Assignment 3: Solutions

TOTAL MARKS: 20

1. (a) Together, these constraints specify that $x_1 + x_2 = 1$ [1 mark]
   (b) The equality reduces the dimension of the feasible set by 1

2. Consider the LP

$$
\begin{array}{lrcrcrcrcr}
(P) & \max z & = & x_1 & - & x_2 & + & 3x_3 & & \\
& \text{subject to} & & 2x_1 & + & x_2 & + & 5x_3 & \leq & 6 \\
& & - & 3x_1 & - & 2x_2 & + & 4x_3 & \leq & -3 \\
& & & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0}
\end{array}
$$

(a) Multiply the second constraint by -1 and add slack variables and we get

	x_1	x_2	x_3	x_4	x_5	b
	2	1	5	1	0	6
	-3	-2	4	0	1	-3
cost row	-1	1	-3	0	0	0

The dual (D) of (P) is

$$
\begin{array}{lrrcrcr}
(D) & \min w = & 6y_1 & - & 3y_2 & & \\
& \text{subject to} & 2y_1 & - & 3y_2 & \geq & 1 \\
& & y_1 & - & 2y_2 & \geq & -1 \\
& & 5y_1 & + & 4y_2 & \geq & 3 \\
& & y_1 & & & \geq & 0 \\
& & & & y_2 & \geq & 0
\end{array}
$$

where y_1 and y_2 are nominally free, but in this case are not because of the last two constraints. You should see how these would arise for any problem where we introduce slack variables. [2 marks]

(b) Solving (P) using the simplex algorithm. Start by introducing non-negative slack variables for each inequality and wite down the tableau. We notice that we need to use simplex phase I as we are not in feasible canonical form. (note that we do have a feasible dual solution, so we also could use dual simplex here)

	x_1	x_2	x_3	x_4	x_5	b
	2	1	5	1	0	6
	-3	-2	4	0	1	-3
cost row	-1	1	-3	0	0	0

Continue by mutiplying the second inequality by -1 to get a non-negative entry in the **b** column. Then establish the u-row.

	x_1	x_2	x_3	x_4	x_5	b
	2	1	5	1	0	6
	$\boxed{3}$	2	-4	0	-1	3
cost row	-1	1	-3	0	0	0
u-row	-5	-3	-1	-1	1	-9

$$\vdots$$

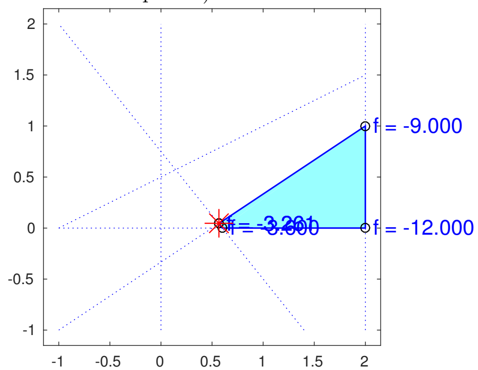
	0	$-\frac{1}{23}$	1	$\frac{3}{23}$	$\frac{2}{23}$	$\frac{12}{23}$
	1	$\frac{14}{23}$	0	$\frac{4}{23}$	$-\frac{5}{23}$	$\frac{39}{23}$
	0	$\frac{34}{23}$	0	$\frac{13}{23}$	$\frac{1}{23}$	$\frac{75}{23}$
	0	0	0	0	0	0

At the end of phase I we have feasible solutions as $u = 0$ and in fact we have the optimal solution

$$x_1^* = \frac{39}{23},\ x_2^* = 0,\ x_3^* = \frac{12}{23},\ z^* = \frac{75}{23}.$$

[4 marks]

(c) The optimal solution of (D) can be obtained in a number of ways
   (i) We could have used Simplex, but that is overkill.
   (ii) We could draw a picture of the feasible region (as this is 2D problem), see below (note though that the right-hand edge is unbounded not bounded – that's just a trick so MATLAB can plot it).



   From this, we see the maximum is at the intersection point is on the vertex defined by the two boundary lines $2y_1 - 3y_2 = 1$ and $5y_1 + 4y_2 = 3$. Thus the optimal solution is

$$y_1^* = \frac{13}{23},\ y_2^* = \frac{1}{23},$$

   (iii) You could use the CSRs themselves (see next question).
   (iv) You might also note that in the final Simplex tableau these values occur in the cost- or c-row of the Tablea in columns 4 and 5, corresponding to the slack variables. This is not a coincidence.

The objective is then

$$w^* = 6y_1^* - 3y_2^* = \frac{78}{23} - \frac{3}{23} = \frac{75}{23} = z^*.$$

[3 marks]

(d) The Complementary Slackness Relations (CSRs) are

$$\begin{aligned} x_1^*(2y_1^* - 3y_2^* - 1) &= 0 \quad \text{as} \quad 2y_1^* - 3y_2^* - 1 = 0 \\ x_2^*(y_1^* - 2y_2^* + 1) &= 0 \quad \text{as} \quad x_2^* = 0 \\ x_3^*(5y_1^* + 4y_2^* - 3) &= 0 \quad \text{as} \quad 5y_1^* + 4y_2^* - 3 = 0 \end{aligned}$$

[3 marks]

3. Consider the LP

$$\begin{array}{llrcrcrcl} (P) & \max z = & - x_1 & + & 2x_2 & - & x_3 & & \\ & \text{subject to} & 2x_1 & + & x_2 & + & 3x_3 & \leq & 2 \\ & & x_1 & + & 4x_2 & + & 2x_3 & \leq & 4 \\ & & \multicolumn{7}{l}{x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0} \end{array}$$

(a) The dual (D) of (P) is

$$\begin{array}{llrcrcl} (D) & \min w = & 2y_1 & + & 4y_2 & & \\ & \text{subject to} & 2y_1 & + & y_2 & \geq & -1 \\ & & y_1 & + & 4y_2 & \geq & 2 \\ & & 3y_1 & + & 2y_2 & \geq & -1 \\ & & \multicolumn{5}{l}{y_1 \geq 0, \ y_2 \geq 0} \end{array}$$

where non-negativity arises as in the previous question.

(b) The Complementary Slackness Relations, for the Primal (P) are

$$\begin{aligned} x_1^*(2y_1^* + y_2^* + 1) &= 0 \\ x_2^*(y_1^* + 4y_2^* - 2) &= 0 \\ x_3^*(3y_1^* + 2y_2^* + 1) &= 0 \end{aligned}$$

(c) Using the optimal solution of the primal (P) given by $x_1^* = x_3^* = 0, \ x_2^* = 1$, we require

$$y_1^* + 4y_2^* - 2 = 0, \quad \text{or} \quad y_1^* = 2 - 4y_2^*,$$

but if the solution is optimal, then

$$w^* = z^* = -x_1^* + 2x_2^* - x_3^* = 2$$

and hence we have that

$$w^* = 2y_1^* + 4y_2^* = 4 - 8y_2^* + 4y_2^* = 2, \quad \text{or} \quad y_2^* = \frac{1}{2}, y_1^* = 0.$$

[0 marks]

4.
   - The statement `x(i,j) = i*j + 1;` has one multiply, and one addition (2 total operations).
   - The outer loop executes n times.
   - The inner loop execute i times (where i varies depending on the inner loop), so the combination is

     $$1 + 2 + \cdots (n-1) + n = n(n-1)/2.$$

   - So the total number of operations is $2 \times n(n-1)/2 = n(n-1)$.

[3 marks]

5. Simplify the following Big-O notations
   (a) We can ignore constant factors $O(n)$
   (b) We only need to consider the largest term, so $O(n^2)$
   (c) We only need to consider the largest term, so $O(n)$
   (d) $O(n \log n)$

[4 marks]