Assignment 4: Due Friday 5th April at 5pm

Late assignments will not be accepted except by prior arrangement (for a good reason)

Please include your student number in your handed up work, as Canvas doesn't give this to me automatically.

1. Suppose that you have a r-regular graph. Show that the vector $\mathbf{1} = (1, 1, \dots, 1)$ is an eigenvector of the adjacency matrix with eigenvalue r.

[1 marks]

2. One of the problems in the Watts-Stogatz small-world network is that the rewiring process can disconnect a node.

   Show that the probability of a node becoming unconnected, starting from an r-regular ring and rewiring links with probability p, is approximately

$$P(r, p) \simeq \left(pe^{-p}\right)^r .$$

[2 marks]

3. For a GER random graph $G(n, p)$, with large n nodes, and probability of a link p such that $np > 1$, we want to calculate the size of the expected giant component, and we shall do so in terms of q, the expected fraction of nodes in the largest connected component.
   (a) Form the network $G(n-1, p)$ and then add the nth node, connecting it to the existing nodes with probability p. The fraction q is the large n asymptotic, so it should be true[1] for $n-1$ nodes as it is for n. Presuming that the new node has degree k, derive an approximation for the large n probability that this new node and the connected component to which it belongs are outside of the existing large connected component?
   (b) Show from this result, and the Poisson approximation to node degree in the GER random graph that the large n probability that this new node and the connected component to which it belongs are outside of the existing large connected component is

$$P(\text{ node's CC is outside large CC}) = e^{-(n-1)pq}.$$

   (c) From this argue that q is (approximately) the solution to the equation

$$q = 1 - e^{-q(n-1)p}.$$

   (d) Plot the shape of the positive solution of q for $n = 50$ as a function of p. Ensure that your plot is in all respects a good visualisation of the phenomena of interest. For instance:
      - Ensure the plot is easily readable, and includes all required information to interpret it.
      - Choose the range of values of p carefully so that a solution exists and is interesting.
      - Highlight relevant features of the plot.
      - You should use simulations or other information to make the plot more informative.

[4 marks]

[1] We might imagine an exception whereby there were actually 2 large connected components, and the new node bridges them, but because $np > 1$ we expect a single large connected component for large n.

4. Write code to
   (a) Generate an ER random network.
      - It should take as input arguments the size of the network, and the parameter p (the probability of an adjacency).
      - It should output the result as a sparse adjacency matrix.

   (b) Generate another random graph model (your choice).

   (c) Test your clustering metric code by calculating clustering for ensembles of random graphs with size 10, 100, 1000 and 10000.
      - Use average node degree for the ER graphs as $\bar{k} = (n-1)p = 2$, and
      - Choose parameters for your second random graph that result in the same average node degree.

   Compare the two with a carefully thought through plot.

[3 marks]