Information Theory and Networks

Lecture 18: Information Theory and the Stock Market

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Part I

The Stock Market

> Put all your eggs in one basket and then watch that basket.
> *Mark Twain, Pudd'nhead Wilson and Other Tales*

Section 1

Basics of the Stock Market

Stock Market

- "Market" referred to is really the *secondary market*
  - *primary market* deals with the issuance of stock
- Consider m assets
  - one asset has the *risk-free rate*: theoretical zero risk
  - our goal: construct a portfolio i.e. allocation of assets with exponential wealth growth
- We assume no
  - Short selling
  - Leveraging

- The risk free rate is the theoretical return with zero risk. In reality, this is often approximated by treasury bonds (although it really depends on the country, e.g. Greece is an exception)
- Short selling refers to the practice of making money from securities that are falling in price. It works as follows: the trader borrows securities from a lender, then immediately sells them off, and buys them back at a later time when the securities are much cheaper than the original sale price, so as to return the securities to the lender. The trader thus profits from the drop in price of the security.
- Leveraging is essentially borrowing money from a lender to invest. The return to the trader is the gain of capital of the security (and associated dividends) minus the interest that has to be paid back to the lender.

Some Definitions

- We look at day-to-day fluctuations of the stock prices
- Stock market $\mathbf{X} = (X_1, X_2, \cdots, X_m)$, $X_i \geq 0$
  - our universe of stocks is m
  - X_i price relative: (price at start of day)/(price at end of day)
  - $F(\mathbf{x})$: underlying distribution of X_is
- The portfolio $\mathbf{b} = (b_1, b_2, \cdots, b_m)$, $b_i \geq 0$, $\sum_{i=1}^{m} b_i = 1$
- The wealth relative $S = \mathbf{b}^T\mathbf{X}$
- Investment period n days results in $S_n = \prod_{i=1}^{n} \mathbf{b}_i^T \mathbf{X}_i$

Section 2

Log-Optimal Portfolios

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Optimising Growth Rate

- Want to maximise $W(\mathbf{b}, F) := E[\log S]$
  - $W^*(F) := \max_{\mathbf{b}} W(\mathbf{b}, F)$
  - portfolio $\mathbf{b}^*$ achieving $W^*(F)$ is the *log-optimal portfolio*
- Suppose price relatives are i.i.d. according to $f(\mathbf{x})$. Assume constant rebalancing with allocation $\mathbf{b}^*$, so $S_n^* = \prod_{i=1}^n \mathbf{b}^{*T}\mathbf{X}_i$. Then,

$$\frac{1}{n}\log S_n^* \to W^* \text{ with probability 1.}$$

- **Implication**: regardless of current wealth, keep allocations between assets constant!
- Can we justify constant rebalancing portfolios beyond i.i.d.? Yes, for stationary markets, conditional allocation

Shannon's Volatility Pumping

- Constant rebalancing portfolio (CRP): suggested by Shannon in a lecture at MIT in the 1960s
- Shannon used geometric Wiener to model the price relatives
- CRPs essentially exploit volatility of the price relatives
  - the higher the price volatilty between assets, the higher the excess returns

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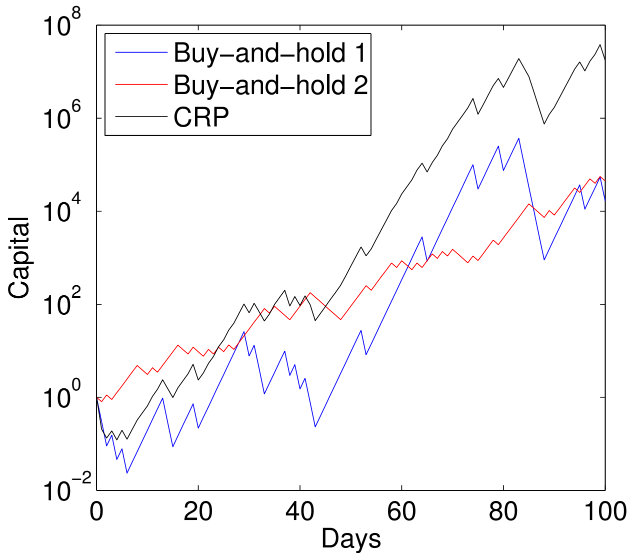
CRP vs. Buy and Hold



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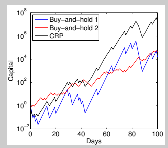

Karush-Kuhn-Tucker Characterisation

- Observe the admissible portfolios form an m-simplex $\mathcal{B}$
- Karush-Kuhn-Tucker (KKT) conditions yield:

$$E\left[\frac{X_i}{\mathbf{b}^{*T}\mathbf{X}}\right] = \begin{cases} 1 & \text{if } b_i^* > 0 \\ 0 & \text{if } b_i^* = 0 \end{cases}$$

- **Implication**: portfolio at least as good as best stock return on average
- KKT conditions also imply:

$$E\left[\log\frac{S}{S^*}\right] \le 0 \text{ for all } S \text{ iff } E\left[\frac{S}{S^*}\right] \le 1 \text{ for all } S.$$

- Also, $E\left[\frac{b_i^* X_i}{\mathbf{b}^{*T}\mathbf{X}}\right] = b_i^* E\left[\frac{X_i}{\mathbf{b}^{*T}\mathbf{X}}\right] = b_i^*$ (c.f. Kelly criterion)

Karush-Kuhn-Tucker Characterisation

Wrong Belief

- In horse racing, side information improves wealth growth rate
- Suppose investor believes underlying distribution is $G(\mathbf{x})$ instead of $F(\mathbf{x})$: what is the impact?
  - end up using allocation $\mathbf{b}_G$ instead of $\mathbf{b}_F$
  - characterise *increase in growth rate*

$$\Delta W = W(\mathbf{b}_F, F) - W(\mathbf{b}_G, G)$$

- Turns out $\Delta W \le D(F\,||\,G)$ (proof: Jensen's inequality and KKT condition)

Wrong Belief

Notice the similarity to the Kelly gambler. The better the estimate of an investor regarding the return distribution of the assets, then better the performance of the portfolio.

Side Information

- Result can be used to show $\Delta W \leq I(\mathbf{X}; Y)$, equality holds if it is the horse race i.e. return due to win or loss
- In real life: private insider trading can significantly increase wealth
  - e.g. buying stock before press release of profit upgrades or sensitive announcement
  - practice is banned in most developed countries
  - insider trading must be declared in public records
- Information asymmetry lead to significant (dis)advantages, not just wealth-wise

Incidentally, looking at when insiders of a company (CEO, CFO, directors, etc.) purchase or sell stocks of their own company can provide signals about the value of the stock; see Chapter 9 of Wesley Gray and Tobias Carlisle, "Quantitative Value", Wiley Finance, 2012. See also Richard Zeckhauser, "Investing in the Unknown and Unknowable", Capitalism and Society, Vol. 1, No. 2, Article 5, 2006 about information asymmetry.

Causality

- Nothing said about causal strategies: in real life, not possible to invest in hindsight
- *Nonanticipating* or *causal* portfolio: sequence of mappings $b_i : \mathbb{R}^{m(i-1)} \to \mathcal{B}$, with the interpretation $b_i(\mathbf{x}_1, \cdots, \mathbf{x}_{i-1})$ used on day i
- Suppose $\mathbf{X}_i$ drawn i.i.d. from $F(\mathbf{x})$, S_n is wealth relative from any causal strategy,

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{S_n}{S_n^*} \leq 0 \text{ with probability 1.}$$

- **Caveat**: theorem does not say for a fixed n, log-optimal portfolio does better than any strategy

Part II

Universal Portfolios

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Universal Portfolios

Background

- Previous discussions assume F is known
- What's the best we can do, if F is not known?
  - use best CRP based on hindsight as benchmark
  - think of something (clever) to approach this benchmark
- Needs to be (somewhat) practical
  - causal strategy
  - universal: distribution free strategy
- **Solution**: adaptive strategy

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Finite Horizon

- Assume n is known in advance, $\mathbf{x}^n = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ is the stock market sequence
- **Theorem**: For any causal strategy $\hat{\mathbf{b}}_i(\cdot)$,

$$\max_{\hat{\mathbf{b}}_i(\cdot)} \min_{\mathbf{x}_1, \cdots, \mathbf{x}_n} \frac{\hat{S}_n(\mathbf{x}^n)}{S_n^*(\mathbf{x}^n)} = V_n$$

- V_n is the normalisation factor, for reasons clearer later on
- Nothing said about the underlying distribution: distribution free!

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Finite Horizon: Big Picture

- **Big Picture**: look at all the outcomes length n, allocate wealth in hindsight, then construct best causal strategy from the optimal
- Has to perform close to optimal under "adversarial" outcomes
  - if $m = 2$, outcomes are $((1,0)^T, (1,0)^T, \cdots, (1,0)^T)$, clearly best hindsight strategy is to allocate only to stock 1
  - without hindsight, might want to "spread" allocation to maximise return, minimise loss
  - $\hat{b} = (1/2, 1/2)$ but will be 2^n away from best strategy, need some form of adaptation

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Finite Horizon: Construction

- By optimality of CRPs, only need to compare the best CRP to the causal strategies
- Consider the case $m = 2$, can generalise from this case
- **Key idea**: convert $S_n(\mathbf{x}^n) = \prod_{i=1}^n \mathbf{b}^T \mathbf{X}_i$ to

$$S_n(\mathbf{x}^n) = \sum_{j^n \in \{1,2\}^n} \prod_{i=1}^n b_{i,j_i} \prod_{i=1}^n x_{i,j_i} = \sum_{j^n \in \{1,2\}^n} w(j^n) x(j^n)$$

- Now, problem is about determining allocation $w(j^n)$ to 2^n stocks

Finite Horizon: Construction II

- With 2 stocks, $w(j^n) = \prod_{i=1}^n b^k (1-b)^{n-k}$, k number of times stock 1 price > stock 2 price
  - what is the optimal allocation b^* for this?
- $\sum_{j^n} w^*(j^n) > 1$ because best CRP has benefit of hindsight: can allocate more to the best sequences
  - causal strategy does not have this hindsight
  - make $\hat{w}(j^n)$ proportional to $w^*(j^n)$ by normalisation (using V_n)
- Then, find the optimal allocation for adversarial sequences
  - what is the best allocation, if at each time step in a sequence, exactly one stock yields non-zero return?
- Putting these two together can show

$$V_n \le \max_{\hat{\mathbf{b}}_i(\cdot)} \min_{\mathbf{x}_1, \cdots, \mathbf{x}_n} \frac{\hat{S}_n(\mathbf{x}^n)}{S_n^*(\mathbf{x}^n)} \le V_n$$

The normalisation factor V_n in [CT91, Theorem 16.7.1] differs from the one in the proof in two ways: 1. the proof V_n is for case $m = 2$; 2. the one in the theorem has been approximated using the asymptotic equipartition property.

Finite Horizon: Sequential

- Finally, need to convert back to the causal portfolio mapping
- For allocation to stock 1 at day i, sum over all sequences with 1 in position i

$$\hat{\mathbf{b}}_{i,1}(\mathbf{x}^{i-1}) = \frac{\sum_{j^{i-1} \in m^{i-1}} \hat{w}(j^{i-1}) x(j^{i-1})}{\sum_{j^i \in m^i} \hat{w}(j^i) x(j^{i-1})}$$

- Algorithm enumerates over all m^n sequences: computationally prohibitive
- Asymptotics yield, for $m = 2$ and all n, $\frac{1}{2\sqrt{n+1}} \leq V_n \leq \frac{2}{\sqrt{n+1}}$
- Observe:

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\hat{S}_n(\mathbf{x}^n)}{S_n^*(\mathbf{x}^n)} = \lim_{n\to\infty} \frac{1}{n} \log V_n = 0$$

  for any $\mathbf{x}^n$

Finite Horizon: Sequential
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Observe:
for any $\mathbf{x}^n$

$w(j^i)$ is the weight placed on all sequences j^n (the full sequence) that start with j^i, while $x(j^{i-1})$ is the corresponding return of those sequences.

Horizon-Free

- Two tier process: think of all CRPs with various $\mathbf{b}$ as mutual funds
- Now, we allocate our wealth according to a distribution $\mu(\mathbf{b})$ to all these funds
  - each fund gets $d\mu(\mathbf{b})$ of wealth
  - some will perform better than others, one is the best CRP in hindsight
- What kind of distribution should one choose? (Hint: think adversarial)

Horizon-Free
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What kind of distribution should one choose? (Hint: think adversarial)

Horizon-Free

- **Idea**: Choose a distribution $\mu(\mathbf{b})$ that spreads over all CRPs to maximise

$$\hat{S}(\mathbf{x}^n) = \int_{\mathcal{B}} S_n(\mathbf{b}, \mathbf{x}^n) d\mu(\mathbf{b})$$

- Choose allocation $\hat{\mathbf{b}}_{i+1}(\mathbf{x}^i) = \frac{\int_{\mathcal{B}} \mathbf{b} S_i(\mathbf{b}, \mathbf{x}^i) d\mu(\mathbf{b})}{\int_{\mathcal{B}} S_i(\mathbf{b}, \mathbf{x}^i) d\mu(\mathbf{b})}$
  - interpretation: numerator is weighted performance of the fund, denominator is total wealth
  - best performing CRP dominates overall, especially as $n \to \infty$
- Allocation results in

$$\frac{\hat{S}_n(\mathbf{x}^n)}{S_n^*(\mathbf{x}^n)} \geq \min_{j^n} \frac{\int_{\mathcal{B}} \prod_{i=1}^n b_{j_i} d\mu(\mathbf{b})}{\prod_{i=1}^n b_{j_i}^*}$$

- With the right distribution, for e.g. the Dirichlet$(\frac{1}{2}, \frac{1}{2})$ for $m = 2$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\hat{S}_n(\mathbf{x}^n)}{S_n^*(\mathbf{x}^n)} = 0$$

Caveats

- There is no assumption on brokerage fees
  - in real life, a commission is charged by the broker for any trade
  - CRP relies on daily(!) rebalancing for best performance
- Optimal for a long enough investment horizon
- Relies on the volatility between stocks
  - simulations show that it performs poorly otherwise
  - need the daily rebalancing to exploit volatility
  - longer horizons such as a month or year less volatile (in general)
- Computationally impractical
  - finite horizon: need to evaluate over all possible m^i sequences on day i, combinatorial explosion
  - horizon free: need to work out the integral of returns over the simplex $\mathcal{B}$

Further reading I

Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, John Wiley and Sons, 1991.