If fighting is sure to result in victory, then you must fight, even though the ruler forbid it;
If fighting will not result in victory, then you must not fight even at the ruler’s bidding.

*Sun Tzu, The Art of War, Chapter 10, 23*
Fixed-Odds Horse Racing

- Pool of money betting on horses
  - odds: expressed as $o$-for-1 or $(o - 1)$-to-1
  - probability of success by probability of failure
  - assume no track take, no commissions
- What’s the best strategy?
  - one-off bet
  - multiple ongoing bets, or parlayed bets

Example

- Here, only bet on horse win (not other bets like place etc.)
- Odds are fixed by a bookie
- We use $o$-for-1 convention

<table>
<thead>
<tr>
<th>Horse</th>
<th>Odds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
Betting Strategies

- One-off bet: all in
  ▶ equivalent: maximizing arithmetic mean
- Parlayed bets: Kelly criterion
  ▶ equivalent: maximizing geometric mean
- What happens with all-in for parlayed bets?
- Note: payout asymmetry most important
- Make sure your capital survives before it can compound

Section 2

The Kelly Criterion
Some History

- Developed by J. L. Kelly at Bell Labs; Shannon reviewed
  - Texan tough guy, gunslinger, daredevil pilot and mathematician!
- Wirelines were used to transmit information between bookies
  - application: placing bets on horses

Formulation

- Assume \( m \) horses, each with i.i.d. probability of winning \( p_i \)
- Assume starting capital \( S_0 = 1 \)
- Odds: \( o_i \), alternative \( (1 + r_i) \), \( r_i \) the rate of return
- Play for \( T \) races
  - allocate \( b_i \) fraction of capital on horse \( i \)
  - capital at \( T \): \( S_T = \prod_{t=1}^{T-1} \prod_{i=1}^{m} b_i o_i \)
- Objective: assuming fully invested, choose allocation \( b_i \geq 0 \), \( \sum_i b_i = 1 \) to maximize \( S_T \)

Recommended read: William Poundstone, “Fortune’s Formula”, 2005, a layman version of the story behind the Kelly criterion, Shannon’s forays into the casino and stock market, and Edward Thorp, a mathematician who figured out card counting for Blackjack and later ran a successful hedge fund Princeton Newport.
Maximising Wealth Growth

- Assume $T \to \infty$
  - maximise $E[\sum_{i=1}^{m} \log b_i o_i]$ subject to constraints
  - doubling rate: $W(b, p) := \sum_{i=1}^{m} p_i \log b_i o_i$
- Solution: the Kelly criterion, or log-optimal wealth growth
  - answer: $b^*_i = p_i$, proportional gambling (for fair odds)
  - solve using standard KKT conditions, or log-sum inequality
- Nature of solution will depend on odds: see [CT91, Exercise 6.2]

Example Run of Kelly’s Strategy

Odds can be classified according to $i \lambda := \sum_{i=1}^{m} \frac{1}{o_i}$. If $\lambda < 1$ these are superfair odds, $\lambda = 1$ are fair odds and $\lambda > 1$ are subfair odds. For subfair odds, proportional betting doesn’t apply as some odds may be so poor that the criterion tells us not to bet. The solution is found via a water filling algorithm. The bottomline, however, is that the Kelly criterion only tells us to bet when the odds are favourable, otherwise don’t bet.
A Simple Bet

- Say a biased coin toss, win if heads, lose if tails
  - heads with probability $p$, $q$ otherwise
  - each round, add $1$ to bet
- Odds: $o$-for-1 (remember: win-lose event)
- Kelly solution: $b^* = \frac{op - qo}{o} = p(o + 1) - 1$
  - what does it mean if $o = q/p$?
  - what does it mean when $b^* < 0$ ($o < q/p$)?
  - what about $b^* > 1$?
- A simple way to remember (for two events)
  \[ b^* = \frac{\text{edge}}{\text{odds}} \]

The case $b^* > 1$ can occur when the Kelly criterion is applied to odds coming from a continuous probability distribution.

Simple Bet: Payoff

\[ p = 0.7, \ o = 2 \]
Simple Bet: Under and Overbetting

- There is no gain in overbetting: growth decreases, risk increases
- Sweet spot: full Kelly for maximum wealth growth
- In practice, partial Kelly more applicable, i.e. $\alpha b_i$
  - with $\alpha$ fraction, only $\alpha^2$ volatility
  - more robust to error in estimating returns
  - lower wealth growth compared to full Kelly

Section 3

Downsides
Caveats

- Strategy is guaranteed to beat any other strategy on wealth growth
- BUT Strategy is asymptotically optimal: assume playing forever
- No guarantee to win in the short term (or at all), just the best chance
- Psychologically unsettling: imagine capital dropping 60% right before tripling!
  ▶ partial Kelly strategies trade smoothness with growth rate
- Guaranteed not to go to ruin
  ▶ BUT assumes capital infinitely divisible
  ▶ capital could be $10^{-10}$ but hey, at least not bankrupt!
  ▶ can show $\lim_{T \to \infty} P(S_T > \epsilon) = 0$, for any $\epsilon > 0$
- Assumes know the probability of winning: not true in real life
  ▶ again, half Kelly strategies help: gives a safety margin
  ▶ estimation methods (e.g. maximum entropy, shrinkage)

Criticism from Modern Finance

- Kelly criterion assumes maximizing growth rate exponent
- Called the log-utility function in finance
- Criticism 1: not everybody would want to maximise growth rate exponent
  ▶ does not take into account risk-averseness (or “sleep test”)
  ▶ definition of risk in finance: volatility
  ▶ different utilities for different folks
- Criticism 2: time horizon, as discussed, need very long term
- Counter-argument: not many people want to do with less money
- “Money can’t buy you happiness, but love can’t get you a Ferrari.”
Approximation of the Stock Market

- Suppose $m$ risky assets, each with random "odds" $r_i$ in one investment period.
- One asset with return $r_0$ is deterministic.
- Assume starting capital $S_0 = 1$.
- The return vector $r$, with $\mu_r = E[r]$, $\Sigma = E[(r - r_01)(r - r_01)^T]$.
  - $\Sigma$ is full rank.
  - Correlations apply only "spatially".
- Derive the optimal allocation $b$ to optimise the wealth doubling rate.
  - Optimise $E[\log(r_0 + b^T(r - r_01))]$.
- Assume no constraints on $b$.
- For what return distribution is this allocation optimal?

Further reading