Information Theory and Networks

Lecture 22: Error Correction Codes

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Part I

Error Correction Codes

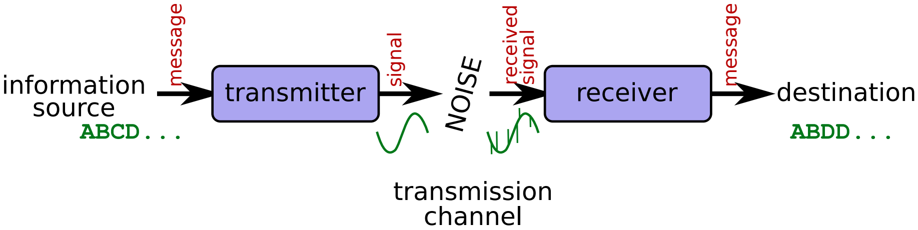
All sorts of computer errors are now turning up. You'd be surprised to know the number of doctors who claim they are treating pregnant men.

Isaac Asimov

Section 1

The Noisy Channel

The basic setup



The setup

- Digital messages could be
  - text
  - audio (digitally coded, e.g., PCM)
  - images (digitally coded, e.g., PNM)
  - video (digitally coded. e.g., MPEG)
  - telemetry (temperature, ...)
  - etc.

  abstract it to be a series of symbols.
- Transmission channel could be
  - a wire (copper or fibre)
  - wireless
  - storage media (transmission over time)

  abstract it with some noise model.

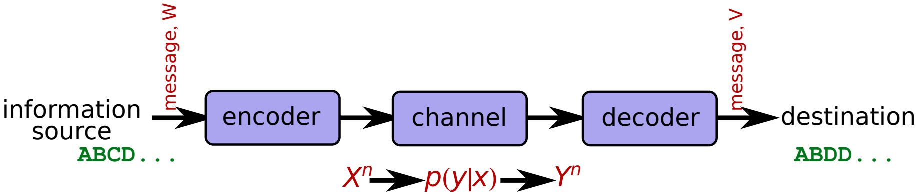
The fundamental questions

Questions: (from Lecture 1)

- Can we have reliable communications?
- How much noise can we tolerate?
- How fast can we transmit? OR How much data can we store?

and how do these three issues interrelate?

Digital Communications Channels



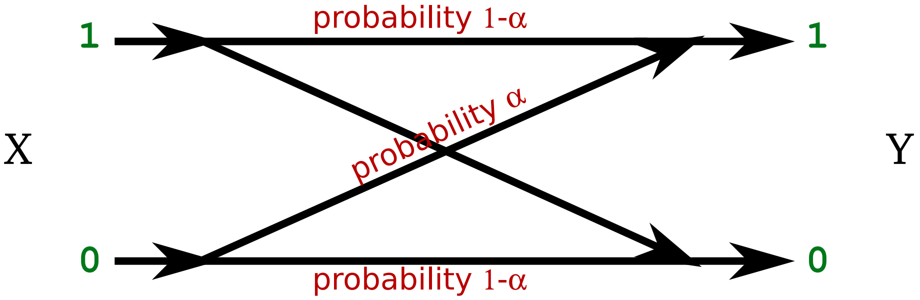
Definition (Discrete Channel)

A discrete channel is a system with an input alphabet $\mathcal{X}$, and output alphabet $\mathcal{Y}$, and a probability transition matrix $p(y|x)$ that describes the probability of observing the output symbol $y \in \mathcal{Y}$ given input $x \in \mathcal{X}$.

Definition

A discrete channel is said to be memoryless if the probability distribution of the output symbols depends only on the current input (and is hence conditionally independent of future and previous inputs or outputs).

Example 1: Binary Symmetric Channel



$$P(Y|X) = \begin{pmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{pmatrix}$$

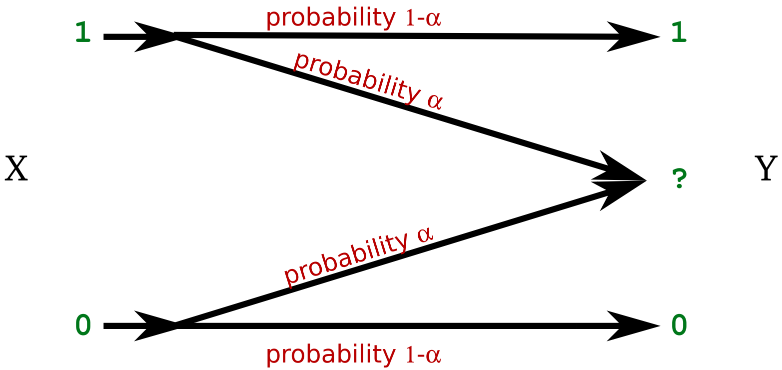
We would say the above channel was noiseless if $\alpha = 0$.

Definition (Noiseless Channel)

If $p(Y|X)$ is the identity then the channel is noiseless.

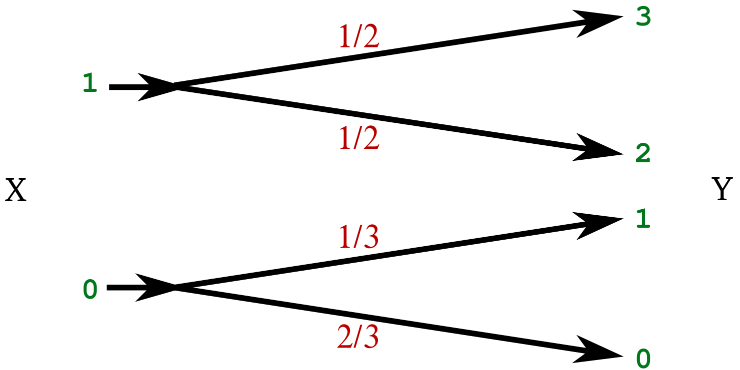
Example 2: Binary Erasure Channel

Some bits are lost (rather than corrupted)



$$P(Y|X) = \begin{pmatrix} 1-\alpha & \alpha & 0 \\ 0 & \alpha & 1-\alpha \end{pmatrix}$$

Example 3: Non-Overlapping Output

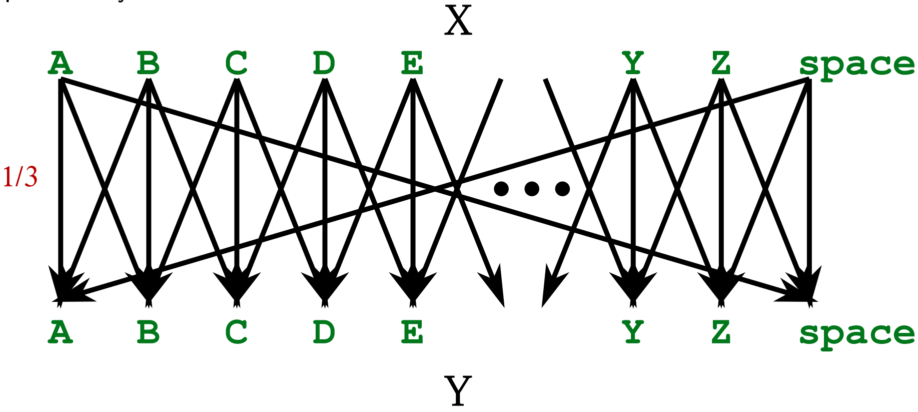


Channel appears to be noisy, but really isn't, as we can exactly determine the input from the output.

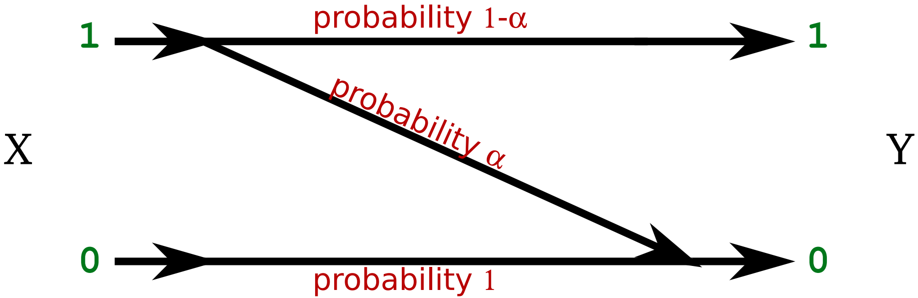
$$P(Y|X) = \begin{pmatrix} 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

Example 4: Noisy Typewriter

$\mathcal{X} = \mathcal{Y} = \{A, B, C, \ldots, Z, space\}$, and we type the correct letter with probability $1/3$, or an adjacent letter on either side, with the same probability.



Example 5: Z Channel



$$P(Y|X) = \begin{pmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{pmatrix}$$

Native ternary codes

- Most codes are actually binary (even if they don't look it)
  - e.g. ASCII makes Letters into binary
  - binary works well with current digital computer systems
  - various attempts at ternary, or other computers rarely seem to work
- One example of native ternary computation is in a TCAM
  - CAM = Content Addressable Memory
    - useful for lots of computation tasks
    - instead of providing, say an array full of data, and having to search it for a particular value (to get the index), you can directly look up the content-value pair in one operation
    - think of as hardware associative array
  - TCAM = Ternary Content Addressable Memory
    - can match contents against 1 or 0 or ? (don't care)
    - useful for routers looking up Internet packet addresses

1?110 matches 11110 or 10110

  - only used for specialised tasks as they are expensive (cost and energy)

Section 2

Channel Capacity

Channel Capacity

Definition (Operational Channel Capacity)

The highest rate of bits we can send per input symbol, with an arbitrarily low probability of error is called the operational channel capacity.

Let's try and work on what this could be.

Channel Capacity: Q0

How much information could we theoretically pump through a channel?

Let's start with units

- Channel capacity C is measured in **bits / input symbol**
  - Remember, with D-ary code, the there are D symbols we can send.
  - With no errors, and fixed length binary codes we have $log_2 D$ bits per symbol
    - e.g., ASCII: there are 256 symbols (the numbers 0-255) with 8 bits per symbol
- Transmission rate T **symbols / second**
- Channel Rate $= C \times T$ **bits / second**
  - commonly there is a tradeoff
  - e.g. the following are equivalent
    - 256 symbols @ 1 per second OR 2 symbols @ 8 per second

Channel Capacity: Q1

Does the input distribution matters?

- think about the Z-Channel example

Channel Capacity: Q1

Does the input distribution matters?

- think about the Z-Channel example
  - if I only send 0's then there will be no errors
  - I should be able to send at a higher rate
  - So we need a model for X – maybe a PMF

Channel Capacity: Q2

What about noiseless coding/compression?

- What might be the best way to improve the rate of information?

Channel Capacity: Q2b

What about noiseless coding/compression?

- What might be the best way to improve the rate of information flow?
  - compress the input before sending it
- What affect does that have on $p_X(x)$?
  - imagine we have (blocks of) original symbols X
  - build code such that $x_1 x_2 \ldots x_n$ has codeword $z_1 z_2 \ldots z_{\ell_k}$
  - now imagine that inputs to the channel are z_i
  - what is $p_Z(z)$?

Huffman Coding Example 1

X	Probability	Codeword $(z_1 z_2 \ldots z_{\ell_k})$	$p(z_i = 0 \mid X = x)$
a	0.25	01	1/2
b	0.25	10	1/2
c	0.2	11	0/2
d	0.15	000	3/3
e	0.15	001	2/3

$$
\begin{aligned}
P(z_i = 0) &= \sum_x p(z_i = 0 \mid X = x) p_X(x) \\
&= \frac{1}{2} p(X = a) + \frac{1}{2} p(X = b) + \frac{0}{2} p(X = c) + \frac{3}{3} p(X = d) + \frac{2}{3} p(
\\
&= 0.5 \\
P(z_i = 1) &= 0.5
\end{aligned}
$$

Channel Capacity: Q2

What about noiseless coding/compression?

- What might be the best way to improve the rate of information flow?
  - compress the input before sending it
  - sort of obvious, because without noise, we are just doing compression
- What affect does that have on $p_X(x)$?
  - imagine we have (blocks of) original symbols X
  - build code such that $x_1 x_2 \ldots x_n$ has codeword $z_1 z_2 \ldots z_{\ell_k}$
  - now imagine that inputs to the channel are z_i
  - what is $p_Z(z)$?
- In general, one of the affects of optimal compression coding is to even out the distribution of symbols.
- You know that this must be true, because otherwise, we might be able to do further compression, so

$$H(Z) \simeq \log_2 D$$

for a close to optimal D-ary code.

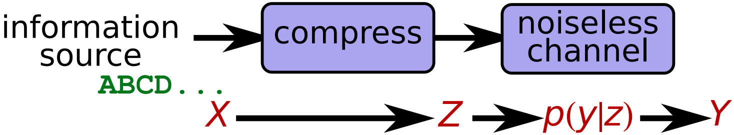
Prefix-free codes, and symbol probability

- We can see this, at least in the dyadic case where optimal binary codeword lengths are
$$\ell_k = -\log_2 p(x_k)$$
- In the dyadic case, the two smallest probabilities must be equal
  - in building the Huffman tree, we sum these to get a new dyadic probability
  - as noted in Huffman proofs, these differ only in the last symbol of the code
  - so WRT to this last digit, there is an equal probability of either
  - the same holds recursively as we build the tree, so at each step, the last digit will be probabilistically balanced
- So for optimal Huffman tree on dyadic probabilities the probability of 0 and 1 in resulting output are exactly balanced
- Obviously this is only approximately true when probabilities aren't dyadic

Channel Capacity: Q3

So for noiseless case, we might think of it as compressing the input. Can we present a more formal case for that?

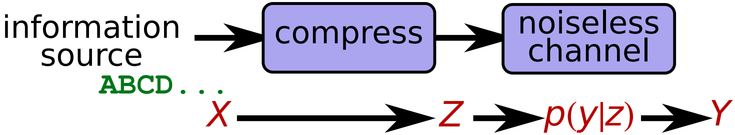
- Assume we compress the input
  - $H(X)$ is entropy of the input, and $H_D(Z) = 1$ is the entropy of the compressed input, which is passed to the channel



- The expected length L of the optimal codewords for a random variable X (for any $\epsilon > 0$ for large enough blocks) satisfies

$$H(X) \leq L < H(X) + \epsilon$$

Channel Capacity: Q3



- Use binary codes for simplicity
  - $Z \in \{0, 1\}$
  - assume channel can send T bits per second
  - entropy $H(Z) \simeq 1$
- Average code length $L \simeq H(X)$
  - can send average $T/L \simeq T/H(X)$ symbols X per second
  - so channel capacity = bits per symbol

$$C = \frac{\text{bits per second}}{\text{symbols per second}} \simeq \frac{T}{T/H(X)} = H(X) \text{ bits per symbol } X$$

Binary Erasure Channel: Q4

In the binary erasure channel, we loose a bit with probability α. What can we do about that?

One approach is to use feedback

- if we don't receive a bit, we retransmit it again until it gets through
- results in a geometric distribution of number of (re)transmissions needed
$$P(N = n) = (1-\alpha)\alpha^{n-1}$$
- average number of transmissions is $1/(1-\alpha)$
- number of bits we can send (on average) with m transmissions is m divided by the number of transmissions per bit $m(1-\alpha)$

$$C = 1 - \alpha \text{ bits per input binary symbol}$$

Channel Capacity: Q5

If we have noise, how would we guess the input symbol from the output?

This is a standard inference problem:

- one approach is maximum *a-posteriori* (MAP) estimation
- given output Y we estimate

$$p(X|Y) = p(Y|X)\frac{P(X)}{P(Y)} = \frac{p(Y|X)P(X)}{\sum_x P(Y|x)P(x)}$$

  by Bayes rule
- So
  - we assume we know $P(Y|X)$
  - we need to have an idea of $P(X)$
  - the bottom line is irrelevant, as it is a constant normalising factor for any given Y

  Choose the x which gives the maximum probability

Example: Binary Symmetric Channel

Take, for example, output $Y = 1$

$$
\begin{aligned}
\operatorname*{argmax}_x p(X = x | Y = 1) &= \operatorname*{argmax}_x p(Y = 1 | X = x) P(X = x) \\
&= \operatorname*{argmax}_x \{\alpha P(X = 0), (1 - \alpha) P(X = 1)\}
\end{aligned}
$$

- assume the errors aren't too big
- assume that the input probabilities aren't too unbalanced

Then this will return $X = 1$, and similarly $X = 0$ when $Y = 0$.

- error rate is α
- with error checking and retransmission, this would look like the binary erasure channel, but the amount of checking creates an overhead, which we need to estimate

Fano's Inequality

Suppose we know a random variable Y and want to guess the value of X

- We want a function $g(Y) = \hat{X}$
- Obviously:
  - Y only helps guess X if $H(X|Y) > 0$
  - When X is completely dependent on Y, its easy to guess

  so we can see the $H(X|Y)$ is important for this problem.
- Fano's inequality formalises the question

Theorem (Fano's Inequality)

Given RVs X and Y related by $p(y|x)$, and an estimate $\hat{X} = g(Y)$, of X based on Y with probability of error $P_e = P(X \neq \hat{X})$ then

$$H(P_e) + P_e \log\left(|\mathcal{X}| - 1\right) \geq H(X|Y)$$

The inequality can be weakened, and rearranged to give

$$P_e \geq \frac{H(X|Y) - 1}{\log|\mathcal{X}|}$$

Channel Capacity: Q6

Is entropy relevant here?

- $H(X)$ is the uncertainty about X before we know the output
  - we know this was important for noiseless channels
- $H(X|Y)$ is the information we gain about X from Y
  - if this is small (e.g. they are nearly completely dependent), then there is little noise, and small errors
  - if this is nearly $= H(X)$ (e.g., they are almost independent), and we learn little about inputs from the outputs
- So why not think of capacity something like?

$$C = H(X) - H(X|Y)$$

which is the amount of information we learn about X by observing Y.

Duality

Duality between data compression and error correction:

- Redundancy in language exists, at least in part, to help correct potential errors
  - redundancy helps because some sequences are impossible or unlikely
  - we can look for a "nearby" one that is more likely
- In compression, we were coding to remove this redundancy
- To do error correction, we need to add back some redundancy
- Later, we will show that the encoder and decoder given above can be separated into two stages:
  - compression
  - error correction coding

Further reading I

Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, John Wiley and Sons, 1991.

David J. MacKay, *Information theory, inference, and learning algorithms*, Cambridge University Press, 2011.