Assignment 0: Solutions

TOTAL MARKS: 19

1. A set $S \subseteq \mathbb{R}^n$ is called a *convex set* if for all $x, y \in S$ and $0 \leq \alpha \leq 1$.

$$x, y \in S \implies \alpha x + (1-\alpha) y \in S,$$

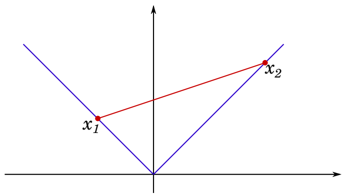
i.e., the line segment joining x and y lies in S.
Proof: Take two points $\mathbf{x}, \mathbf{y} \in \mathcal{C}_1 \cap \mathcal{C}_2$, then form a third $\mathbf{z} = \alpha \mathbf{x} + (1-\alpha)\mathbf{y}$ for some $\alpha \in [0,1]$. As $\mathbf{x}$ and $\mathbf{y}$ are in the intersection of the two sets, they are also in the sets $\mathcal{C}_i$. The fact that the $\mathcal{C}_i$ are convex means that $\mathbf{z} \in \mathcal{C}_i$ for $i = 1, 2$. Hence $\mathbf{z} \in \mathcal{C}_1 \cap \mathcal{C}_2$, and hence this set is convex.

[1 mark]

2. A function $f : S \to \mathbb{R}$ is a *convex function* if for all $x, y \in S$ and $0 \leq \alpha \leq 1$

$$\alpha f(x) + (1-\alpha) f(y) \geq f(\alpha x + (1-\alpha) y)$$

i.e., chords don't lie below the function.
Take two points x_1 and x_2. If they have the same sign (or one is zero), then the the chord between them lies exactly on top of $|x|$, so the proof is trivial. WLOG take $x_1 > 0$ and $x_2 < 0$, then the chord between them can be sketched as follows, and evidently lies above the function.



Mathematically, we demonstrate this by noting that by the triangle inequality $|x+y| \leq |x| + |y|$

$$|\alpha x_1 + (1-\alpha) x_2| \leq |\alpha x_1| + |(1-\alpha) x_2| = \alpha |x_1| + (1-\alpha)|x_2|,$$

when $\alpha \in [0,1]$, which is exactly what we need to show.

[1 mark]

3. (a) A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is *linearly independent* if the equation

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_r \mathbf{v}_r = \mathbf{0},$$

has only the trivial solution

$$x_1 = x_2 = \cdots = x_r = 0.$$

Otherwise the vectors are *linearly dependent.*

(b) If we have $n+1$ vectors in R^n, the the above equation has $n+1$ variables x_i, but is equivalent to n equations.
The trivial solution is always possible, so the equations aren't inconsistent, so there will always be an infinite number of solutions, and hence the vectors must be linearly dependent.

[1 mark]

4. $\left(MM^T\right)^T = \left(M^T\right)^T M^T = MM^T$, *i.e.,* the transpose equals the matrix, so it is symmetric.

[1 mark]

5.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{array}\right]$$

[1 mark]

6.

(a)	$x_1 + 4x_2x_3 + 5x_3 = 1$	no	non-linear
(b)	$3x_1 - x_2 + x_3 = 0$	yes	linear
(c)	$x_1 \leq x_2$	no	linear but an inequality
(d)	$x_1 - x_2 + \sqrt{x_3} = 6$	no	non-linear

[2 marks]

7. (a) Two systems of equations are *equivalent* iff they have exactly the same solutions.
(b) The first set of equations has solution $(4, 2, 3)$, as does the second, so they are equivalent.

[1 mark]

8. (a) pivot(3,3), which is made up of $R_2' \leftarrow R_2 - 3R_3$ [1 mark]
(b) pivot(1,1), which is made up of $R_2' \leftarrow R_2 - R_1$

9. *Reduced row echelon form* has

- All non-zero rows are above any rows of all zeros.
- The leading coefficient of each non-zero row (the first non-zero coefficient) is always strictly to the right of the leading coefficient of the row above it.
- The leading coefficients must be one.
- The leading coefficient is the only non-zero element of its column.

So the answers are
(a) yes [1 mark]
(b) no, but it can by dividing row 1 by 2.
(c) no, but it can be by swapping R_3 with R_4. [1 mark]
(d) no, but it can be by performing a pivot(3,3), *i.e.,* by taking $R_1' \leftarrow R_1 - 5R_3$.
Note the last one *is* in row echelon form (which doesn't require the last condition, which makes it reduced form).

10. (a) The augmented matrix and row-operations are

0	2	-8	1	0	0	20	
-3	12	-3	0	1	0	-36	
2	-8	-6	0	0	1	14	
-3	12	-3	0	1	0	-36	swap $R_1 \leftrightarrow R_2$
0	2	-8	1	0	0	20	
2	-8	-6	0	0	1	14	
1	-4	1	0	-1/3	0	12	$R_1 \leftarrow R_1/(-3)$
0	2	-8	1	0	0	20	
0	0	-8	0	2/3	1	-10	$R_3 \leftarrow R_3 + 2R_1/3$
1	0	-15	2	-1/3	0	52	$R_1 \leftarrow R_1 + 2R_2$
0	1	-4	1/2	0	0	10	$R_2 \leftarrow R_2/2$
0	0	-8	0	2/3	1	-10	
1	0	0	2	-19/12	-15/8	283/4	$R_1 \leftarrow R_1 - 15R_3/8$
0	1	0	1/2	-1/3	-1/2	5	$R_2 \leftarrow R_2 - 4R_3/8$
0	0	1	0	-1/12	-1/8	5/4	$R_3 \leftarrow R_3/(-8)$

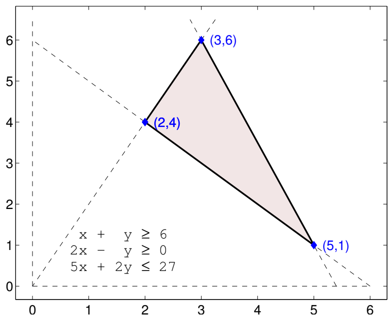
where all row operations in a block are performed using the rows from the block above.

(b) $A' = I$

(c) The matrix products work, and $W = A^{-1}$. [1 mark]

(d) We can immediately read off the solution $\mathbf{x} = (283/4, 15, 5/4)$.

11. Sketch of region [1 mark]



(a) The region is convex, and bounded, with vertices shown.

(b) Including one slack variable for each inequality, the initial tableau is

$$\begin{array}{rrrrr|r} -1 & -1 & 1 & 0 & 0 & -6 \\ -2 & 1 & 0 & 1 & 0 & 0 \\ 5 & 2 & 0 & 0 & 1 & 27 \end{array}$$

We can immediately draw a solution with $x_1 = 0$ and $x_2 = 0$, and basic variables x_3, x_4 and x_5 given by the right-hand column, *i.e.,* $(0, 0, -6, 0, 27)$. Note that this is not feasible because one of the slack variables is negative, so this is not a vertex.

(c) There are potentially $\binom{5}{2} = 10$ basic solutions. [1 mark]

(d) The basic solutions, and their values of f and g are given in the table:

basic solution	$f(x,y)$	g(x,y)
(3,6,3,0,0)	-3	12
(5,1,0,9,0)	-14	11
(2,4,0,0,9)	-2	8

So f is maximised at $(2, 4)$ and g is minimised at the same point.

12. The variables are the numbers of lamps x_A and x_B, so

(objective function)

$$\max\ z = 15x_A + 10x_B$$

subject to

$$\begin{aligned} 20x_A + 30x_B &\leq 6000 \text{ (Asterix's minutes)} \\ 20x_A + 10x_B &\leq 4800 \text{ (Pauls's minutes)} \end{aligned}$$

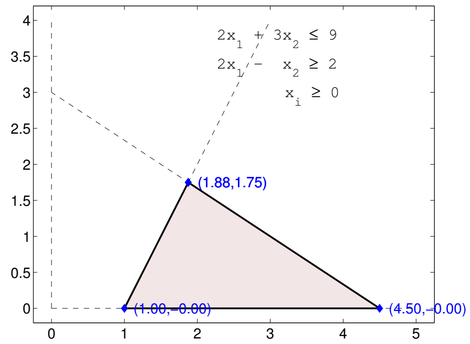
(nonnegativity)

$$x_A \geq 0, \qquad x_B \geq 0,$$

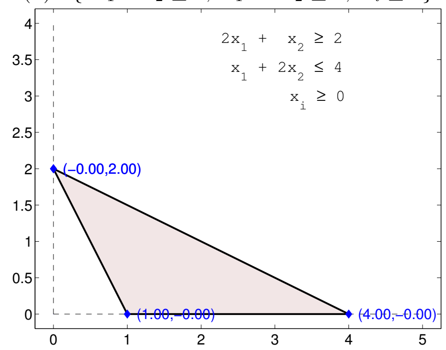
[2 marks]

13.

(i) $\{2x_1 + 3x_2 \leq 9,\ 2x_1 - x_2 \geq 2,\ x_i \geq 0\}$



(ii) $\{2x_1 + x_2 \geq 2,\ x_1 + 2x_2 \leq 4,\ x_i \geq 0\}$



[1 mark]

14. Let A be an $n \times n$ matrix.

(a) An eigenvector of A with eigenvalue λ satisfies $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x} \neq 0$

(b) The characteristic polynomial of A is $|\lambda I - A|$.

(c) **(i)**

$$\begin{aligned} |\lambda I - A_1| &= \begin{vmatrix} \lambda & -3 \\ -6 & \lambda + 3 \end{vmatrix} \\ &= \lambda(\lambda + 3) - 18 \\ &= \lambda^2 + 3\lambda - 18 \\ &= (\lambda + 6)(\lambda - 3) \end{aligned}$$

So the two eigenvalues of A_2 are $\lambda = -6, 3$. [1 mark]

(ii)

$$\begin{aligned} |\lambda I - A_2| &= \begin{vmatrix} \lambda - 2 & -1 & 0 \\ -1 & \lambda - 3 & -1 \\ 0 & -1 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 2) \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 2)\left[(\lambda - 3)(\lambda - 2) - 1\right] - (\lambda - 2) \\ &= (\lambda - 2)\left[(\lambda - 3)(\lambda - 2) - 2\right] \\ &= (\lambda - 2)\left[\lambda^2 - 5\lambda + 4\right] \\ &= (\lambda - 2)(\lambda - 1)(\lambda - 4) \end{aligned}$$

So the three eigenvalues of A_2 are $\lambda = 2, 1, 4$. [1 mark]

15. Calculate

(a) $\displaystyle\sum_{n=0}^{10} n = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$. (Cheap trick $\sum_{n=1}^{10} n = n(n+1)/2$).

(b) $\displaystyle\sum_{m=3}^{6} km = k \sum_{m=3}^{6} m = k(3 + 4 + 5 + 6) = 2.5 \times 18 = 45.$