic range (6 dB per bit).
  - overloaded, clipping occurs, and SNR drops.

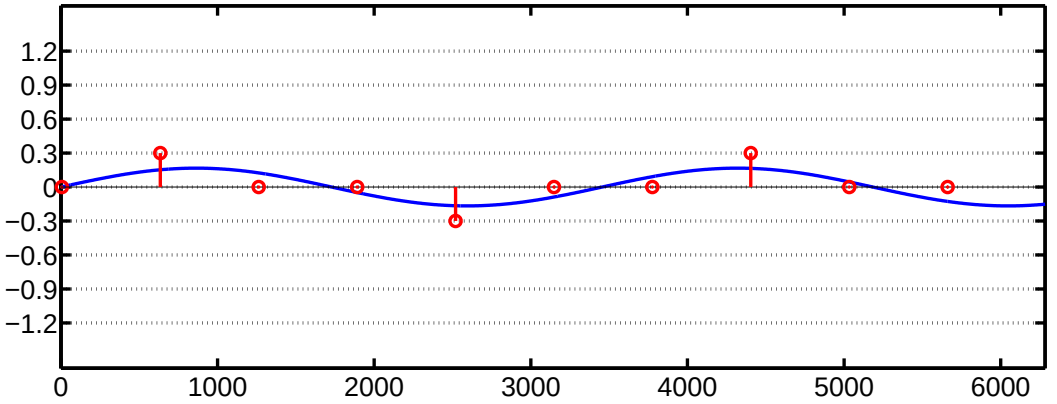
More accurate calculations in "Understanding Digital Signal Processing", Lyons.

Quantization noise notes

When the signal fully loads range of possible values, the maximum amplitude of the signal will be $(2^b - 1)\delta$

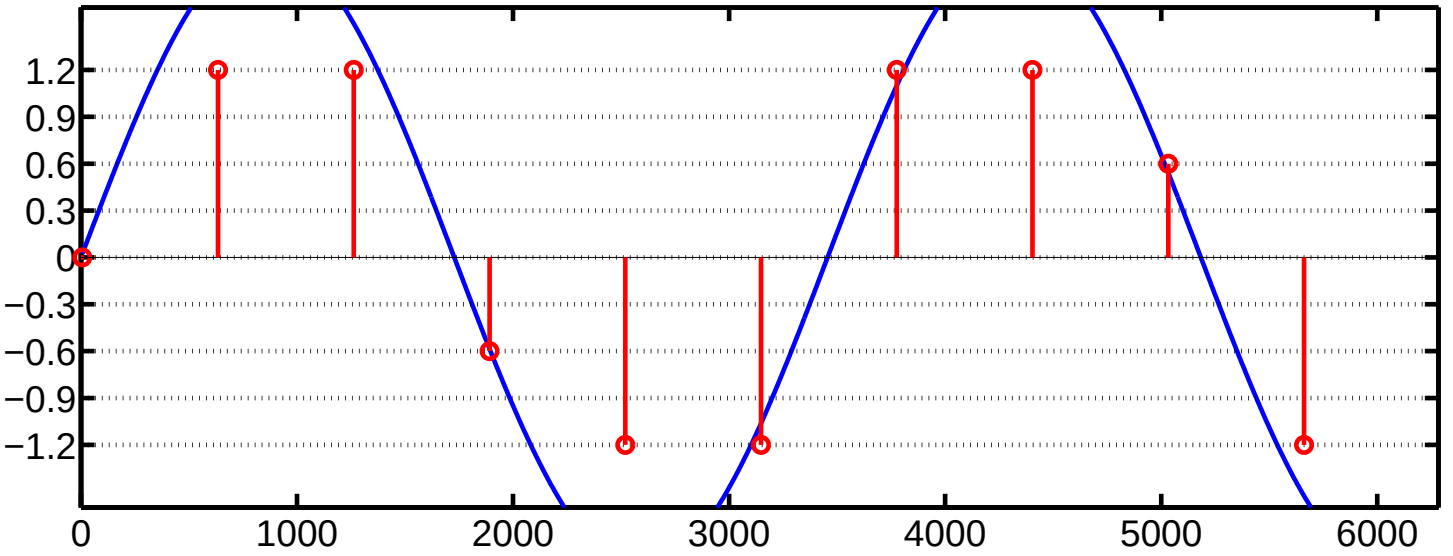


As long as clipping doesn't occur, then the errors will be of order δ, but this is **relatively** larger for small signals



Clipping

If the signal is too large we get clipping, which results in large amounts of quantization noise, e.g.



Sometimes clipping is used deliberately to alter sounds, for example in a guitar amp, clipping is used to produce distortion (e.g. for heavy-metal music). However, clipping is usually very bad.

Example

Compact Discs (CDs) are recorded

- using 16 bits
- 44.1 kHz
- so that they record sound frequencies up to 22.05 kHz with a theoretical dynamic range $\simeq$ 96dB.
- Human hearing goes up to about 15 kHz
- LPs have at most 70 dB dynamic range, so CDs should be effectively perfect.
  - audiophiles argue about this
  - some say you lose upper harmonics (not audible but effect tone), or perhaps you loose transient?
  - but I can't tell the difference

Example: extreme audio

Some audio formats propose 96 kHz sampling, at 24 bits. Ignoring audiophile fantasies, why would I want better digital recordings?

- Even if you can't hear it, what about in the studio. In mixing, noise from multiple inputs could add to increase noise floor.
- When an audio signal is dithered to remove structure from the quantization noise, this adds a little noise, so its helpful to have a lower noise floor to start with when recording audio.
- Stereo imaging: requires very finely adjusted time-of-arrival of wavefronts which might be distorted by sampling???

Discrete Fourier Transform

Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.

Jean Baptiste Joseph Fourier

Discrete transformation

Discrete-time transformation

- Discrete Fourier transform
- Discrete Cosine (and sin) transforms
- Discrete Wavelet transform
- Z-transform

Discrete-value transformation

- Probability generating function

Discrete Fourier Transformation

Continuous Fourier transform $F(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st}\, dt$

But note that for a finite length, discrete-time signal, it can be written as

$$x(t) = \sum_{n=0}^{N-1} f(nt_s)\delta(t - nt_s)$$

The Fourier transform can then be written

$$X(s) = \sum_{n=0}^{N-1} f(nt_s) e^{-i2\pi snt_s}$$

The result is simpler to compute, but its still redundant.

Discrete Fourier Transformation

If we have N data points, we would like a (frequency domain) representation that only needs N data points as well. Hence no redundancy.

Use $s = \frac{k}{Nt_s}$ for $k = 0, 1, \ldots, N-1$ and we get

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N},$$

where $x(n)$ are the N discrete samples from the continuous time process.

This is the Discrete Fourier Transform **(DFT)**

Inverse DFT

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N},$$

Inverse DFT (IDFT)

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{i2\pi kn/N},$$

Examples (i)

Take $x(n) = (1,0,0,0)$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N}$$

$$\begin{aligned} X(0) &= e^{-i2\pi 0/4} = 1 \\ X(1) &= e^{-i2\pi 0/4} = 1 \\ X(2) &= e^{-i2\pi 0/4} = 1 \\ X(3) &= e^{-i2\pi 0/4} = 1 \end{aligned}$$

So $X(k) = (1,1,1,1)$

Examples (i) IDFT

Take $X(k) = (1,1,1,1)$

$$
\begin{aligned}
x(n) &= \tfrac{1}{N}\textstyle\sum_{n=0}^{N-1} X(k) e^{i2\pi kn/N} \\
x(0) &= \tfrac{1}{4}\left(e^{-i2\pi 0/4} + e^{-i2\pi 0/4} + e^{-i2\pi 0/4} + e^{-i2\pi 0/4}\right) \\
&= \tfrac{1}{4}(1+1+1+1) &= 1 \\
x(1) &= \tfrac{1}{4}\left(e^{-i2\pi 0/4} + e^{-i2\pi 1/4} + e^{-i2\pi 2/4} + e^{-i2\pi 3/4}\right) \\
&= \tfrac{1}{4}(1+i-1-i) &= 0 \\
x(2) &= \tfrac{1}{4}\left(e^{-i2\pi 0/4} + e^{-i2\pi 2/4} + e^{-i2\pi 4/4} + e^{-i2\pi 6/4}\right) \\
&= \tfrac{1}{4}(1-1+1-1) &= 0 \\
x(3) &= \tfrac{1}{4}\left(e^{-i2\pi 0/4} + e^{-i2\pi 3/4} + e^{-i2\pi 6/4} + e^{-i2\pi 9/4}\right) \\
&= \tfrac{1}{4}(1-i-1+i) &= 0
\end{aligned}
$$

So $x(n) = (1,0,0,0)$

Examples (ii)

Take $x(n) = (0, 1, 0, 0)$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N}$$

$$\begin{aligned}
X(0) &= e^{-i2\pi 0/4} &&= 1 \\
X(1) &= e^{-i2\pi 1/4} &&= e^{-i\pi/2} &&= -i \\
X(2) &= e^{-i2\pi 2/4} &&= e^{-i\pi} &&= -1 \\
X(3) &= e^{-i2\pi 3/4} &&= e^{-i\pi 3/2} &&= i
\end{aligned}$$

So $X(k) = (1, -i, -1, i)$

Examples (iii)

Take $x(n) = (1,1,0,0)$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N}$$

$$\begin{aligned}
X(0) &= e^{-i2\pi 0/4} + e^{-i2\pi 0/4} = 1+1 = 2 \\
X(1) &= e^{-i2\pi 0/4} + e^{-i2\pi 1/4} = e^{0} + e^{-i\pi/2} = 1-i \\
X(2) &= e^{-i2\pi 0/4} + e^{-i2\pi 2/4} = e^{0} + e^{-i\pi} = 0 \\
X(3) &= e^{-i2\pi 0/4} + e^{-i2\pi 3/4} = e^{0} + e^{-i\pi 3/2} = 1+i
\end{aligned}$$

So $X(k) = (2, 1-i, 0, 1+i)$

DFT basis

Once again we are simply changing basis, when we perform the transform (or its inverse).

The basis functions are a discrete set of sin and cosine functions.

Note, now we are operating in a finite dimensional space $\mathbb{R}^N$, so we can write the transform as

$$X = Ax \qquad \text{analysis}$$

The inverse transform is just

$$x = A^{-1}X \qquad \text{synthesis}$$

Where both x and X are just vectors in $\mathbb{R}^N$.

DFT transform matrix

$$X = Ax$$

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i2\pi 1/N} & e^{-i2\pi 2/N} & \cdots & e^{-i2\pi(N-1)/N} \\ 1 & e^{-i2\pi 2/N} & e^{-i2\pi 4/N} & \cdots & e^{-i2\pi 2(N-1)/N} \\ 1 & e^{-i2\pi 3/N} & e^{-i2\pi 6/N} & \cdots & e^{-i2\pi 3(N-1)/N} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{-i2\pi(N-1)/N} & e^{-i2\pi 2(N-1)/N} & \cdots & e^{-i2\pi(N-1)(N-1)/N} \end{pmatrix}$$

Examples (i)

Take $x(n) = (1,0,0,0)$

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-i2\pi 1/4} & e^{-i2\pi 2/4} & e^{-i2\pi 3/4} \\ 1 & e^{-i2\pi 2/4} & e^{-i2\pi 4/4} & e^{-i2\pi 6/4} \\ 1 & e^{-i2\pi 3/4} & e^{-i2\pi 6/4} & e^{-i2\pi 9/4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Frequency resolution

Frequencies of basis functions are $k = 0, 1, 2, \ldots, (N-1)$ cycles over the data set. If the data set has N samples at sampling frequency f_s, then its duration is $T = N/f_s$. To convert from data units to absolute units, we take $k/T = \frac{kf_s}{N}$

Frequency resolution is $\frac{f_s}{N}$

- higher sampling frequencies reduce frequency resolution
- longer data, improves frequency resolution

Getting units right

Note that absolute frequency depends on sample frequency f_s, so we need to convert.
The component $X(m)$ will correspond to frequency

$$X(m) \equiv F\left(\frac{m f_s}{N}\right)$$

Output magnitude of DFT will be amplitude of sin wave signal A times $N/2$. Alternative definitions of DFT exist

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-i2\pi kn/N}, \qquad x(n) = \sum_{n=0}^{N-1} X(k)e^{i2\pi kn/N}$$

$$X(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x(n)e^{-i2\pi kn/N}, \qquad x(n) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X(k)e^{i2\pi kn/N}$$

Matlab

Note, indexes in Matlab run from 1 to N (not 0 to $N-1$).

$$\text{fft}(x(n)) = X(k) = \sum_{n=1}^{N} x(n) e^{-i2\pi(k-1)(n-1)/N}, \quad k = 1, \ldots, N.$$

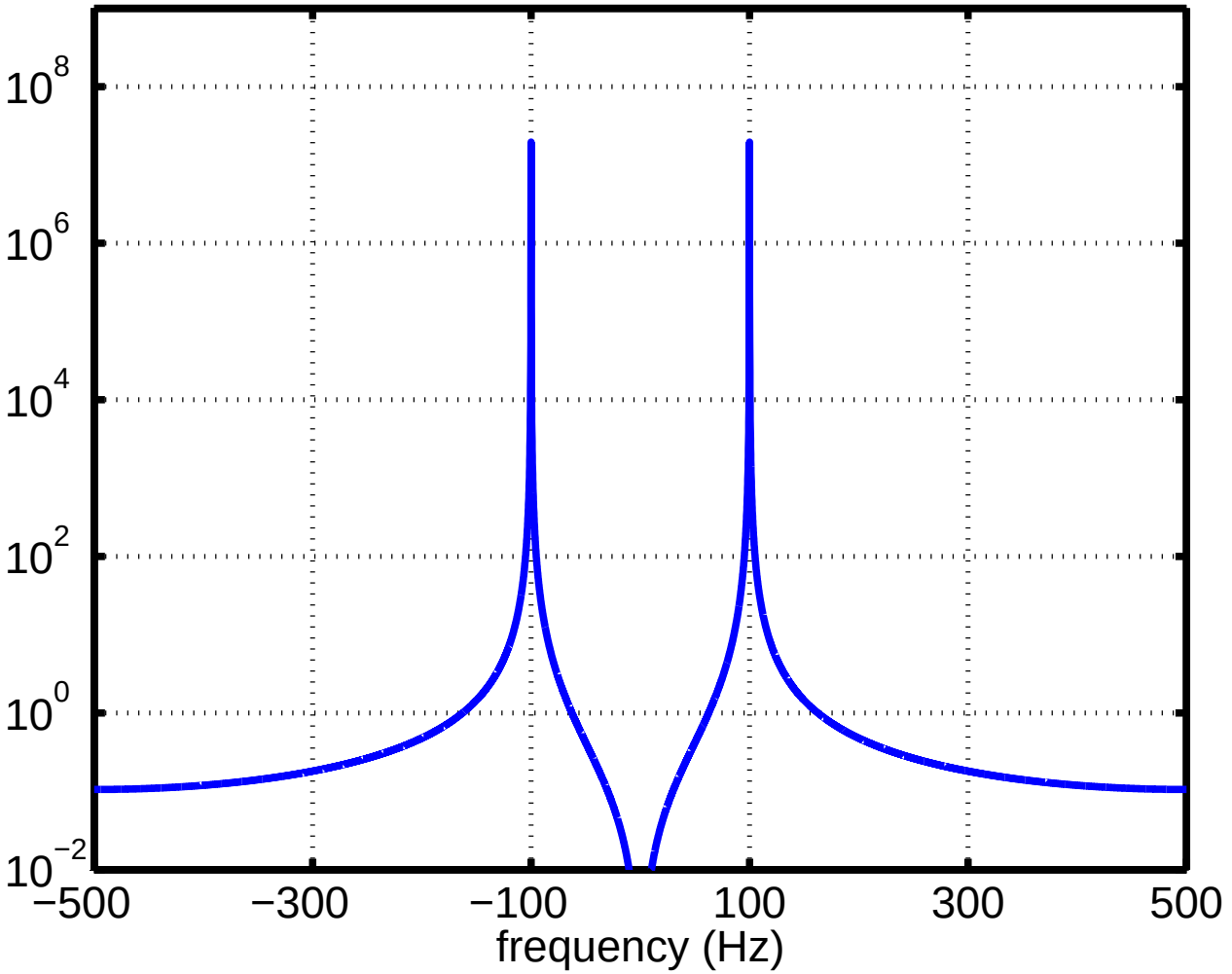
$$\text{ifft}(X(k)) = x(n) = \frac{1}{N} \sum_{k=1}^{N} X(k) e^{i2\pi(k-1)(n-1)/N}, \quad n = 1, \ldots, N.$$

$X(1)$ is the DC term, $X(n)$ is the f_s term. To plot symmetric power spectrum use, e.g.

```
f_s = 1000;
f_0 = 100;
x = 1:1/f_s:10;
y = sin(2*pi*f_0*x);
semilogy(-f_s/2+f_s/N:f_s/N:f_s/2, abs(fftshift(fft(y))).^2);
set(gca, 'ylim', 10.^[-2 9]);
xlabel('frequency (Hz)');
```

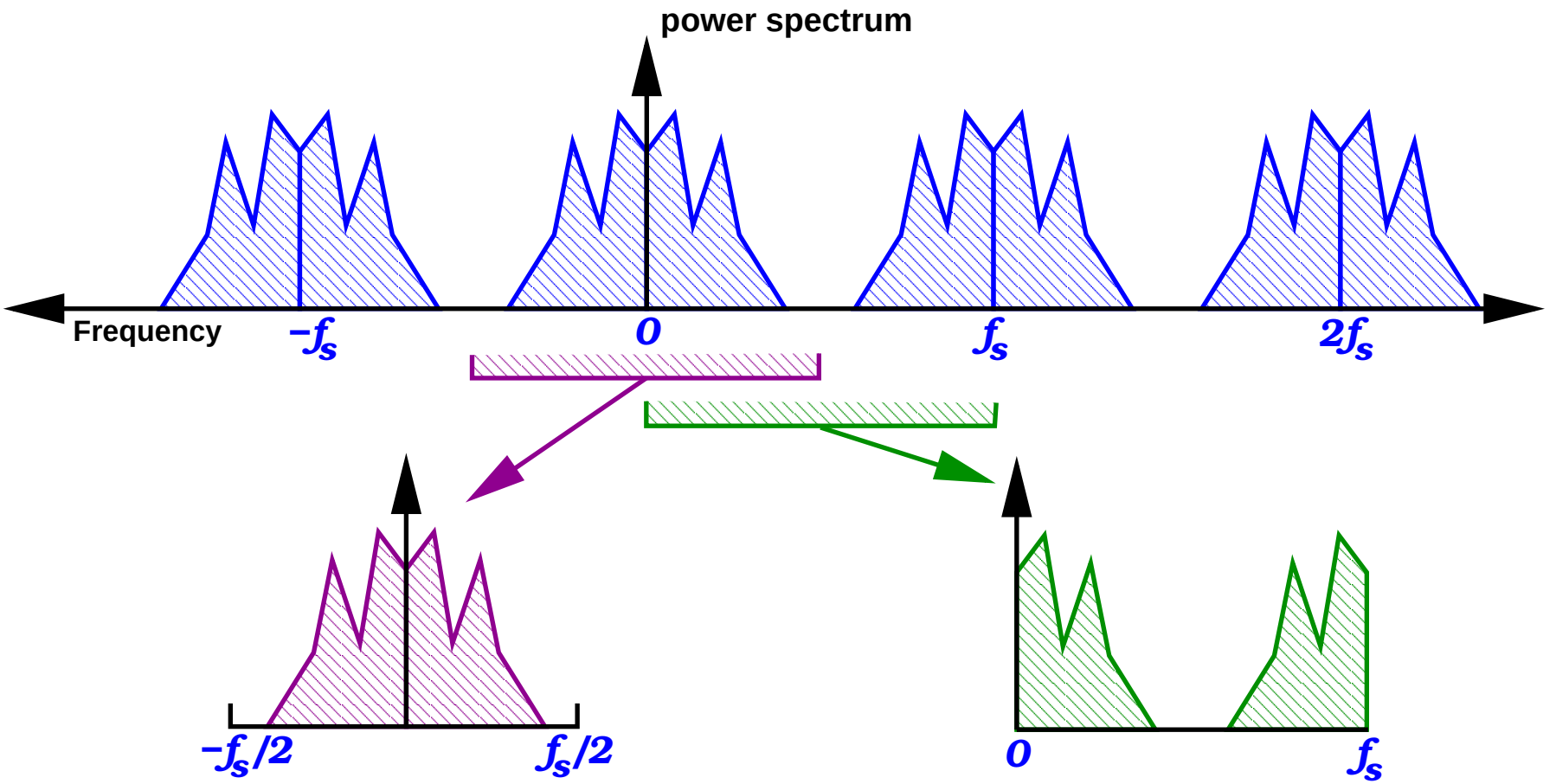
Matlab example

`matlab_ex_1.m`



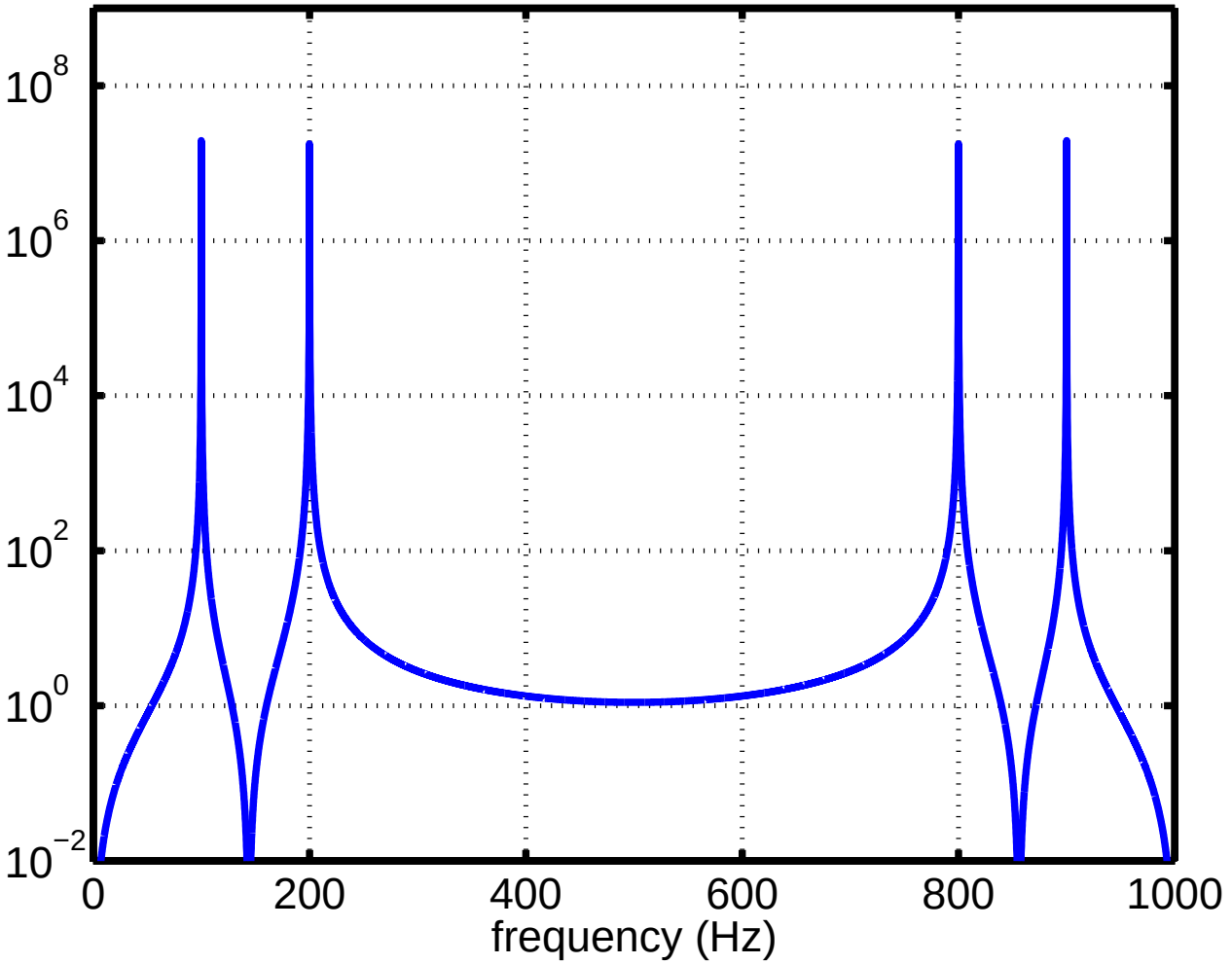
Symmetry

Discrete power spectrum is **even** and **periodic** so we can display in a number of ways.



Matlab example 2

`matlab_ex_2.m`



Properties of the DFT

Mostly the same as Continuous FT

- invertible
- no redundancy so it is efficient
- Linearity: $ax_1(n) + bx_2(n) \longrightarrow aX_1(k) + bX_2(k)$
- Time shift: $x(n - n_0) \longrightarrow X(k)e^{-i2\pi kn_0}$
- Time scaling: a bit more complicated!
- Duality: a bit more complicated!
- Frequency shift: $x(n)e^{-i2\pi k_0 n} \longrightarrow X(k - k_0)$
- Convolution: $x_1(n) * x_2(n) \longrightarrow X_1(k)X_2(k)$

Now n and k are integers, with the result that we are missing properties related to derivatives.

Duality and the DFT

The duality property is a little changed from before: given a signal $x(n)$ for $n = 0, \ldots, N-1$, with DFT $X(k)$ for $k = 0, \ldots, N-1$, then the DFT of $X(n)$ is

$$\begin{aligned} DFT(X;k) &= \begin{cases} Nx(0), & \text{for } k = 0 \\ Nx(N-k), & \text{for } k \neq 0 \end{cases} \\ &= Nx(N-k \bmod N) \end{aligned}$$

The result is similar to previous duality results if we think of the points cyclically, i.e.

$$x(-k \bmod N) = x(N-k \bmod N)$$

That works well with the periodic representation of frequency spectrum that we get for a sampled signal.

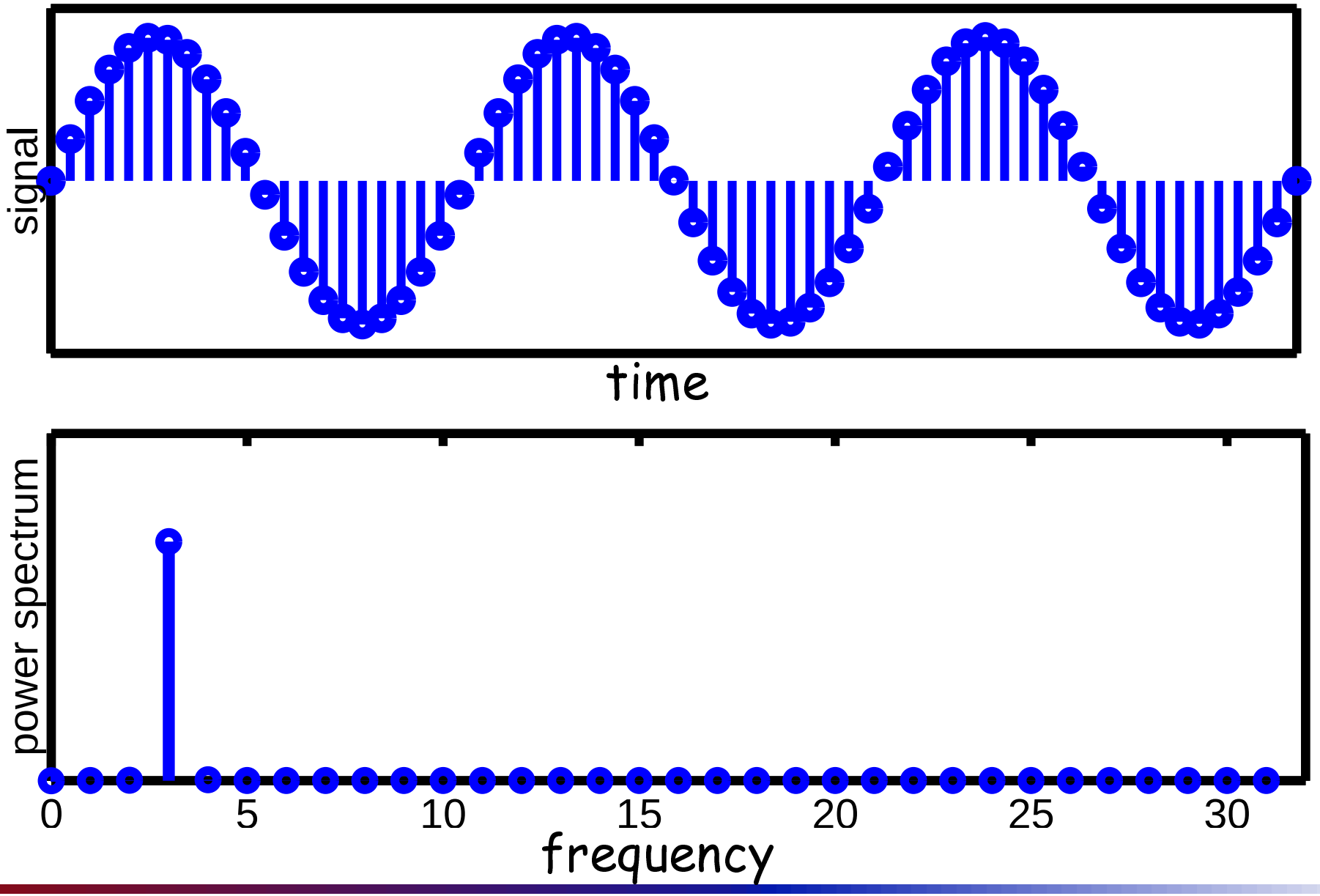
Properties of the DFT

There are some new properties unique to DFTs

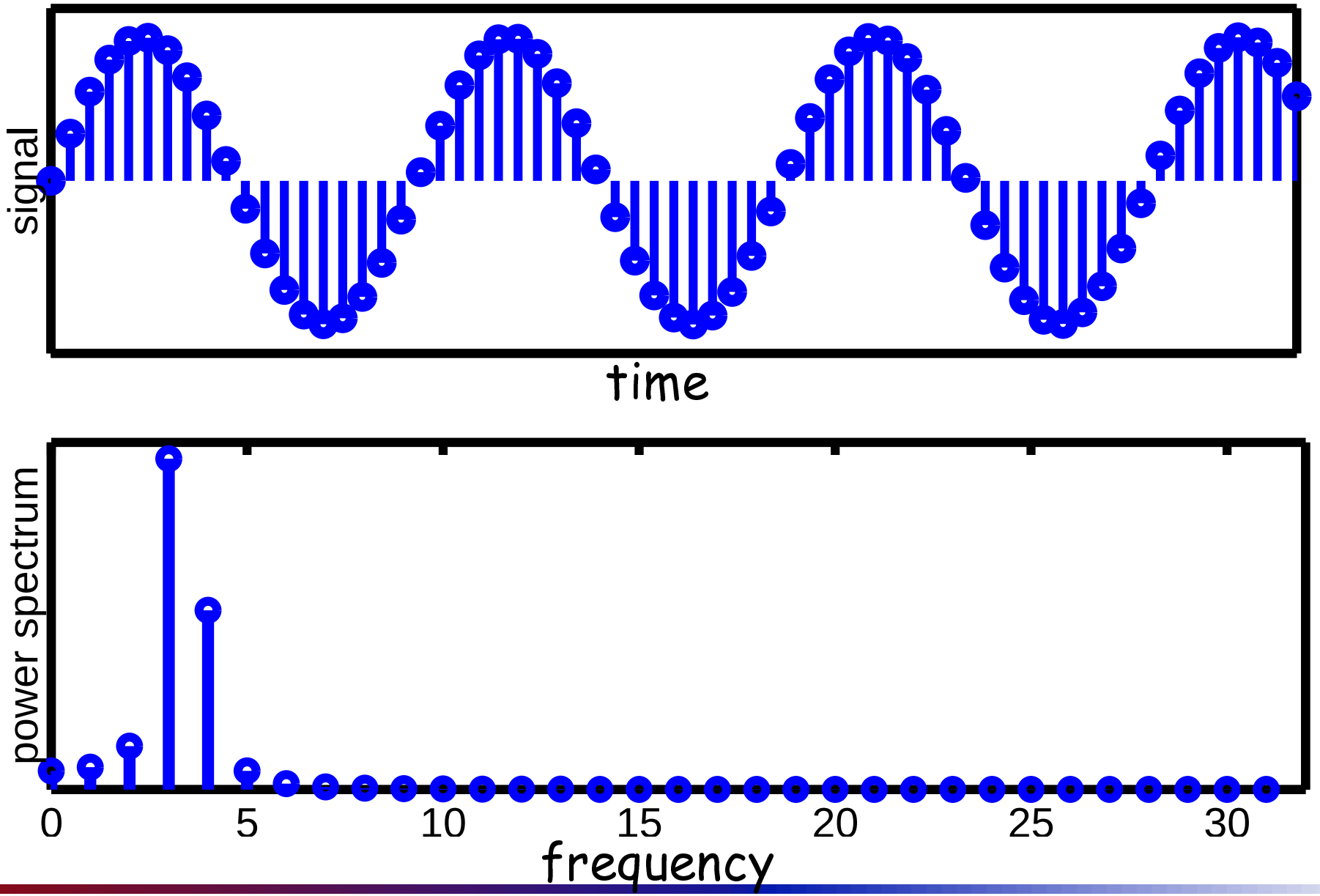
- Leakage that fits exactly our discrete frequencies
- Padding (packing)
- Similarity (discrete version of time scaling)

See below for details.

Leakage example



Leakage example



Properties of the DFT: Leakage

DFT is different from the continuous time FT is that the DFT suffers from **Leakage**.

- Unlike Continuous transform, DFT uses a finite number of frequencies.
- Not all signals fit this mold exactly: what happens to sinusoids with non-integral frequencies?
- Their power is spread over a few frequencies.
- Note we are representing the signal by a series of numbers $X(k)$ which represent the correlation of the signal to a particular sinusoid with freq. k/N,
- Note that, as the data gets longer, the frequency resolution improves

DFT properties: padding

We can pad (or pack) a sequence with zeros to extend its length

$$y(n) = \begin{cases} x(n), & \text{if } 0 \leq n \leq N-1 \\ 0, & \text{if } N \leq n < KN \end{cases}$$

The resulting DFT is

$$\mathcal{F}\{y\} = Y(k) = X\left(\frac{k}{K}\right)$$

Padding (packing) example (ii)

Data $x(n) = (0,1,0,0)$ with transform $X(k) = (1,-i,-1,i)$
Pad to get $y(n) = (0,1,0,0,0,0,0,0)$ then the DFT

$$Y(k) = \sum_{n=0}^{N-1} y(n) e^{-i2\pi kn/N}$$

$$\begin{aligned}
Y(0) &= e^{-i2\pi 0/8} &&= 1 \\
Y(1) &= e^{-i2\pi 1/8} &&= e^{-i\pi/4} &&= (1-i)/\sqrt{2} \\
Y(2) &= e^{-i2\pi 2/8} &&= e^{-i\pi/2} &&= -i \\
Y(3) &= e^{-i2\pi 3/8} &&= e^{-i\pi 3/4} &&= (-1-i)/\sqrt{2} \\
Y(4) &= e^{-i2\pi 4/8} &&= e^{-i\pi} &&= -1 \\
Y(5) &= e^{-i2\pi 5/8} &&= e^{-i\pi 5/4} &&= (-1+i)/\sqrt{2} \\
Y(6) &= e^{-i2\pi 6/8} &&= e^{-i\pi 3/2} &&= i \\
Y(7) &= e^{-i2\pi 7/8} &&= e^{-i\pi 7/4} &&= (1+i)/\sqrt{2}
\end{aligned}$$

Padding (packing) example (ii)

Data $x(n) = (0, 1, 0, 0)$ with transform $X(k) = (1, -i, -1, i)$
Pad to get $y(n) = (0, 1, 0, 0, 0, 0, 0, 0)$ then the DFT

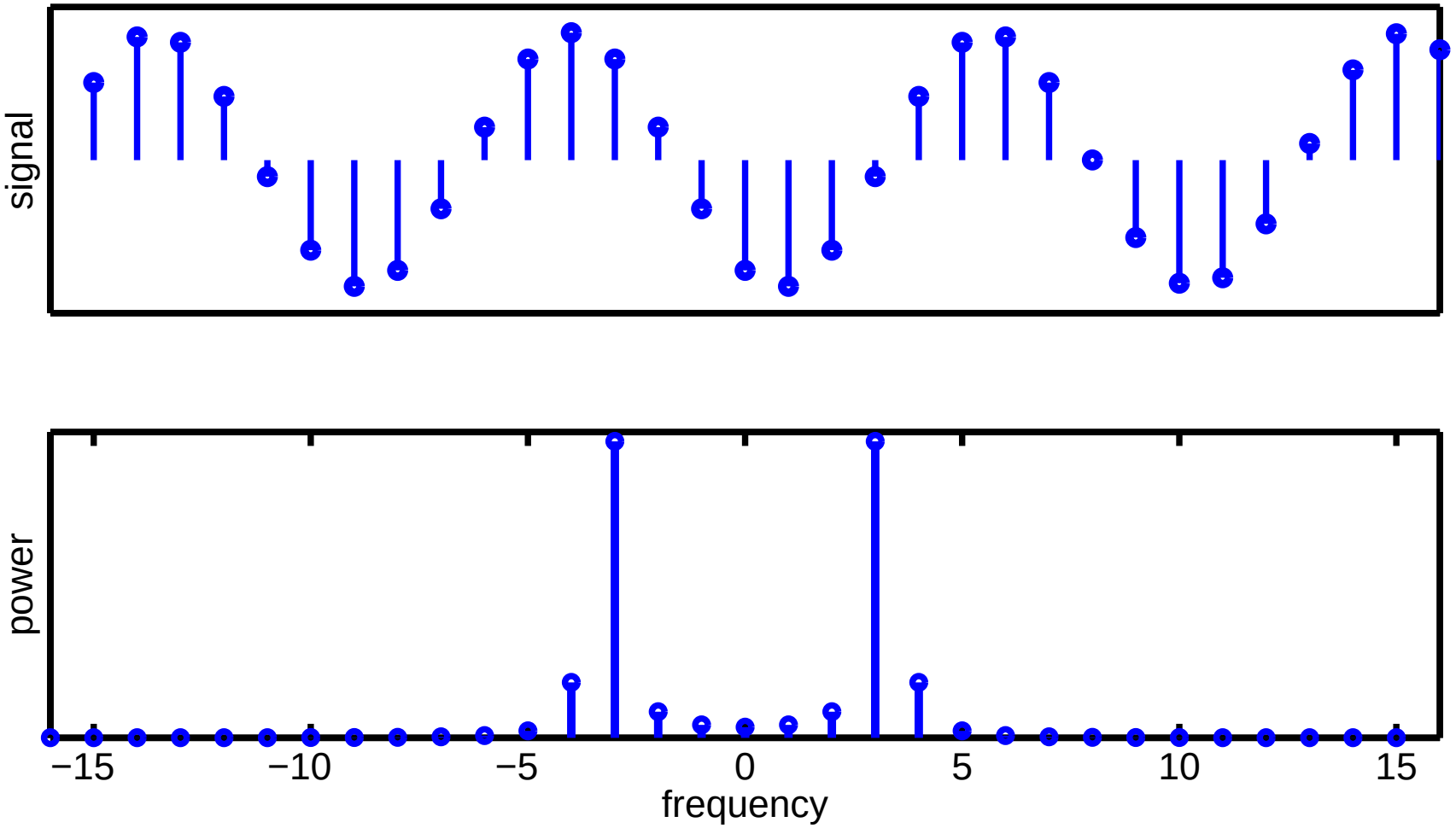
$$\begin{aligned} Y(0) &= X(0) \\ Y(2) &= X(1) \\ Y(4) &= X(2) \\ Y(6) &= X(3) \end{aligned}$$

So the relationship $Y(k) = X(k/2)$ holds, with $K = 2$, for even values of k.

Note we cannot derive $Y(k)$ for odd values of k, or if K is not an integer, but the relationship still tells us how to scale the frequency units, when we pad.

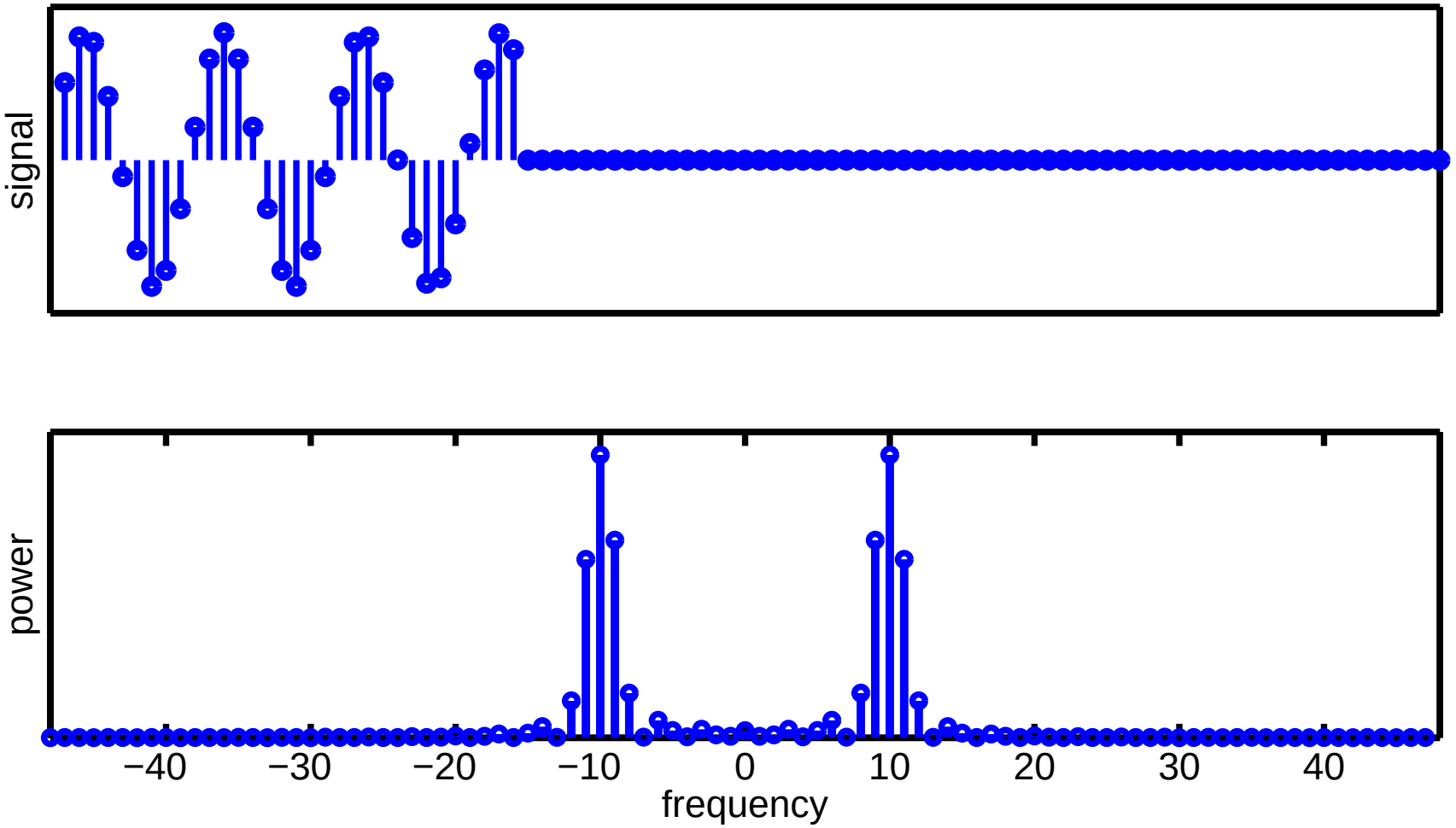
Padding (packing) example

Original data length $N = 32$ (frequency = 3.333)



Padding (packing) example

$K = 3$, new sequence length $KN = 96$. (frequency = 10/K)



DFT properties: similarity

We can interleave a sequence with zeros, e.g.

$$y(n) = \begin{cases} x(n/K), & \text{if } n = 0, K, 2K, \ldots, (N-1)K \\ 0, & \text{otherwise} \end{cases}$$

The resulting DFT is

$$\mathcal{F}\{y\} = Y(k) = \begin{cases} X(k) & k = 0, \ldots, N-1 \\ X(k-N) & k = N, \ldots, 2N-1 \\ \vdots & \\ X(k-(K-1)N) & k = (K-1)N, \ldots, KN-1 \end{cases}$$

Similarity example (ii)

Data $x(n) = (0,1,0,0)$ with transform $X(k) = (1,-i,-1,i)$
Interleave zeros to get $y(n) = (0,0,1,0,0,0,0,0)$ then

$$
\begin{aligned}
Y(k) &= \textstyle\sum_{n=0}^{N-1} y(n) e^{-i2\pi kn/N} \\
Y(0) &= e^{-i2\pi 0/8} &&= 1 \\
Y(1) &= e^{-i2\pi 2/8} &&= e^{-i\pi/2} &&= -i \\
Y(2) &= e^{-i2\pi 4/8} &&= e^{-i\pi} &&= -1 \\
Y(3) &= e^{-i2\pi 6/8} &&= e^{-i\pi 3/2} &&= i \\
Y(4) &= e^{-i2\pi 8/8} &&= e^{-i2\pi} &&= 1 \\
Y(5) &= e^{-i2\pi 10/8} &&= e^{-i\pi 5/2} &&= -i \\
Y(6) &= e^{-i2\pi 12/8} &&= e^{-i\pi 3} &&= -1 \\
Y(7) &= e^{-i2\pi 14/8} &&= e^{-i\pi 7/2} &&= i
\end{aligned}
$$

So $Y(k) = (1,-i,-1,i,1,-i,-1,i)$ (or $X(k)$ repeated twice)

Sampling, Quantization, Dithering and Half-toning

The properties we have just seen leed to some direct applications. In particular, we don't always get a signal in the form we want it, so we may have to change its sampling rate, or quantization, and we can exploit our new mathematically derived intuition to start work out how to do this (we'll see more later).

Similarity application

Practical use: upsampling (interpolation)

> We have a sequence sampled every t_s seconds, e.g. at a rate $f_s = 1/t_s$, but we need a sequence sampled at rate $K f_s$.

Approach: produce a new sequence with $K-1$ zeros interleaved between each original data point.

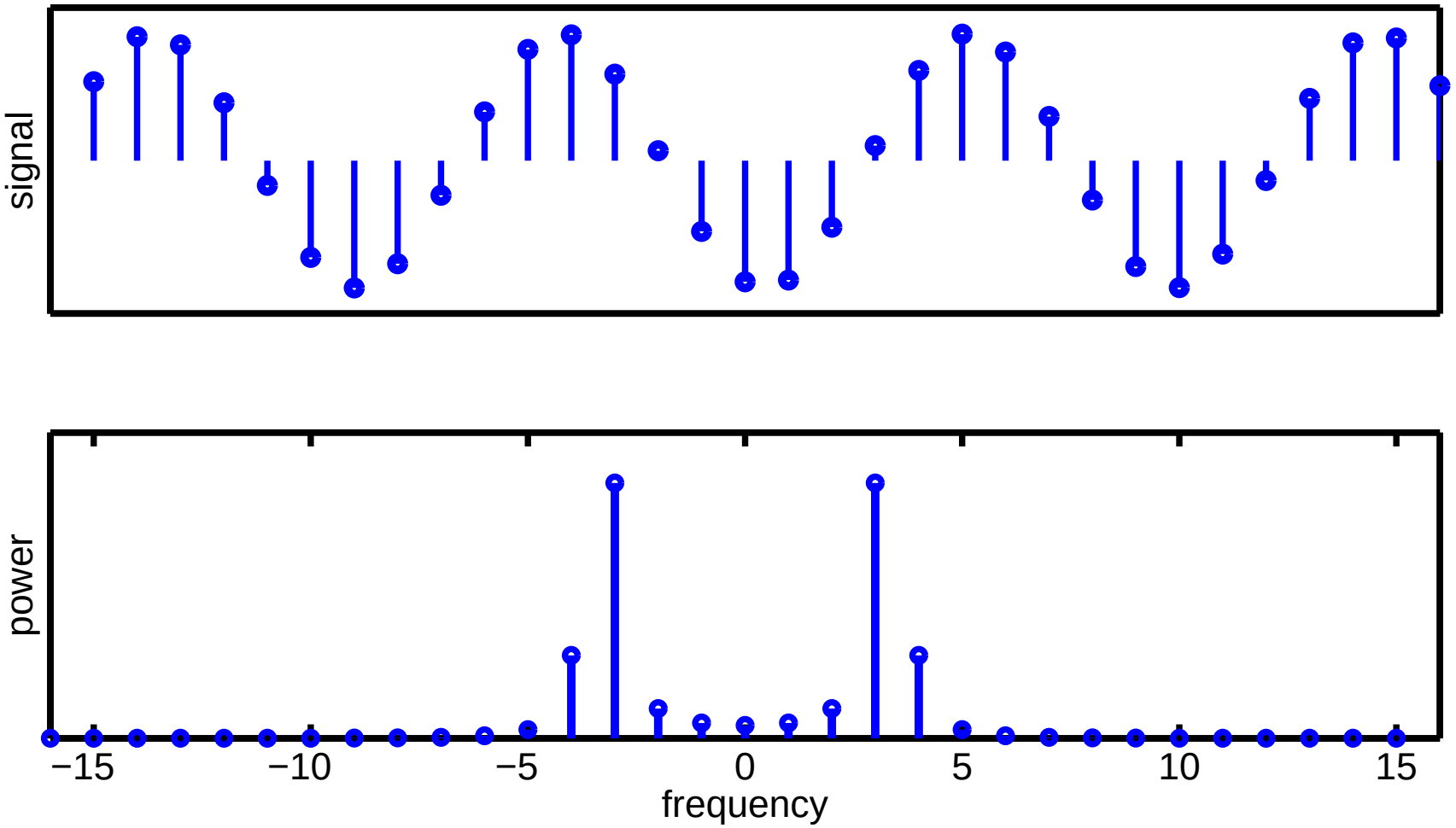
Similarity application: upsampling

Given $K-1$ zeros interleaved between each original sample.

- max frequency in original data is $f_s/2$, with frequency resolution f_s/N, and $N/2$ points in frequency domain.
- upsampled data has max frequency $Kf_s/2$, with frequency resolution f_s/N, and $KN/2$ points in frequency domain.
- the frequency resolution doesn't change, but now we have K repeats of the original spectrum at intervals f_s/N.
- to get a signal with the same original band-limited power-spectrum, we apply a low-pass filter, smoothing the data.

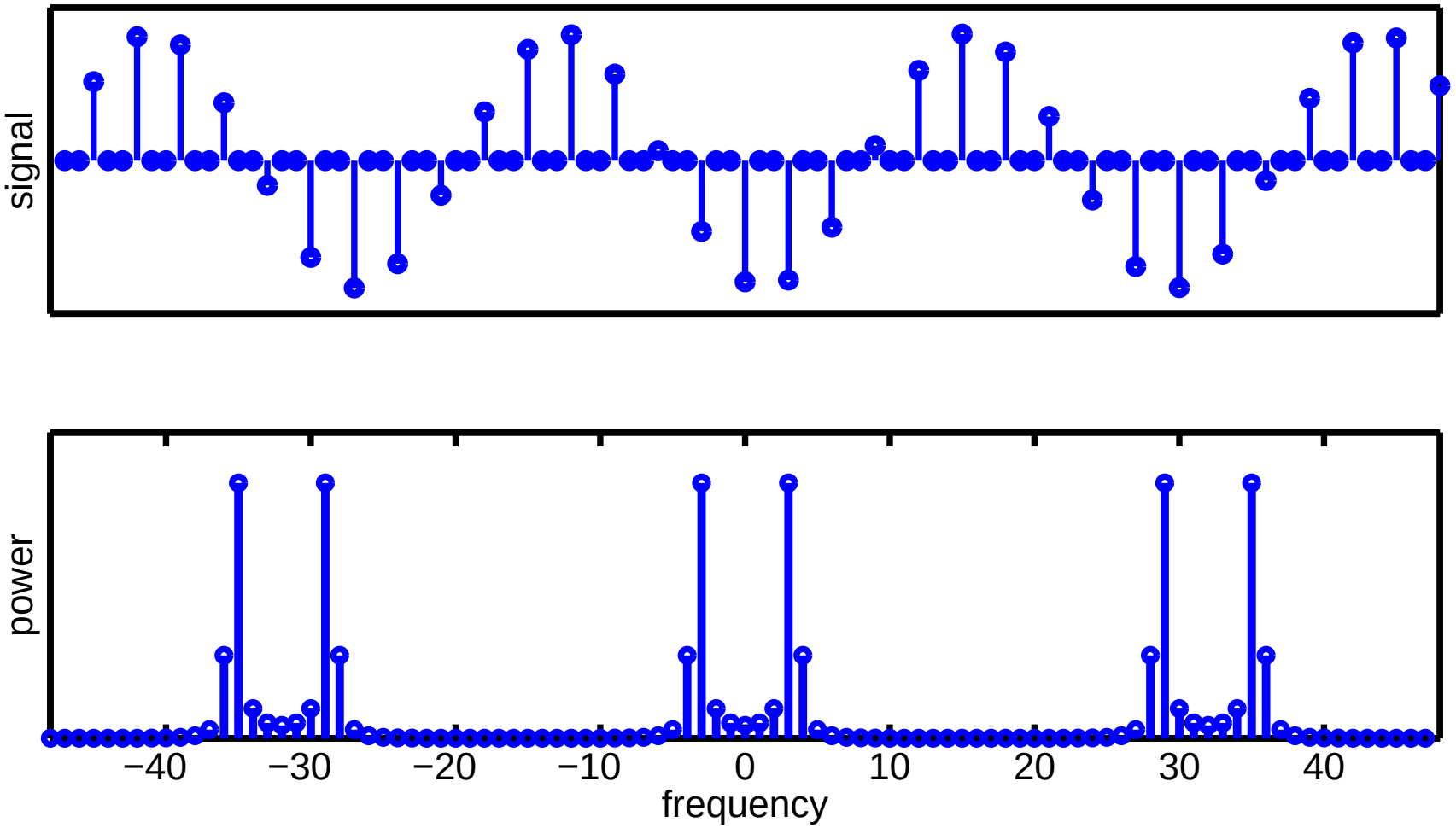
Upsampling example

32 samples (frequency 3.4 cycles)



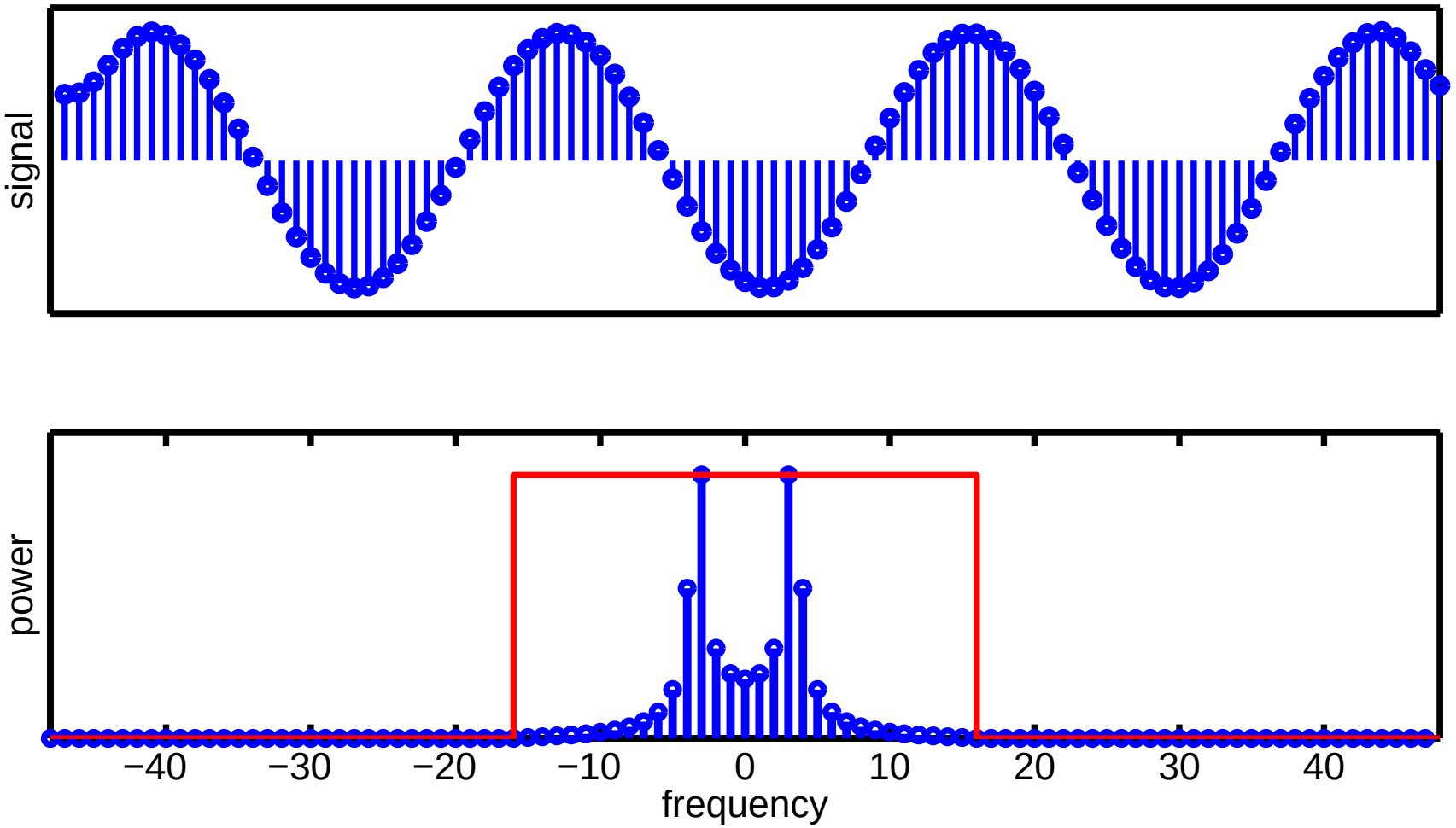
Upsampling example

3 $\times$'s upsampled (96 samples)



Upsampling example

low pass filter, then IDFT



Upsampling tricks

Trick of the day: low-pass before upsampling.

- notionally, the filtering occurs after upsampling
- If filtering in the time domain however, $K - 1/K$ proportion of multiplies in the filter are by zero.
- can ignore these, but this is the same as low-pass before upsampling.

Let's revisit this later (after discussing filtering in more detail).

Upsampling applications: audio

Oversampling CD or DVD players

- digital components are cheap
- analogue components are more expensive
- Digital to Analogue Conversion (DAC) is required in CD player
- want to make this as cheap as possible (for a given quality)

The trick

- upsample in the digital domain (where it is cheap)
- when we convert to analogue, we can use a simpler, cheaper analogue filter, to get the same results