Optimisation and Operations Research

Lecture 10: Empirical Sensitivity Analysis

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Section 1

Sensitivity Analysis

Errors in Linear Program Formulation

- All data has errors, or *noise*
  - artefacts of measurement
  - we might only have estimates of hard-to-measure quantities
- Optimisation often considers the future
  - we base objectives and constraints on *predictions*
- Formulating a LP often involves *approximation*
  - quadratic cost approximated as linear
  - complex boundary approximated by linear segments

All of these factors mean that the LP that we solve today, might be different from the *real* problem we aim to solve

Errors in Linear Program Formulation

- All LPs have errors in their formulation
- Do these errors matter?
  - maybe a small error doesn't really change the result?
  - maybe even a large error only has a small effect?
- *Sensitivity analysis* is the process of learning about how *sensitive* or *robust* our LP is to such errors

Types of errors

We can have errors in several components of the problem

- The objective function coefficients $\mathbf{c}$
- The constraint coefficients A
- The constraint coefficients $\mathbf{b}$
- We might want to add a constraint
- We might want to add a variable

Effects of errors

1. changes in $\mathbf{c}$ can
   1. change the vertex of the optimal solution
   2. keep the same vertex, but change the objective function value
   3. have no effect at all
2. changes in A and $\mathbf{b}$ perturb the shape of the feasible region
   1. this could toggle feasibility of the problem
   2. it could change the vertex of the optimal solution
   3. it could change the location of the optimal vertex
   4. it might have no effect at all

Formal Sensitivity Analysis

- We can calculate these things formally
  - Use clever matrix analysis
  - Avoids costly recalculations
  - See lectures 20 and 21
- We'll start a bit simpler

Section 2

Empirical Sensitivity Analysis

Empirical Sensitivity Analysis

Fundamental idea

- if you are worried about the effect of an error
- try it out!

An error in the objective coefficients $\mathbf{c}$

Example $z = [c_1, c_2]^T \mathbf{x}$

- Consider alternative objectives, *e.g.*,
  - $[c_1 + \delta, c_2 - \delta]$
  - $[c_1, c_2 + \delta]$

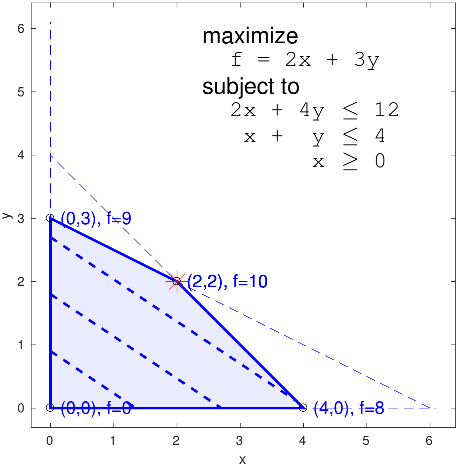
  your choice depends on what you know about the errors, or how you want to examine their effect

- Now solve the new problem, *e.g.*,

$$\begin{aligned} \max \quad & z = [c_1 + \delta, c_2 - \delta]^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

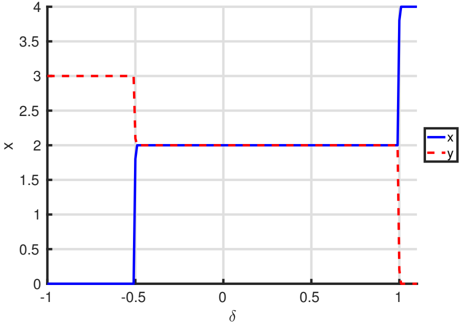
  for a range of values of δ and plot the results

Example



Take $f = (2 + \delta)x + 3y$

Example



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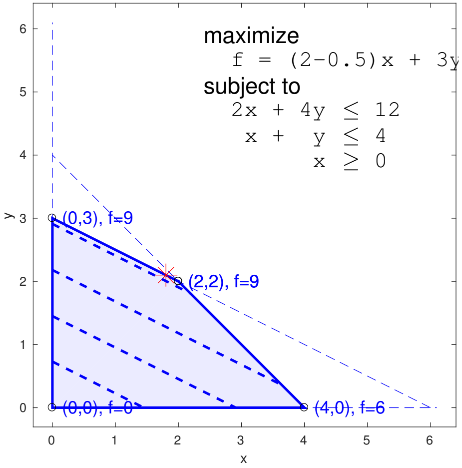

Take $f = (2 + \delta)x + 3y$

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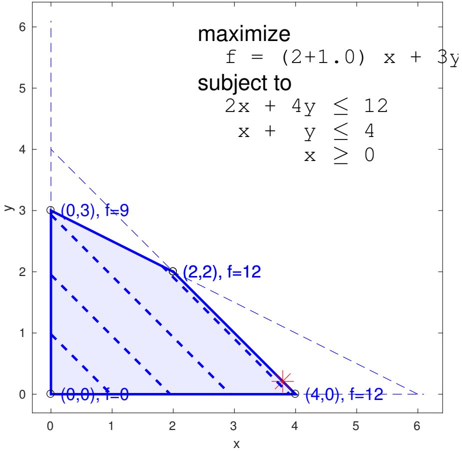


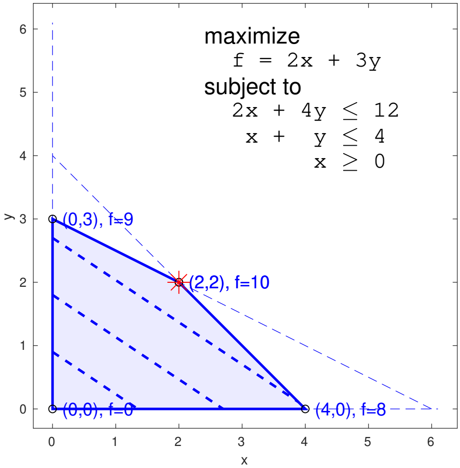
Take $f = (2+\delta)x + 3y$

An error in the constraints

- As before, formulate the error model
- Substitute the new values in the problem and solve
  - *e.g.*, take $\mathbf{b}_\delta$ as the constraint values plus error of size δ and solve

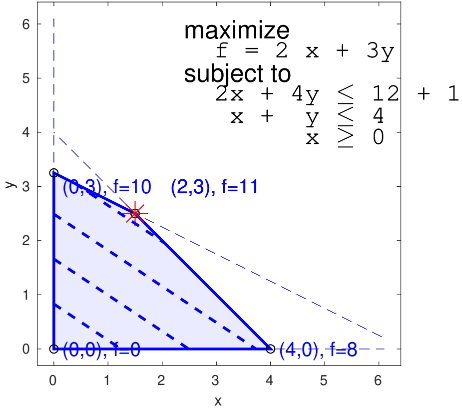
$$\begin{aligned} \max \quad & z = \mathbf{c}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}_\delta \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Example



Use constraint $2x + 4y \leq 12 + \delta$

Example



Use constraint $2x + 4y \leq 12 + \delta$

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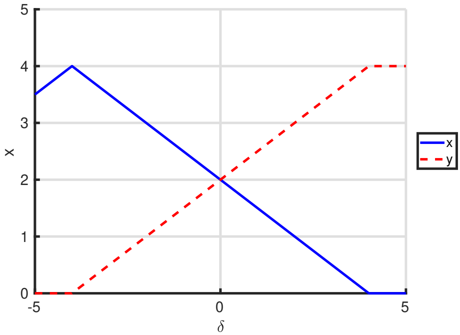

Use constraint $2x + 4y \leq 12 + \delta$

Error models

The hard part is choosing an error model:

1. As above, choose a set of simple changes to the coefficient vectors, and explore the effect of a range of values
    1. use simple scenarios to explore the space
2. Alternatively, add random noise to coefficients
    1. relatively easy
    2. scale/size of errors can be controlled by standard deviation

    but this might be unrealistic: *e.g.*, might end up with negative b_i

The key is in understanding your problem well.

Section 3

Interrogating a problem

One of the hardest things in mathematics is transcribing a problem into mathematics in the first place.

- Customers and managers don't have the terminology to tell you what you want
  - they speak a different language, literally
- They sometimes don't know what the problem is
  - because they don't know what is possible
- The data they have is usually a mess
  - The age "Big Data" didn't change that, it just meant there was more mess

Interrogating the Problem

We'll start simple, with *interrogating the problem*

- Assume the problem is known, and has been expressed in words
- We need to learn how to extract a mathematical description of the problem
- It will require
  - a bit of linguistics
  - some puzzle solving
  - some approximations
  - a clear understanding of where we are trying to get to

Interrogating the Problem

We've seen simpler examples

> *A manufacturing company makes three types of items: desks, chairs, and bed-frames, using metal and wood. A single desk requires 2 hours of labour, 1 unit of metal and 3 units of wood, ... In a given time period, there are 225 hours of labour available, 117 units of metal and 420 units of wood. The profit on one desk is $13, ... Choose the right number of items to produce to maximise the company's profit.*

Interrogating the Problem

1. Read the question right through!
2. Identify the variables
3. Identify the objective
4. Formulate the constraints

L$\tilde{\odot}\tilde{\odot}$k carefully.

Interrogating the Problem

> A manufacturing company makes three types of items: desks, chairs, and bed-frames, using metal and wood. A single desk requires 2 hours of labour, 1 unit of metal and 3 units of wood, ... In a given time period, there are 225 hours of labour available, 117 units of metal and 420 units of wood. The profit on one desk is \$13, ... Choose the right number of items to produce to maximise the company's profit.

Variables: look for

- words like "choose" or "decide"
- repeated words:
  - these *might* be related to variables
- we are looking for something numerical that we can control
- identify *units*
  - these are *required* but also give clues

Interrogating the Problem

A manufacturing company makes three types of items: desks, chairs, and bed-frames, using metal and wood. A single desk requires 2 hours of labour, 1 unit of metal and 3 units of wood, ... In a given time period, there are 225 hours of labour available, 117 units of metal and 420 units of wood. The profit on one desk is \$13, ... Choose the right number of items to produce to maximise the company's profit.

Objective: look for

- words like "maximise" or "minimise" or "profit" or "cost"
- we are looking for something numerical that we will optimise
- it should be written in terms of the variables
  - so this is another clue about variables
- then find the coefficients **c**
  - if its profit or loss, units (of **c**) should be \$s per unit variable
  - otherwise, look for the values/numbers related to the objective

Interrogating the Problem

A manufacturing company makes three types of items: desks, chairs, and bed-frames, using metal and wood. A single desk requires 2 hours of labour, 1 unit of metal and 3 units of wood, ... In a given time period, there are 225 hours of labour available, 117 units of metal and 420 units of wood. The profit on one desk is \$13, ... Choose the right number of items to produce to maximise the company's profit.

Constraints: look for

- words like "less than" or "no more" or "at least"
- repeated words:
  - these *might* be related to constraints
- the find the coefficients A and $\mathbf{b}$
  - look for the values/numbers related to each constraints

Interrogating the Problem

The hardest parts are often *implicit* constraints

- No-one states that we can't have "negative" chairs
  - Likewise, there can be other constraints that seem so obvious (to the person setting the problem) that they don't state them
- Other constraints are in the problem statement, but spread out, and never explicitly stated (see following example)
- Often constraints require some reasoning
  - think about the meaning behind the words
  - think about "physics"
  - use common sense
- One BIG clue is that we are doing *Linear Programming*, so all of your constraints will be either linear inequalities, or linear equations

Interrogating the Problem

You have $1 million to spend on a new coin collection. There are a variety of coins available. Each has characteristics of interest: rarity, age, and condition. You want a balance in your collection. That is, you want a certain number of coins that are rare, and a number that are old, a number in good condition, and so on. And you wish to maximise the total number of coins in the collection.

(actual numbers omitted for brevity)

- Variables: whether or not to purchase each possible coin.
  - notice that these are *binary* variables
- Objective: maximise the number of coins
- Constraints
  - the obvious, explicit constraints concern rarity, age, and condition of the overall collection
  - implicitly, each coin has a cost, and you can't spend more than $1 million.

Interrogating the Problem

Takeaways

- All LPs contain errors
- Sensitivity analysis is used to see the effect of these errors
- Interrogating a problem
  - this is the HARDEST bit of mathematics

Further reading I