Optimisation and Operations Research
Lecture 21: Sensitivity Analysis

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Section 1

Sensitivity Analysis

Sensitivity analysis

Often, when a business or industry is modelling its operations, there is insufficient information available to determine constants correctly.

For instance, in the telecommunications industry, the traffic matrices used to estimate numbers of calls between two points is just that: an estimate. Consequently, the optimal solution of an LP might be based on data that is not quite accurate. That is, we have the optimal solution to a model, but the model might be based on flawed data.

Businesses then need to know how that optimal solution is affected, if the model changes; how robust is the optimal solution to errors? This can be termed "post-optimal analysis".

We have seen that once B, A_B and $\mathbf{c}_B$ are known and given some initial data for M, we can directly get $\widehat{M}$ without going through all the steps of the Simplex Method.

Post-optimal analysis is based on this idea, where we investigate how changes in the model affect optimality and feasibility.

Errors in Linear Program Formulation

- All data has errors, or *noise*
    - artefacts of measurement
    - we might only have estimates of hard-to-measure quantities
- Optimisation often considers the future
    - we base objectives and constraints on *predictions*
- Formulating a LP often involves *approximation*
    - quadratic cost approximated as linear
    - complex boundary approximated by linear segments

All of these factors mean that the LP that we solve today, might be different from the *real* problem we aim to solve

Errors in Linear Program Formulation

- All LPs have errors in their formulation
- Do these errors matter?
    - maybe a small error doesn't really change the result?
    - maybe even a large error only has a small affect?
- *Sensitivity analysis* is the process of learning about how *sensitive* or *robust* our LP is to such errors

Types of errors

We can have errors in several components of the problem

- The objective function coefficients **c**
- The constraint coefficients A
- The constraint coefficients **b**
- We might want to add a constraint
- We might want to add a variable

Changes may effect

- optimality (the size and location of optimal value)
- feasibility

Section 2

Sensitivity Analysis: changes in the objective

Changes in the objective function coefficients

Use the matrix view of Simplex. If $\mathbf{c}$ changes, to $\breve{\mathbf{c}}$, then only the last row of the simplex tableau $\widehat{M}$ changes, as

$$
\widehat{M} = \left[\begin{array}{c|c} A_B^{-1} & \mathbf{0} \\ \hline \breve{\mathbf{c}}_B^T A_B^{-1} & 1 \end{array} \right] \left[\begin{array}{c|c|c} A_B & A_N & \mathbf{b} \\ \hline -\breve{\mathbf{c}}_B^T & -\breve{\mathbf{c}}_N^T & z_0 \end{array} \right]
$$

$$
= \left[\begin{array}{c|c|c} I & A_B^{-1} A_N & A_B^{-1} \mathbf{b} \\ \hline \mathbf{0}^T & \breve{\mathbf{c}}_B^T A_B^{-1} A_N - \breve{\mathbf{c}}_N^T & \breve{\mathbf{c}}_B^T A_B^{-1} \mathbf{b} + z_0 \end{array} \right].
$$

We can determine if the new solution should have the same basic variables, but checking to see if the last row has negative elements now, *i.e.,* is

$$
\breve{\mathbf{c}}_B^T A_B^{-1} A_N - \breve{\mathbf{c}}_N^T \geq 0.
$$

Changes in the objective function coefficients

Example

Suppose that the furniture manufacturer from our earlier problem

$$\max z = 13x_1 + 12x_2 + 17x_3$$

subject to

$$
\begin{aligned}
2x_1 + x_2 + 2x_3 &\leq 225 \\
x_1 + x_2 + x_3 &\leq 117 \\
3x_1 + 3x_2 + 4x_3 &\leq 420 \\
x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 &\geq 0
\end{aligned}
$$

has determined that with a change in the economy, he would now make a profit of \$$z$ given by the objective function

$$12.75x_1 + 12.25x_2 + 17x_3.$$

1. How would this affect production?
2. Will his new optimal profit be more or less than before?

Changes in the objective function coefficients

Recall that the final (optimal) tableau from the original furniture manufacturing problem was

x_1	x_2	x_3	x_4	x_5	x_6	z	$\mathbf{b}$
1	-1	0	2	0	-1	0	30
0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	0	$4\frac{1}{2}$
0	$\frac{3}{2}$	1	$-\frac{3}{2}$	0	1	0	$82\frac{1}{2}$
0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	4	1	$1792\frac{1}{2}$

with $B = \{x_1, x_5, x_3\}$ and $N = \{x_2, x_4, x_6\}$

We first rearrange the Tableau into the desired format with the basic variables on the left.

Changes in the objective function coefficients

x_1	x_5	x_3	x_2	x_4	x_6	z	$\mathbf{b}$
1	0	0	-1	2	-1	0	30
0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$4\frac{1}{2}$
0	0	1	$\frac{3}{2}$	$-\frac{3}{2}$	1	0	$82\frac{1}{2}$
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	4	1	$1792\frac{1}{2}$

and so

$$A_B^{-1}A_N = \begin{bmatrix} -1 & 2 & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix}$$

Changes in the objective function coefficients

With the new objective function coefficients

$$\breve{\mathbf{c}}_B^T = [12.75 \quad 0 \quad 17] \text{ and } \breve{\mathbf{c}}_N^T = [12.25 \quad 0 \quad 0],$$

we have that $\breve{\mathbf{c}}_B^T A_B^{-1} A_N - \breve{\mathbf{c}}_N^T$

$$= [12.75 \quad 0 \quad 17] \begin{bmatrix} -1 & 2 & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix} - [12.25 \quad 0 \quad 0]$$

$$= [\tfrac{1}{2} \quad 0 \quad 4.25] \qquad \geq \qquad \mathbf{0}$$

and since all entries are still nonnegative, this solution will still be optimal with new value $z^* = 30(\$12.75) + 82.5(\$17) = \$1785$.

Changes in the objective function coefficients

Example

Now the furniture manufacturer has determined that he would now make a profit of \$$z$ given by
$$12.5x_1 + 12.5x_2 + 17x_3.$$

So with $B = \{x_1, x_5, x_3\}$ and $N = \{x_2, x_4, x_6\}$ as before, we have

$$\check{\mathbf{c}}_B^T = [12.5 \quad 0 \quad 17] \quad \text{and} \quad \check{\mathbf{c}}_N^T = [12.5 \quad 0 \quad 0].$$

and so we again consider

$$\check{\mathbf{c}}_B^T A_B^{-1} A_N - \check{\mathbf{c}}_N^T.$$

Changes in the objective function coefficients

$$[12.5 \ 0 \ 17] \begin{bmatrix} -1 & 2 & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix} - [12.5 \ 0 \ 0] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 4.5 \end{bmatrix} \not\geq \mathbf{0}$$

and we no longer have an optimal basic solution. The tableau is

$$\widehat{M} =$$

	x_1	x_5	x_3	x_2	x_4	x_6	z	$\mathbf{b}$
	1	0	0	-1	$\boxed{2}$	-1	0	30
	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$4\frac{1}{2}$
	0	0	1	$\frac{3}{2}$	$-\frac{3}{2}$	1	0	$82\frac{1}{2}$
	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$4\frac{1}{2}$	1	$1777\frac{1}{2}$

Changes in the objective function coefficients

One more G-J pivot step on the $(1,5)$ entry at this stage gives the optimal tableau.

$$\widehat{M} =$$

x_1	x_5	x_3	x_2	x_4	x_6	z	$\mathbf{b}$
$\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	15
$\frac{1}{4}$	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	12
$\frac{3}{4}$	0	1	$\frac{3}{4}$	0	$\frac{1}{4}$	0	105
$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$4\frac{1}{4}$	1	1785

with the new basic variables being $B = \{x_4, x_5, x_3\}$.

Changes in the objective function coefficients

> ### Example
>
> Suppose we can increase the profit $\$c_3$ on each bedframe by an amount Δ, provided we decrease the profit on each desk and chair by the same amount.

We can find limits on Δ, such that in our optimal solution, our basic variables remain unchanged. For the new $\breve{\mathbf{c}}_B^T = [13 - \Delta, 0, 17 + \Delta]$ and $\breve{\mathbf{c}}_N^T = [12 - \Delta, 0, 0]$, we have

$$
\breve{\mathbf{c}}_B^T A_B^{-1} A_N - \breve{\mathbf{c}}_N^T = [(13 - \Delta), \ 0, \ (17 + \Delta)] \begin{bmatrix} -1 & 2 & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix} - [(12 - \Delta) \ 0 \ 0]
$$

$$
= \left[\tfrac{1}{2} + \tfrac{7}{2}\Delta, \quad \tfrac{1}{2} - \tfrac{7}{2}\Delta, \quad 4 + 2\Delta \right] \quad \geq \mathbf{0} \quad \text{for} \quad \Delta \in \left[-\tfrac{1}{7}, \tfrac{1}{7} \right].
$$

Section 3

Sensitivity analysis: Changes in the constraint coefficients

Changes in the constraint coefficients (a_{ij})

If (a_{ij}) is associated with a basic variable $x_j \in B$, then A_B will change, and hence so will A_B^{-1}.

Often it requires as much work recalculating A_B^{-1} as it does reworking the problem from scratch and so we will only consider the case where the coefficient a_{ij} being changed is associated with a non-basic variable.

In this case, we can use a duality argument to determine if the change affects the present basic optimal solution.

Recall that if the current basic solution **x** is optimal, then so is the associated dual solution **y**. In fact, the current primal solution is optimal if and only if the associated dual solution is dual-feasible.

Changes in the constraint coefficients (a_{ij})

Recall that corresponding to every non-basic variable, there is a dual constraint.

If we change a coefficient a_{ij} associated with a non-basic variable x_j, we also change the associated dual constraint.

- The present basic solution **x** will still satisfy the changed primal constraint.

  - That this is true is because the changed a_{ij} is being multiplied by zero, since x_j is non-basic.

- If the **new** dual solution satisfies the **changed** dual constraint, then we still have the optimal solution!

Changes in the constraint coefficients (a_{ij})

Example

Suppose now, a brand new design for a chair is proposed to increase their popularity. In the original design, each chair consumed 1 unit of labour, 1 unit of metal, and 3 units of wood. The new design however, consumes $a_{12} = 4$ units of labour, $a_{22} = 0$ units of metal, and $a_{32} = 2\frac{1}{2}$ units of wood.

Is the new profit still optimal?

Corresponding to the variable x_2 (the number of chairs), is a dual constraint that has changed from

$$y_1 + y_2 + 3y_3 \geq 12 \qquad \text{to} \qquad 4y_1 + 2\frac{1}{2}y_3 \geq 12,$$

but what is the dual solution corresponding to our basic solution?

Changes in the constraint coefficients (a_{ij})

Recall that from our original Primal LP, we had
$x_1 = 30, x_2 = 0, x_3 = 82.5, x_4 = 0, x_5 = 4.5, x_6 = 0$, giving us the following inequalities from the CSRs

$$
\begin{aligned}
2y_1 + y_2 + 3y_3 &= 13 \\
2y_1 + y_2 + 4y_3 &= 17 \\
y_2 &= 0.
\end{aligned}
$$

The dual solution corresponding to this basic solution is therefore given by $\mathbf{y} = [\frac{1}{2} \; 0 \; 4]$, which satisfies all of the original dual constraints. We note that it also satisfies the new dual constraint

$$
4y_1 + 2\frac{1}{2}y_3 = 4 \times \frac{1}{2} + 2\frac{1}{2} \times 4 = 12 \; (\geq 12)
$$

and therefore $\mathbf{x}$ is still optimal.

Changes in the constraint coefficients (a_{ij})

If, however, we had a chair design that used 1 unit of labour, 2 units of metal, and 2 units of wood, then calculating $-\mathbf{c'_N}^T = \mathbf{c}_B^T A_B^{-1} A_N - \mathbf{c'_N}^T$, the tableau $\widehat{M}$ becomes

$$\widehat{M} =$$

	x_1	x_2	x_3	x_4	x_5	x_6	z	$\mathbf{b}$
	1	0	0	2	0	-1	0	30
	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	0	0	$4\frac{1}{2}$
	0	$\frac{1}{2}$	1	$-\frac{3}{2}$	0	1	0	$82\frac{1}{2}$
	0	$-\frac{7}{2}$	0	$\frac{1}{2}$	0	4	1	$1792\frac{1}{2}$

and the dual solution is no longer feasible, and so $\mathbf{x}$ is not optimal in this case. That is, $\quad y_1 + 2y_2 + 2y_3 = \frac{1}{2} + 2 \times 4 = 8\frac{1}{2}$ ($\not\geq 12$).

Changes in the constraint coefficients (a_{ij})

Since the dual solution is no longer feasible, we perform two more G-J
pivot steps, to get the new optimal solution given by the new tableau

$$\widehat{M} = $$

	x_1	x_2	x_3	x_4	x_5	x_6	z	$\mathbf{b}$
	$\frac{1}{2}$	0	0	1	0	$-\frac{1}{2}$	0	15
	$\frac{1}{6}$	1	0	0	$\frac{2}{3}$	$-\frac{1}{6}$	0	8
	$\frac{2}{3}$	0	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	101
	$\frac{1}{3}$	0	0	0	$\frac{7}{3}$	$\frac{11}{3}$	1	1813

So this chair design would increase profit to \$1813, with a production
including 8 chairs.
That is, $x_1 = 0, x_2 = 8, x_3 = 101, x_4 = 15, x_5 = 0, x_6 = 0$.

Section 4

Sensitivity analysis: Adding a new variable / dual constraint

Adding a new variable / dual constraint

Example

In an effort to improve business, the furniture manufacturer decides to make video cabinets as well. Each video cabinet needs 1.5 units of labour, 1 unit of metal, and 2 units of wood; the profit from each cabinet sold is $14.

If we let x_7 be the number of video cabinets produced. Then the new problem is

$$
\begin{aligned}
(P) \quad \max z = 13x_1 + 12x_2 + 17x_3 + 14x_7 \\
2x_1 + x_2 + 2x_3 + 1.5x_7 &\leq 225 \\
x_1 + x_2 + x_3 + x_7 &\leq 117 \\
3x_1 + 3x_2 + 4x_3 + 2x_7 &\leq 420
\end{aligned}
$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, x_7 \geq 0.$$

Adding a new variable / dual constraint

In the dual, there will be one more dual constraint,

$$1.5y_1 + y_2 + 2y_3 \geq 14.$$

Checking to see if the original dual solution $\mathbf{y} = [\frac{1}{2} \; 0 \; 4]$ satisfies the new dual constraint, we find that it is infeasible for the dual. That is,

$$1.5y_1 + y_2 + 2y_3 = 1.5 \times \frac{1}{2} + 2 \times 4 = 8\frac{3}{4} \quad \not\geq \quad 14.$$

and so the original primal solution is no longer optimal. Hence we know that we can increase our profit by including some video cabinets in the production schedule.

To determine the new production schedule, we can use our knowledge of how to update M from the beginning of this section.

Adding a new variable / dual constraint

$$\widehat{M} \;=\; \left[\begin{array}{c|c} A_B^{-1} & \mathbf{0} \\ \hline \mathbf{c}_B^T A_B^{-1} & 1 \end{array}\right] \left[\begin{array}{c|c|c} A_B & A_N & \mathbf{b} \\ \hline -\mathbf{c}_B^T & -\mathbf{c}_N^T & z_0 \end{array}\right]$$

$$=\; \left[\begin{array}{c|c|c} I & A_B^{-1} A_N & A_B^{-1}\mathbf{b} \\ \hline \mathbf{0}^T & \mathbf{c}_B^T A_B^{-1} A_N - \mathbf{c}_N^T & \mathbf{c}_B^T A_B^{-1}\mathbf{b} + z_0 \end{array}\right].$$

Since x_7 is a non-basic variable in the current solution, only A_N and $\mathbf{c}_N$ have been modified.

We therefore do not need to recalculate A_B^{-1} and $\mathbf{c}_B^T A_B^{-1}$, but only need to evaluate $A_B^{-1} A_N$ and $\left(\mathbf{c}_B^T A_B^{-1}\right) A_N - \mathbf{c}_N^T$ for the new A_N and $\mathbf{c}_N$ to get the current tableau inclusive of the variable x_7.

Adding a new variable / dual constraint

x_1	x_2	x_3	x_4	x_5	x_6	x_7	**b**
1	-1	0	2	0	-1	1	30
0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	$\boxed{\frac{1}{4}}$	$4\frac{1}{2}$
0	$\frac{3}{2}$	1	$-\frac{3}{2}$	0	1	$-\frac{1}{4}$	$82\frac{1}{2}$
0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	4	$-\frac{21}{4}$	$1792\frac{1}{2}$
1	-3	0	$\boxed{4}$	-4	-1	0	12
0	2	0	-2	4	0	1	18
0	2	1	-2	1	1	0	87
0	11	0	-10	21	4	0	1887
$\frac{1}{4}$	$-\frac{3}{4}$	0	1	-1	$-\frac{1}{4}$	0	3
$\frac{1}{2}$	$\frac{1}{2}$	0	0	2	$-\frac{1}{2}$	1	24
$\frac{1}{2}$	$\frac{1}{2}$	1	0	-1	$\frac{1}{2}$	0	93
$2\frac{1}{2}$	$3\frac{1}{2}$	0	0	11	$1\frac{1}{2}$	0	1917

The optimal solution is $z^* = 1917$, at $x_1 = 0, x_2 = 0, x_3 = 93, x_7 = 24$.

Section 5

Sensitivity analysis: changes to feasibility

Changes to feasibility

We will now consider changes that affect feasibility:

(i) changes in the right-hand side, *i.e.*, **b**.

(ii) including a new constraint.

Section 6

Feasibility, changes in **b**

Changes to feasibility

Recall that

$$\widehat{M} \;=\; \left[\begin{array}{c|c|c} I & A_B^{-1}A_N & A_B^{-1}\mathbf{b} \\ \hline \mathbf{0}^T & \mathbf{c}_B^T A_B^{-1}A_N - \mathbf{c}_N^T & \mathbf{c}_B^T A_B^{-1}\mathbf{b} + z_0 \end{array} \right].$$

from which we see that the optimal basis B remains optimal for changing $\mathbf{b}$ if $A_B^{-1}\mathbf{b} \geq \mathbf{0}$ with the new $\mathbf{b}$.

Changes in the right-hand side, *i.e.*, **b**

Example

Let's determine bounds on the amount of wood available in **Our First Example** such that the original basis remains optimal.

Recall our final original tableau.

x_1	x_2	x_3	x_4	x_5	x_6	z	**b**
1	-1	0	2	0	-1	0	30
0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	0	$4\frac{1}{2}$
0	$\frac{3}{2}$	1	$-\frac{3}{2}$	0	1	0	$82\frac{1}{2}$
0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	4	1	$1792\frac{1}{2}$

with $B = \{x_1, x_5, x_3\}$ and $N = \{x_2, x_4, x_6\}$.

Changes in the right-hand side, *i.e.,* **b**

Let Δ be the number of units of wood available, then with $B = \{x_1, x_5, x_3\}$

$$
A_B^{-1}\mathbf{b} \;=\; \begin{bmatrix} 2 & 0 & -1 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 225 \\ 117 \\ \Delta \end{bmatrix} = \begin{bmatrix} 450 - \Delta \\ 4\frac{1}{2} \\ \Delta - 337\frac{1}{2} \end{bmatrix}
$$

$$
\geq \;\; \mathbf{0} \text{ when } \Delta \in \left[337\frac{1}{2} \, , \, 450 \right].
$$

Example

Suppose now that the amount of available metal decreases from 117 units to 113 units.

Changes in the right-hand side, *i.e.,* **b**

Then $A_B^{-1}\mathbf{b} = \begin{bmatrix} 2 & 0 & -1 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 225 \\ 113 \\ 420 \end{bmatrix} = \begin{bmatrix} 30 \\ \frac{1}{2} \\ 82\frac{1}{2} \end{bmatrix} \geq \mathbf{0}.$

So our present optimal basis is still optimal, with the same optimal value

$$\mathbf{c}_B^T A_B^{-1}\mathbf{b} + z_0 = [13, 0, 17] \begin{bmatrix} 2 & 0 & -1 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 225 \\ 113 \\ 420 \end{bmatrix} = \$1792.5$$

but with $x_1 = 30$, $x_5 = \frac{1}{2}$, $x_3 = 82.5$.

$x_5 = \frac{1}{2}$ represents the fact that there is $117 - 113 = 4$ less units of metal left over in this situation (x_5 was previously $4\frac{1}{2}$), because we started with 4 units less and manufactured the same quantities of furniture.

Changes in the right-hand side, *i.e.,* **b**

Example

Suppose now that the amount of available metal decreases from 117 units to 112 units (5 units less)

Then $A_B^{-1}\mathbf{b} = \begin{bmatrix} 2 & 0 & -1 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 225 \\ 112 \\ 420 \end{bmatrix} = \begin{bmatrix} 30 \\ -\frac{1}{2} \\ 82\frac{1}{2} \end{bmatrix} \not\geq \mathbf{0}$,

and so the solution (not surprisingly) would no longer be feasible and hence the basis $B = \{x_1, x_5, x_3\}$ is no longer optimal).

In this case, we can use a method called the "dual simplex method" to determine a new optimal solution (to be introduced soon).

$B = \{x_4, x_1, x_3\}$ with $(x_4, x_1, x_3) = (1, 28, 84)$, optimal value \$1792.

Section 7

Sensitivity analysis: feasibility, including a new constraint

Including a new constraint

- If the new constraint is satisfied by the original optimal solution, then that constraint is obviously redundant, and the original optimal solution is still valid.

- If the original optimal solution does **not** satisfy the new constraint, then more work needs to be done to find the new optimal solution. In this case, the original optimal solution is outside the feasible region...

  the new constraint has "cut it off"

- The "dual simplex method" can be used to determine the new optimal solution.
    - start from current dual solution
    - work towards solution that is primal feasible

Takeaways

- Optimisers should not assume that data is perfect
  - it could contain errors or unknowns
  - sensitivity analysis is how we can see whether the solutions depend strongly on these errors
- The matrix view of optimisations gives us a way to calculate sensititivity without having the redo the optimisation again, and again.
- Often, even if the solution should change, we don't have to go back to the start

Further reading I