An Algebraic Approach to Internet Routing
Day 1

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22 June, 2011
Definition (Semigroup)

A semigroup \((S, \oplus)\) is a non-empty set \(S\) with a binary operation such that

\[
\text{ASSOCIATIVE} : \quad a \oplus (b \oplus c) = (a \oplus b) \oplus c
\]

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\oplus)</th>
<th>() where</th>
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</thead>
<tbody>
<tr>
<td>(\mathbb{N}^\infty)</td>
<td>(\min)</td>
<td>((abc \circ de = abcde))</td>
</tr>
<tr>
<td>(\mathbb{N}^\infty)</td>
<td>(\max)</td>
<td>((a \text{ left } b = a))</td>
</tr>
<tr>
<td>(2^W)</td>
<td>(\cup)</td>
<td>((a \text{ right } b = b))</td>
</tr>
<tr>
<td>(2^W)</td>
<td>(\cap)</td>
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<tr>
<td>(S^*)</td>
<td>(\circ)</td>
<td></td>
</tr>
<tr>
<td>(S)</td>
<td>(\text{left})</td>
<td></td>
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<tr>
<td>(S)</td>
<td>(\text{right})</td>
<td></td>
</tr>
</tbody>
</table>
Special Elements

**Definition**

- \( \alpha \in S \) is an **identity** if for all \( a \in S \)

\[
\alpha \oplus a = a = a \oplus \alpha
\]

- A semigroup is a **monoid** if it has an identity.

- \( \omega \) is an **annihilator** if for all \( a \in S \)

\[
\omega = \omega \oplus a = a \oplus \omega
\]

<table>
<thead>
<tr>
<th>S</th>
<th>( \oplus )</th>
<th>( \alpha )</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{N}^\infty )</td>
<td>min</td>
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<td>0</td>
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<tr>
<td>( 2^W )</td>
<td>+</td>
<td>0</td>
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</tr>
<tr>
<td>( 2^W )</td>
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<tr>
<td>( S^* )</td>
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<tr>
<td>( S )</td>
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</tbody>
</table>

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## Important Properties

**Definition (Some Important Semigroup Properties)**

- **COMMUTATIVE**: \( a \oplus b = b \oplus a \)
- **SELECTIVE**: \( a \oplus b \in \{ a, b \} \)
- **IDEMPOTENT**: \( a \oplus a = a \)

<table>
<thead>
<tr>
<th>( S )</th>
<th>( \oplus )</th>
<th>COMMUTATIVE</th>
<th>SELECTIVE</th>
<th>IDEMPOTENT</th>
</tr>
</thead>
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<tr>
<td>( \mathbb{N}_\infty )</td>
<td>\text{min}</td>
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<td>( \mathbb{N}_\infty )</td>
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<td>( \mathbb{N}_\infty )</td>
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<tr>
<td>( 2^W )</td>
<td>( \cup )</td>
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<td>( 2^W )</td>
<td>( \cap )</td>
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<tr>
<td>( S^* )</td>
<td>( \circ )</td>
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<tr>
<td>( S )</td>
<td>\text{left}</td>
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<td>( S )</td>
<td>\text{right}</td>
<td></td>
<td>∗</td>
<td>∗</td>
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</tbody>
</table>

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Order Relations

We are interested in order relations $\leq \subseteq S \times S$

**Definition (Important Order Properties)**

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>REFLEXIVE</td>
<td>$a \leq a$</td>
</tr>
<tr>
<td>TRANSITIVE</td>
<td>$a \leq b \land b \leq c \rightarrow a \leq c$</td>
</tr>
<tr>
<td>ANTISYMMETRIC</td>
<td>$a \leq b \land b \leq a \rightarrow a = b$</td>
</tr>
<tr>
<td>TOTAL</td>
<td>$a \leq b \lor b \leq a$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>pre-order</th>
<th>partial order</th>
<th>preference order</th>
<th>total order</th>
</tr>
</thead>
<tbody>
<tr>
<td>REFLEXIVE</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>TRANSITIVE</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>ANTISYMMETRIC</td>
<td>*</td>
<td></td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td></td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>
Suppose $\oplus$ is commutative.

**Definition (Canonical pre-orders)**

- $a \trianglelefteq^R b \equiv \exists c \in S : b = a \oplus c$
- $a \trianglelefteq^L b \equiv \exists c \in S : a = b \oplus c$

**Lemma ( Sanity check)**

**Associativity of $\oplus$ implies that these relations are transitive.**

**Proof.**

Note that $a \trianglelefteq^R b$ means $\exists c_1 \in S : b = a \oplus c_1$, and $b \trianglelefteq^R c$ means $\exists c_2 \in S : c = b \oplus c_2$. Letting $c_3 = c_1 \oplus c_2$ we have $c = b \oplus c_2 = (a \oplus c_1) \oplus c_2 = a \oplus (c_1 \oplus c_2) = a \oplus c_3$. That is, $\exists c_3 \in S : c = a \oplus c_3$, so $a \trianglelefteq^R c$. The proof for $\trianglelefteq^L$ is similar.
Definition (Canonically Ordered Semigroup)

A commutative semigroup \((S, \oplus)\) is canonically ordered when \(a \preceq_R c\) and \(a \preceq_L c\) are partial orders.

Definition (Groups)

A monoid is a group if for every \(a \in S\) there exists a \(a^{-1} \in S\) such that \(a \oplus a^{-1} = a^{-1} \oplus a = \alpha\).
Canonically Ordered Semigroups vs. Groups [Car79, GM08]

Lemma (THE BIG DIVIDE)

Only a trivial group is canonically ordered.

Proof.

If \( a, b \in S \), then 
\[
a = \alpha \oplus a = (b \oplus b^{-1}) \oplus a = b \oplus (b^{-1} \oplus a) = b \oplus c,
\]
for \( c = b^{-1} \oplus a \), so \( a \leq_L b \). In a similar way, \( b \leq_R a \). Therefore \( a = b \).
Natural Orders

Definition (Natural orders)

Let \((S, \oplus)\) be a semigroup.

\[
\begin{align*}
a \leq_L b & \equiv a = a \oplus b \\
a \leq_R b & \equiv b = a \oplus b
\end{align*}
\]

Lemma

If \(\oplus\) is commutative and idempotent, then \(a \leq_D b \iff a \leq_D b\), for \(D \in \{R, L\}\).

Proof.

\[
\begin{align*}
a \leq_R b & \iff b = a \oplus c = (a \oplus a) \oplus c = a \oplus (a \oplus c) \\
& = a \oplus b \iff a \leq_R b \\

a \leq_L b & \iff a = b \oplus c = (b \oplus b) \oplus c = b \oplus (b \oplus c) \\
& = b \oplus a = a \oplus b \iff a \leq_L b
\end{align*}
\]
Special elements and natural orders

Lemma (Natural Bounds)

- If $\alpha$ exists, then for all $a$, $a \leq^{L} \alpha$ and $\alpha \leq^{R} a$
- If $\omega$ exists, then for all $a$, $\omega \leq^{L} a$ and $a \leq^{R} \omega$
- If $\alpha$ and $\omega$ exist, then $S$ is bounded.

\[ \omega \leq^{L} a \leq^{L} \alpha \]
\[ \alpha \leq^{R} a \leq^{R} \omega \]

Remark (Thanks to Iljitsch van Beijnum)

Note that this means for $(\min, +)$ we have

\[ 0 \leq_{\min}^{L} a \leq_{\min}^{L} \infty \]
\[ \infty \leq_{\min}^{R} a \leq_{\min}^{R} 0 \]

and still say that this is bounded, even though one might argue with the terminology!
Examples of special elements

<table>
<thead>
<tr>
<th>$S$</th>
<th>$\oplus$</th>
<th>$\alpha$</th>
<th>$\omega$</th>
<th>$\leq^L$</th>
<th>$\leq^R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{N} \cup {\infty}$</td>
<td>min</td>
<td>$\infty$</td>
<td>0</td>
<td>$&lt;$</td>
<td>$&lt;$</td>
</tr>
<tr>
<td>$\mathbb{N} \cup {\infty}$</td>
<td>max</td>
<td>0</td>
<td>$\infty$</td>
<td>$&lt;$</td>
<td>$&lt;$</td>
</tr>
<tr>
<td>$\mathcal{P}(W)$</td>
<td>$\cup$</td>
<td>$\emptyset$</td>
<td>$W$</td>
<td>$\subseteq$</td>
<td>$\subseteq$</td>
</tr>
<tr>
<td>$\mathcal{P}(W)$</td>
<td>$\cap$</td>
<td>$W$</td>
<td>$\emptyset$</td>
<td>$\subseteq$</td>
<td>$\subseteq$</td>
</tr>
</tbody>
</table>
Let $D \in \{R, L\}$.

1. IDEMPOTENT $((S, \oplus)) \iff \text{REFLEXIVE}((S, \leq_D))$
2. COMMUTATIVE $((S, \oplus)) \implies \text{ANTISYMMETRIC}((S, \leq_D))$
3. SELECTIVE $((S, \oplus)) \iff \text{TOTAL}((S, \leq_D))$

**Proof.**

1. $a \leq_D a \iff a = a \oplus a$,
2. $a \leq_L b \land b \leq_D a \iff a = a \oplus b \land b = b \oplus a \implies a = b$
3. $a = a \oplus b \lor b = a \oplus b \iff a \leq_D b \lor b \leq_D a$
Direct Product of Semigroups

Let \((S, \oplus_S)\) and \((T, \oplus_T)\) be semigroups.

**Definition (Direct product semigroup)**

The direct product is denoted \((S, \oplus_S) \times (T, \oplus_T) = (S \times T, \oplus)\), where
\[
\oplus = \oplus_S \times \oplus_T
\]
is defined as

\[
(s_1, t_1) \oplus (s_2, t_2) = (s_1 \oplus_S s_2, t_1 \oplus_T t_2).
\]
Lexicographic Product of Semigroups

Definition (Lexicographic product semigroup (from [Gur08]))

Suppose $S$ is commutative idempotent semigroup and $T$ be a monoid. The lexicographic product is denoted $(S, \oplus_S) \times (T, \oplus_T) = (S \times T, \oplus)$, where $\oplus = \oplus_S \times \oplus_T$ is defined as

$$(s_1, t_1) \oplus (s_2, t_2) = \begin{cases} 
(s_1 \oplus_S s_2, t_1 \oplus_T t_2) & s_1 = s_1 \oplus_S s_2 = s_2 \\
(s_1 \oplus_S s_2, t_1) & s_1 = s_1 \oplus_S s_2 \neq s_2 \\
(s_1 \oplus_S s_2, t_2) & s_1 \neq s_1 \oplus_S s_2 = s_2 \\
(s_1 \oplus_S s_2, \bar{0}_T) & \text{otherwise}.
\end{cases}$$
Semirings

\[(S, \oplus, \otimes, 0, 1)\] is a semiring when

- \((S, \oplus, 0)\) is a commutative monoid
- \((S, \otimes, 1)\) is a monoid
- \(0\) is an annihilator for \(\otimes\)

and distributivity holds,

\[
\begin{align*}
\text{LD} & : a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \\
\text{RD} & : (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)
\end{align*}
\]
A few examples

<table>
<thead>
<tr>
<th>name</th>
<th>$S$</th>
<th>$\ominus$,</th>
<th>$\otimes$</th>
<th>$\overline{0}$</th>
<th>$\overline{1}$</th>
<th>possible routing use</th>
</tr>
</thead>
<tbody>
<tr>
<td>sp</td>
<td>$\mathbb{N}^\infty$</td>
<td>min</td>
<td>+</td>
<td>$\infty$</td>
<td>0</td>
<td>minimum-weight routing</td>
</tr>
<tr>
<td>bw</td>
<td>$\mathbb{N}^\infty$</td>
<td>max</td>
<td>min</td>
<td>0</td>
<td>$\infty$</td>
<td>greatest-capacity routing</td>
</tr>
<tr>
<td>rel</td>
<td>$[0, 1]$</td>
<td>max</td>
<td>$\times$</td>
<td>0</td>
<td>1</td>
<td>most-reliable routing</td>
</tr>
<tr>
<td>use</td>
<td>${0, 1}$</td>
<td>max</td>
<td>min</td>
<td>0</td>
<td>1</td>
<td>usable-path routing</td>
</tr>
<tr>
<td></td>
<td>$2^W$</td>
<td>$\cup$</td>
<td>$\cap$</td>
<td>$\emptyset$</td>
<td>$W$</td>
<td>shared link attributes?</td>
</tr>
<tr>
<td></td>
<td>$2^W$</td>
<td>$\cap$</td>
<td>$\cup$</td>
<td>$W$</td>
<td>$\emptyset$</td>
<td>shared path attributes?</td>
</tr>
</tbody>
</table>
Encoding path problems

- \((S, \oplus, \otimes, 0, 1)\) a semiring
- \(G = (V, E)\) a directed graph
- \(w \in E \rightarrow S\) a weight function

Path weight

The weight of a path \(p = i_1, i_2, i_3, \cdots, i_k\) is

\[
w(p) = w(i_1, i_2) \otimes w(i_2, i_3) \otimes \cdots \otimes w(i_{k-1}, i_k).
\]

The empty path is given the weight \(1\).

Adjacency matrix \(A\)

\[
A(i, j) = \begin{cases} 
  w(i, j) & \text{if } (i, j) \in E, \\
  0 & \text{otherwise}
\end{cases}
\]
The general problem of finding globally optimal paths

Given an adjacency matrix $A$, find $R$ such that for all $i, j \in V$

$$R(i, j) = \bigoplus_{p \in P(i, j)} w(p)$$

How can we solve this problem?
Powers and closure

- $(S, \oplus, \otimes, \bar{0}, \bar{1})$ a semiring

### Powers, $a^k$

\[
a^0 = \bar{1} \\
 a^{k+1} = a \otimes a^k
\]

### Closure, $a^*$

\[
 a^{(k)} = a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \\
 a^* = a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \oplus \cdots
\]

### Fun Facts [Con71]

\[
 (a^*)^* = a^* \\
 (a \oplus b)^* = (a^* b)^* a^* \\
 (ab)^* = \bar{1} \oplus a(ba)^* b
\]
Stability

Definition (q stability)
If there exists a $q$ such that $a^{(q)} = a^{(q+1)}$, then $a$ is $q$-stable. Therefore, $a^* = a^{(q)}$, assuming $\oplus$ is idempotent.

Fact 1
If $\bar{1}$ is an annihilator for $\oplus$, then every $a \in S$ is 0-stable!
Lift semiring to matrices

- \((S, \oplus, \otimes, \bar{0}, \bar{1})\) a semiring
- Define the semiring of \(n \times n\)-matrices over \(S\) : \((\mathbb{M}_n(S), \oplus, \otimes, J, I)\)

\(\oplus\) and \(\otimes\)

\[
(A \oplus B)(i, j) = A(i, j) \oplus B(i, j)
\]

\[
(A \otimes B)(i, j) = \bigoplus_{1 \leq q \leq n} A(i, q) \otimes B(q, j)
\]

\(J\) and \(I\)

\[
J(i, j) = \bar{0}
\]

\[
I(i, j) = \begin{cases} 
\bar{1} & \text{(if } i = j) \\
\bar{0} & \text{(otherwise)} 
\end{cases}
\]
$M_n(S)$ is a semiring!

Check (left) distribution

$$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$$

$$(A \otimes (B \oplus C))(i, j)$$

$$= \bigoplus_{1 \leq q \leq n} A(i, q) \otimes (B \oplus C)(q, j)$$

$$= \bigoplus_{1 \leq q \leq n} A(i, q) \otimes (B(q, j) \oplus C(q, j))$$

$$= \bigoplus_{1 \leq q \leq n} (A(i, q) \otimes B(q, j)) \oplus (A(i, q) \otimes C(q, j))$$

$$= ((A \otimes B) \oplus (A \otimes C))(i, j)$$
On the matrix semiring

Matrix powers, $A^k$

$$A^0 = I$$

$$A^{k+1} = A \otimes A^k$$

Closure, $A^*$

$$A^{(k)} = I \oplus A^1 \oplus A^2 \oplus \ldots \oplus A^k$$

$$A^* = I \oplus A^1 \oplus A^2 \oplus \ldots \oplus A^k \oplus \ldots$$

Note: $A^*$ might not exist (sum may not converge)
Fact 2

If $S$ is 0-stable, then $\mathbb{M}_n(S)$ is $(n-1)$-stable. That is,

$$A^* = A^{(n-1)} = I \oplus A^1 \oplus A^2 \oplus \cdots \oplus A^{n-1}$$
Computing optimal paths

- Let \( P(i, j) \) be the set of paths from \( i \) to \( j \).
- Let \( P^k(i, j) \) be the set of paths from \( i \) to \( j \) with exactly \( k \) arcs.
- Let \( P^{(k)}(i, j) \) be the set of paths from \( i \) to \( j \) with at most \( k \) arcs.

**Theorem**

\[
\begin{align*}
(1) \quad & A^k(i, j) = \bigoplus_{p \in P^k(i, j)} w(p) \\
(2) \quad & A^{(k+1)}(i, j) = \bigoplus_{p \in P^{(k)}(i, j)} w(p) \\
(3) \quad & A^*(i, j) = \bigoplus_{p \in P(i, j)} w(p)
\end{align*}
\]
Proof of (1)

By induction on $k$. Base Case: $k = 0$.

\[ P^0(i, i) = \{ \epsilon \}, \]

so \( A^0(i, i) = 1 = I(i, i) = 1 = w(\epsilon) \).

And $i \neq j$ implies \( P^0(i, j) = \{ \} \). By convention

\[ \bigoplus_{p \in \{ \}} w(p) = 0 = I(i, j). \]
Proof of (1)

Induction step.

\[
A^{k+1}(i,j) = (A \otimes A^k)(i,j)
\]

\[
= \bigoplus_{1 \leq q \leq n} A(i, q) \otimes A^k(q, j)
\]

\[
= \bigoplus_{1 \leq q \leq n} A(i, q) \otimes \bigoplus_{p \in P^k(q, j)} w(p)
\]

\[
= \bigoplus_{1 \leq q \leq n} \bigoplus_{p \in P^k(q, j)} A(i, q) \otimes w(p)
\]

\[
= \bigoplus_{(i, q) \in E} \bigoplus_{p \in P^k(q,j)} w(i, q) \otimes w(p)
\]

\[
= \bigoplus_{p \in P^{k+1}(i, j)} w(p)
\]
Example with \((2\{a, b, c\}, \cap, \cup)\)

We want matrix \(A^*\) to solve this global optimality problem:

\[
A^*(i, j) = \bigcap_{p \in P(i, j)} w(p),
\]

where \(w(p)\) is now the union of all edge weights in \(p\).

For \(x \in \{a, b, c\}\), interpret \(x \in A^*(i, j)\) to mean that every path from \(i\) to \(j\) has at least one arc with weight containing \(x\).
\((2\{a, b, c\}, \cap, \cup)\) continued ...

The matrix \(A^*\)

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & \{\} & \{\} & \{b\} & \{b\} & \{\} \\
2 & \{\} & \{\} & \{b\} & \{b\} & \{\} \\
3 & \{b\} & \{b\} & \{\} & \{b\} & \{b\} \\
4 & \{b\} & \{b\} & \{b\} & \{\} & \{b\} \\
5 & \{\} & \{\} & \{b\} & \{b\} & \{\} \\
\end{bmatrix}
\]
Partition Equation (left)

\[ X = (AX) \oplus I \]

\[
\begin{pmatrix}
  X_{1,1} & X_{1,2} \\
  X_{2,1} & X_{2,2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  (A_{1,1}X_{1,1}) \oplus (A_{1,2}X_{2,1}) \oplus I_{1,1} & (A_{1,1}X_{1,2}) \oplus (A_{1,2}X_{2,2}) \\
  (A_{2,1}X_{1,1}) \oplus (A_{2,2}X_{2,1}) & (A_{2,1}X_{1,2}) \oplus (A_{2,2}X_{2,2}) \oplus I_{2,2}
\end{pmatrix}
\]
We now have four (left) equations

\[
\begin{align*}
X_{1,1} &= (A_{1,1}X_{1,1}) \oplus (A_{1,2}X_{2,1}) \oplus I_{1,1} \\
X_{2,1} &= (A_{2,1}X_{1,1}) \oplus (A_{2,2}X_{2,1}) \\
X_{1,2} &= (A_{1,1}X_{1,2}) \oplus (A_{1,2}X_{2,2}) \\
X_{2,2} &= (A_{2,1}X_{1,2}) \oplus (A_{2,2}X_{2,2}) \oplus I_{2,2}
\end{align*}
\]

- Solve for \(X_{2,1}\) with \(A_{2,2}^{-1}A_{2,1}X_{1,1}\)
- Therefore

\[
X_{1,1} = (A_{1,1}X_{1,1}) \oplus (A_{1,2}A_{2,2}^{-1}A_{2,1}X_{1,1}) \oplus I_{1,1} \\
X_{1,1} = (A_{1,1} \oplus A_{1,2}A_{2,2}^{-1}A_{2,1})X_{1,1} \oplus I_{1,1}
\]

- Solve for \(X_{1,1}\) with \((A_{1,1} \oplus A_{1,2}A_{2,2}^{-1}A_{2,1})^*\)
- So \(X_{2,1}\) is solved with \(A_{2,2}^{-1}A_{2,1}(A_{1,1} \oplus A_{1,2}A_{2,2}^{-1}A_{2,1})^*\)
- In a similar way, solve for \(X_{1,2}\) and \(X_{2,2}\)
This gives a partition of $A^*$ [Con71]

\[
\begin{pmatrix}
\frac{(A_{1,1} \oplus A_{1,2}A^*_{2,2}A_{2,1})^*}{A^*_{2,2}A_{2,1}(A_{1,1} \oplus A_{1,2}A^*_{2,2}A_{2,1})^*} & A^*_{1,1}A_{1,2}(A_{2,2} \oplus A_{2,1}A^*_{1,1}A_{1,2})^* \\
\end{pmatrix}
\]
Trivial example of forwarding = routing + mapping

<table>
<thead>
<tr>
<th>matrix</th>
<th>solves</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^*$</td>
<td>$R = (A \otimes R) \oplus I$</td>
</tr>
<tr>
<td>$A^*M$</td>
<td>$F = (A \otimes F) \oplus M$</td>
</tr>
</tbody>
</table>

$M = \begin{bmatrix} d_1 & d_2 \\ 1 & \infty & \infty \\ 2 & 3 & \infty \\ 3 & \infty & \infty \\ 4 & \infty & 1 \\ 5 & 2 & 3 \end{bmatrix}$

Forwarding matrix

$F = \begin{bmatrix} d_1 & d_2 \\ 1 & 5 & 6 \\ 2 & 3 & 7 \\ 3 & 5 & 5 \\ 4 & 9 & 1 \\ 5 & 2 & 3 \end{bmatrix}$

Mapping matrix
Routing Matrix vs. Forwarding Matrix (see [BG09])

- Inspired by the Locator/ID split work
  - See Locator/ID Separation Protocol (LISP)
- Let's make a distinction between infrastructure nodes $V$ and destinations $D$.
- Assume $V \cap D = \{\}$
- $\mathbf{M}$ is a $V \times D$ mapping matrix
  - $\mathbf{M}(v, d) \neq \infty$ means that destination (identifier) $d$ is somehow attached to node (locator) $v$
More Interesting Example: Hot-Potato Idiom

Mapping matrix:

\[
M = \begin{bmatrix}
\infty & \infty \\
(0, 3) & \infty \\
\infty & \infty \\
\infty & (0, 1) \\
(0, 2) & (0, 3)
\end{bmatrix}
\]

Forwarding matrix:

\[
F = \begin{bmatrix}
(2, 3) & (4, 3) \\
(0, 3) & (4, 3) \\
(3, 2) & (3, 3) \\
(7, 2) & (0, 1) \\
(0, 2) & (0, 3)
\end{bmatrix}
\]
General Case

\[ G = (V, E), \ n \text{ is the size of } V. \]

A \( n \times n \) (left) **routing matrix** \( L \) solves an equation of the form

\[ L = (A \otimes L) \oplus I, \]

over semiring \( S \).

\( D \) is a set of destinations, with size \( d \).

A \( n \times d \) **forwarding matrix** is defined as

\[ F = L \triangleright M, \]

over some structure \((N, \square, \triangleright)\), where \( \triangleright \in (S \times N) \to N \).
forwarding = routing + mapping

Does this make sense?

\[ F(i, d') = (L \triangleright M)(i, d') = \sum_{q \in V} L(i, q) \triangleright M(q, d'). \]

- Once again we are leaving paths implicit in the construction.
- Forwarding paths are best routing paths to egress nodes, selected with respect \(\square\)-minimality.
- \(\square\)-minimality can be very different from selection involved in routing.
When we are lucky ...

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</tr>
<tr>
<td>$A^* \triangleright M$</td>
<td>$F = (A \triangleright F) \Box M$</td>
</tr>
</tbody>
</table>

When does this happen?

When $(N, \Box, \triangleright)$ is a (left) semi-module over the semiring $S$. 
(left) Semi-modules

- \((S, \oplus, \otimes, 0, 1)\) is a semiring.

A (left) semi-module over \(S\)

Is a structure \((N, □, \triangleright, 0_N)\), where

- \((N, □, 0_N)\) is a commutative monoid
- \(\triangleright\) is a function \(\triangleright \in (S \times N) \rightarrow N\)
- \((a \otimes b) \triangleright m = a \triangleright (b \triangleright m)\)
- \(0 \triangleright m = 0_N\)
- \(s \triangleright 0_N = 0_N\)
- \(\overline{1} \triangleright m = m\)

and distributivity holds,

\[
\begin{align*}
LD : \quad s \triangleright (m \square n) &= (s \triangleright m) \square (s \triangleright n) \\
RD : \quad (s \oplus t) \triangleright m &= (s \triangleright m) \square (t \triangleright m)
\end{align*}
\]
Example: Hot-Potato

$S$ idempotent and selective

\[
S = (S, \oplus_S, \otimes_S) \\
T = (T, \oplus_T, \otimes_T) \\
\triangleright_{\text{fst}} \in S \times (S \times T) \rightarrow (S \times T) \\
S_1 \triangleright_{\text{fst}} (S_2, t) = (S_1 \otimes_S S_2, t)
\]

Hot($S$, $T$) = ($S \times T$, $\vec{\oplus}$, $\triangleright_{\text{fst}}$),

where $\vec{\oplus}$ is the (left) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\triangleright_{\text{hp}}$ on matrices

\[
(L \triangleright_{\text{hp}} M)(i, d) = \sum_{q \in V} L(i, q) \triangleright_{\text{fst}} M(q, d)
\]
Sanity Check: does this implement hot-potato?

Define $M$ to be **simple** if either $M(v, d) = (1_S, t)$ or $M(v, d) = (\infty_S, \infty_T)$.

$$
(L \triangleright_{hp} M) (i, d) = \sum_{q \in V} (L(i, q) \triangleright_{fst} M(q, d))
$$

$$
= \sum_{q \in V} (L(i, q) \otimes_S s, t)
$$

$$
= \sum_{q \in V} (L(i, q), t) \quad \text{(if } M \text{ is simple})
$$
Example of *hot-potato* forwarding

\[
\begin{bmatrix}
  d_1 & d_2 \\
  1 & \infty & \infty \\
  2 & (0, 3) & \infty \\
  3 & \infty & \infty \\
  4 & \infty & (0, 1) \\
  5 & (0, 2) & (0, 3)
\end{bmatrix}
\]

**Mapping matrix**

\[
\begin{bmatrix}
  d_1 & d_2 \\
  1 & (2, 3) & (4, 3) \\
  2 & (0, 3) & (4, 3) \\
  3 & (3, 2) & (3, 3) \\
  4 & (7, 2) & (0, 1) \\
  5 & (0, 2) & (0, 3)
\end{bmatrix}
\]

**Forwarding matrix**

\[
\begin{bmatrix}
  d_1 & d_2 \\
  1 & \infty & \infty \\
  2 & (0, 3) & \infty \\
  3 & \infty & \infty \\
  4 & \infty & (0, 1) \\
  5 & (0, 2) & (0, 3)
\end{bmatrix}
\]

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</tr>
<tr>
<td>(A^* \triangleright_{hp} M)</td>
<td>(F = (A \triangleright_{hp} F) \oplus M)</td>
</tr>
</tbody>
</table>
Example : Cold-Potato

$T$ idempotent and selective

\[
\begin{align*}
S &= (S, \oplus_S, \otimes_S) \\
T &= (T, \oplus_T, \otimes_T) \\
\triangleright_{fst} &\in S \times (S \times T) \to (S \times T) \\
s_1 \triangleright_{fst} (s_2, t) &= (s_1 \otimes_S s_2, t)
\end{align*}
\]

\[
\text{Cold}(S, T) = (S \times T, \tilde{\oplus}, \triangleright_{fst}),
\]

where $\tilde{\oplus}$ is the (left) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\triangleright_{cp}$ on matrices

\[
(L \triangleright_{cp} M)(i, d) = \sum_{q \in V} L(i, q) \triangleright_{fst} M(q, d)
\]
Example of cold-potato forwarding

\[
\begin{bmatrix}
  d_1 & d_2 \\
  1 & \infty & \infty \\
  2 & (0, 3) & \infty \\
  3 & \infty & \infty \\
  4 & \infty & (0, 1) \\
  5 & (0, 2) & (0, 3)
\end{bmatrix}
\]

Matrix solves

\[
A^* \
A^* \triangleright_{\text{cp}} M
\]

L = (A \otimes L) \oplus I

\[
F = A \triangleright_{\text{cp}} F \tilde{\oplus} M
\]

Mapping matrix

\[
F =
\begin{bmatrix}
  d_1 & d_2 \\
  1 & (4, 2) & (5, 1) \\
  2 & (4, 2) & (9, 1) \\
  3 & (3, 2) & (4, 1) \\
  4 & (7, 2) & (0, 1) \\
  5 & (0, 2) & (7, 1)
\end{bmatrix}
\]

Forwarding matrix
A simple example of route redistribution

We will use the routing and mapping of $G_2$ to construct a forwarding $F_2$, that will be passed as a mapping to $G_1$ ...
A simple example of route redistribution

- $G_2$ is routing with the bandwidth semiring $bw$
- $G_2$ is forwarding with $\text{Cold}(bw, sp)$
- $G_1$ is routing with the bandwidth semiring $sp$
- $G_1$ is forwarding with $\text{Hot}(sp, \text{Cold}(bw, sp))$
First, construct $F_2$
First, construct $F_2$

$$F_2 = L_2 \triangleright_{cp} M_2 = \begin{bmatrix} d_1 & d_2 \\ 6 & (30, 2) & (30, 1) \\ 7 & (20, 2) & (40, 1) \\ 8 & (\infty, 2) & (\infty, 1) \\ 9 & (20, 2) & (\infty, 1) \end{bmatrix}$$
Now, ship it over to $G_2$ as a mapping matrix, using $B_{1,2}$

$$B_{1,2} = \begin{bmatrix}
1 & \infty & \infty & \infty & \infty \\
2 & \infty & \infty & \infty & \infty \\
3 & \infty & \infty & \infty & \infty \\
4 & (0, (\infty, 0)) & \infty & (0, (\infty, 0)) & \infty \\
5 & \infty & \infty & (0, (\infty, 0)) & \infty \\
\end{bmatrix}$$
Now, ship it over to $G_2$ as a mapping matrix, using $B_{1,2}$

$$
M_1 = B_{1,2} \triangleleft_{hp} F_2 =
$$

$$
\begin{bmatrix}
1 & \infty & \infty \\
2 & \infty & \infty \\
3 & \infty & \infty \\
4 & (0, (30, 2)) & (0, (30, 1)) \\
5 & (0, (20, 2)) & (0, (40, 1))
\end{bmatrix}
$$
Finally, construct a forwarding matrix $F_1$ for $G_1$.
Finally, construct a forwarding matrix $F_1$ for $G_1$

$$F_1 = L_1 \triangleright_{hp} M_1 =$$

<table>
<thead>
<tr>
<th></th>
<th>$d_1$</th>
<th>$d_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(5, (30, 2))$</td>
<td>$(5, (40, 1))$</td>
</tr>
<tr>
<td>2</td>
<td>$(2, (30, 2))$</td>
<td>$(2, (30, 1))$</td>
</tr>
<tr>
<td>3</td>
<td>$(4, (30, 2))$</td>
<td>$(4, (40, 1))$</td>
</tr>
<tr>
<td>4</td>
<td>$(0, (30, 2))$</td>
<td>$(0, (30, 1))$</td>
</tr>
<tr>
<td>5</td>
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