Information Theory and Networks

Lecture 33: Information Theory, the Universe and Everything

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Part I

Information Theory, the Universe, and Everything

> If you take a pack of cards as it comes from the maker and shuffle it for a few minutes, all traces of the original systematic order disappears. The order will never come back however long you shuffle. There is only one law of nature — the second law of thermodynamics — which recognises a distinction between the past and the future. Its subject is the random element in a crowd. A practical measure of the random element which can increase in the universe but never decrease is called entropy.
>
> *Arthur Eddington, The Nature of the Physical World, 1928*

Section 1

More on the Second Law

The second law of thermodynamics

In a closed system, entropy cannot decrease.

- An operation is dissipative if it turns useful forms of energy into useless ones, such as heat energy
- Arrow of time implicit in this
  - most physical laws are reversible
    - they don't have a "natural" direction for time
    - you couldn't tell if a "video" of physical events at the microscopic level was running forward or backwards
  - yet most macroscopic processes are not
    - largely due to the 2nd law
  - 2nd law creates idea of causality?
    - one things in "caused" by another in linear time
    - our consciousness perceives it that way because we are also subject to the second law

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Entropy is guaranteed not to decrease, but stays equal only in highest state of disorder, so it effectively increases.

Weak-nuclear force can violate time symmetry very rarely.

Some problems

- Large-scale Universe
  - big-bang
    - where does local organization come from?
  - heat death might be OK, but big crunch reverses it
    - so long-term entropy works out the same?
- Black holes have no hair
  - if black holes evaporate (Hawking's radiation)
  - what happens to information that drops into a black hole?
  - so black holes have entropy

$$S_{BH} = \frac{k_B A}{4\ell_P^2}$$

  - where
    - A is the area of the event horizon
    - ℓ_P is the Plank length

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The Second Law and Markov Chains

- Model an isolated system as a Markov chain
  - transitions according to physical laws governing the system
  - future of system independent of past (except through current state)
- 2nd law (in naive form) doesn't work
  - entropy can decrease
  - e.g., consider a case where the
    - initial distribution is uniform (max entropy)
    - stationary distribution is non-uniform
  - we could just chalk this up to Markov chains not really being covered by thermodynamics, but
- Four different interpretations of 2nd law [CT91, pp.34-36]
  1. relative entropy decreases with n
  2. relative entropy decreases RE stationary distribution
  3. entropy increases if the stationary distribution is uniform
  4. conditional entropy increases with n

The problem (it seems to me) arises because a Markov chain is often a representation of a system that is closed in principle (e.g., people might not enter or leave), but not in the strict sense of thermodynamics.

The Second Law and Markov Chains: 1

- For the Markov chain discussed above
- Consider relative entropy $D(\mu_n \| \mu'_n)$ where
  1. μ_n and μ'_n are two probability distributions at time n
  2. μ_{n+1} and μ'_{n+1} are corresponding distributions at time $n+1$
- Then

$$D(\mu_n \| \mu'_n) \geq D(\mu_{n+1} \| \mu'_{n+1})$$

  1. relative entropy decreases with time
- Think of $D(\cdot)$ as a distance
  1. the two probability distributions get closer together as the system evolves
  2. remember $D(\cdot)$ has a lower-bound of zero, so a limit must exist

Derivation just uses the chain rule for relative entropy.

The Second Law and Markov Chains: 2

- For the Markov chain discussed above
- Consider $D(\mu_n \| \mu)$
  1. μ_n is a probability distributions at time n
  2. μ is the stationary distribution
- Then

$$D(\mu_n \| \mu) \geq D(\mu_{n+1} \| \mu)$$

  1. relative entropy decreases with time
- Think of $D(\cdot)$ as a distance
  1. the probability distribution gets closer and close to the stationary distribution
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The Second Law and Markov Chains: 3

- For the Markov chain discussed above
  1. consider the case where the stationary distribution is uniform
- We can write the relative entropy as

$$D(\mu_n \| \mu) = \log |\Omega| - H(\mu_n)$$

  1. We know $D(\mu_n \| \mu)$ can't increase
  2. so $H(\mu_n)$ can't decrease
- So for this case $H(\mu_n) \leq H(\mu_{n+1})$
  1. this makes sense for physical systems
  2. in equilibrium, microstates are equally likely (uniform stationary distribution)
  3. so this kind-of handles the transition to equilibrium
- A nice example is a shuffle
  - a crude idea of a shuffle is a random permutation, with ultimately uniform distribution of all cards

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Theorem

A Markov chain will have uniform stationary distribution iff its probability transition matrix is doubly stochastic, i.e., all its rows and columns sum to one.

In some sense this equates to reversible physical processes, such as often considered in thermodynamics, as in this case, the transpose of the transition matrix is also a valid transition matrix.

The Second Law and Markov Chains: 4

- For the Markov chain discussed above
  1. make the additional assumption that it is stationary
  2. consider the conditional entropy $H(X_n|X_1)$

$$H(X_n|X_1) \leq H(X_{n+1}|X_1)$$

- The conditional uncertainty about the future increases
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We didn't require stationarity of the Markov chain before, except implicitly in talking about stationary distributions.

Proof follows from conditioning reducing entropy, and then the Markov property, or more simply from the data processing inequality.

Section 2

Landauer's Principle

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Landauer's Principle

- Any calculation must involve some exchange of energy
  - so there is a lower bound on per bit calculation
  - any logically irreversible manipulation (e.g., erasure of a bit) is accompanied by an increase in entropy
- Landauer limit
  - minimum possible energy required to change one bit

    $$= k_B T \ln 2$$

    where k_B =Boltzmann's constant and T is temperature (in K)
  - modern computers use millions of times this energy
- Practical lower bound given by $T = 3K$ cosmic background radiation

$$k \simeq 1.38 \times 10^{23} J/K$$

Explanation

- Imagine a hypothetical efficient computer
  - never wastes energy
  - it's isolated (no energy comes in or out)
  - any
    - ★ logical state (binary bits in computer) is a macrostate
    - ★ represented by some number of microstates (physical states of electrons, magnetic particles, etc.)
  - we can imagine either
    - ★ keep track of logical state
    - ★ of not
- Irreversible computation
  - two or more logical states map to a single state
  - not invertible

Explanation

(1) Don't keep track of logical state

- Irreversible calculation implies that the number of possible logical states of the computer decreases
  - in erasing a bit, we have reduced no. of states by factor of 2
  - All else equal (equal probabilities)

$$H(X) = \log_2 |\Omega|$$

    - so if we reduce state space by factor of two
    - $H(X)$ is reduced by 1 bit
- But entropy can't decrease in isolated system
  - there must be some other increase
  - number of physical microstates corresponding to the macrostate (or the logical bits), must have increased to compensate
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- Irreversible calculation doesn't change the number of possible states (just the actual state)
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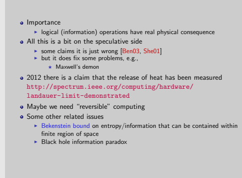
- Importance
  - logical (information) operations have real physical consequence
- All this is a bit on the speculative side
  - some claims it is just wrong [Ben03, She01]
  - but it does fix some problems, e.g.,
    - Maxwell's demon
- 2012 there is a claim that the release of heat has been measured http://spectrum.ieee.org/computing/hardware/landauer-limit-demonstrated
- Maybe we need "reversible" computing
- Some other related issues
  - Bekenstein bound on entropy/information that can be contained within finite region of space
  - Black hole information paradox



Further reading I

Charles H. Bennett, *Notes on landauer's principle, reversible computation and maxwell's demon*, arXiv:physics/0210005v2, 2003, http://arxiv.org/abs/physics/0210005.

Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, John Wiley and Sons, 1991.

Neri Merhav, *Information theory and statistical physics – lecture notes*, arXiv:1006.1565v1, June 2010.

Orly R. Shenker, *Logic and entropy*, http://philsci-archive.pitt.edu/115/, 2001.