

Puzzle: Is $4^{545} + 545^4$ prime or composite?

Solution: The fact that one of the summands is a fourth power brings to mind the identity:

$$x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 = (x^2 - \sqrt{2}xy + y^2)(x^2 + \sqrt{2}xy + y^2).$$

Of course, the square roots mean that this identity does not provide an integer factorization for sums of fourth powers of integers.

However we can write

$$\begin{aligned} 4^{545} + 545^4 &= \left((\sqrt{2})^{545}\right)^4 + 545^4 \\ &= (2^{545} + \sqrt{2}(\sqrt{2})^{545} 545 + 545^2)(2^{545} - \sqrt{2}(\sqrt{2})^{545} 545 + 545^2) \\ &= (2^{545} + 2^{273} 545 + 545^2)(2^{545} - 2^{273} 545 + 545^2). \end{aligned}$$

These integer factors can be evaluated using Maple if desired.