Localisation of Rayleigh-Bloch waves and stability of resonant loads on arrays of bottom-mounted cylinders with respect to positional disorder

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Resonant loads on an array of cylinders

Maniar & Newman (1997, JFM)

\[ \psi = 0 \]
\[ N = 9 \]
\[ d/a = 4 \]
Resonant loads on an array of cylinders

\[ kd = \frac{2\pi}{3} \]

\[ kd = 2.7814 \]

\[ kd = \pi \]

Loads

Neumann trapped modes

\[ kd = 2.7826 \]
Since Maniar & Newman (1997)

177 citations, including...

- Porter & Evans (1998, Quarterly Journal on Mechanics & Applied Mathematics)
- Walker & Eatock-Taylor (2005, Ocean Engineering)
- Thompson, Linton & Porter (2007, Quarterly Journal on Mechanics & Applied Mathematics)
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Mathematical model: solitary cylinder

Linear potential-flow theory:

\[
\text{velocity field} = \nabla \cdot \text{Re}\left\{ \frac{g}{i\omega} \Phi(x, y, z)e^{-i\omega t} \right\}
\]

where \( \phi = \) velocity potential; \( \omega = \) frequency; \( g = \) gravity.
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where \( \phi = \) velocity potential; \( \omega = \) frequency; \( g = \) gravity.

\[\nabla^2 \Phi = 0\]

\[\Phi_z = 0\]

\[z = -h\]
Mathematical model: solitary cylinder

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where \( \phi = \) velocity potential; \( \omega = \) frequency; \( g = \) gravity.

\[ \Phi_z = \frac{\omega^2 \Phi}{g} \]

\[ \nabla^2 \Phi = 0 \]

\[ \Phi_z = 0 \]

\[ z = -h \]
Vertical eigenfunctions

\[ \Phi = \phi(x, y) \cosh k(z + h) + \sum_{n=1}^{\infty} \phi_n(x, y) \cos k_n(z + h) \]

where

\[ k \tanh(kh) = \frac{\omega^2}{g} \]
Vertical eigenfunctions

\[ \Phi = \phi(x, y) \cosh k(z + h) + \sum_{n=1}^{\infty} \phi_n(x, y) \cos k_n(z + h) \]

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\[ k \tanh(kh) = \frac{\omega^2}{g} \]
Vertical eigenfunctions

\[ \Phi = \phi(x, y) \cosh k(z + h) \]

\[ \phi_{inc}(x, y : \psi) \cosh k(z + h) \rightarrow \Phi = \phi(x, y) \cosh k(z + h) \]
Problem in horizontal plane

\[ \phi_n = 0 \]

\[ \nabla^2 \phi + k^2 \phi = 0 \]
Problem in horizontal plane

\[ \nabla^2 \phi + k^2 \phi = 0 \]
Problem in horizontal plane

\[ \phi_{n} = 0 \]

\[ \nabla^{2} \phi + k^{2} \phi = 0 \]

\[ \phi = \phi_{\text{inc}} + \phi_{\text{sca}} \]
### Horizontal eigenfunctions

- Local polar coordinates \((r, \theta)\)

#### Bessel-Fourier Expansions

<table>
<thead>
<tr>
<th>Field Type</th>
<th>Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incident Field</td>
<td>[ \phi_{\text{inc}}(r, \theta) = \sum_{m=-\infty}^{\infty} b_m J_m(kr) e^{im\theta} ]</td>
</tr>
<tr>
<td>Scattered Field</td>
<td>[ \phi_{\text{sca}}(r, \theta) = \sum_{m=-\infty}^{\infty} Z_m b_m H_m(kr) e^{im\theta} ]</td>
</tr>
</tbody>
</table>

- \(J_m\) and \(H_m\) are Bessel and Hankel functions of 1st kind, resp.
- \(Z_m = -J'_m(ka)/H'_m(ka)\)
Multiple cylinders: general setting

- Scattered waves force (become incident) on other cylinders.
- E.g. below:
  \[ \phi_{\text{inc}}^{(1)} = \phi_{\text{amb}} + \phi_{\text{sca}}^{(2)} + \phi_{\text{sca}}^{(3)} + \phi_{\text{sca}}^{(4)} \]

Assign local polars \((r_p, \theta_p)\) to \(C_p\).

Map scattered to incident:

\[
\phi_{\text{sca}}^{(p)} = \sum_{j=\infty}^{\infty} Z_j b_j^{(p)} H_j(kr_p) e^{ij\theta_p} \\
= \sum_{j=\infty}^{\infty} Z_j b_j^{(p)} \times \sum_{m=\infty}^{\infty} g_{jm}^{(p1)} J_m(kr_1) e^{im\theta_1} \\
\text{for known } g_{jm}^{(p1)}
\]
Multiple cylinders: general setting

- Scattered waves force (become incident) on other cylinders.
- E.g. below:
  \[ \phi^{(1)}_{\text{inc}} = \phi_{\text{amb}} + \phi^{(2)}_{\text{sca}} + \phi^{(3)}_{\text{sca}} + \phi^{(4)}_{\text{sca}} \]

Assign local polars \((r_p, \theta_p)\) to \(C_p\).

Map scattered to incident:

\[ \phi^{(p)}_{\text{sca}} = \sum_{j=-\infty}^{\infty} Z_j b_j^{(p)} H_j(kr_p) e^{ij\theta_p} \]

\[ = \sum_{m=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{\infty} Z_j b_j^{(p)} \times \right\} J_m(kr_1) e^{im\theta_1} \]
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- E.g. below: \( \phi_{\text{inc}}^{(1)} = \phi_{\text{amb}} + \phi_{\text{sca}}^{(2)} + \phi_{\text{sca}}^{(3)} + \phi_{\text{sca}}^{(4)} \)

Then express ambient field

\[
\phi_{\text{amb}} = \sum_{m=-\infty}^{\infty} c_m^{(1)} J_m(k r_1) e^{im \theta_1}
\]

Matching modal amplitudes gives

\[
b_m^{(1)} = c_m^{(1)} + \sum_{p=2}^{4} \sum_{j=-\infty}^{\infty} Z_j b_j^{(p)} g_{jm}^{(p1)}
\]

for \( m = 0, \pm 1, \pm 2, \ldots \).
Multiple cylinders: general setting

- Scattered waves force (become incident) on other cylinders.
- E.g. below: \[ \phi_{inc}^{(1)} = \phi_{amb} + \phi_{sca}^{(2)} + \phi_{sca}^{(3)} + \phi_{sca}^{(4)} \]

- Repeat for \( C_2, C_3 \) and \( C_4 \) and solve.
- But, computational limit is \( O(100) \) cylinders.
Infinite line of eqispaced cylinders

Quasi-periodicity implies there exists $Q$ such that

$$b_m^{(p)} = e^{iQdp} b_m.$$ 

Hence

$$b_m = c_m^{(?)} + \sum_{j=-\infty}^{\infty} Z_j b_j \sigma_{j-m}(Qd)$$

where

$$\sigma_n(Qd) = \sum_{p=1}^{\infty} \left\{ (-1)^n e^{ipQd} - e^{-ipQd} \right\} H_n(kp)$$
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Forced problem

\[ T_0 e^{ik(x \cos \psi_0 + y \sin \psi_0)} \]

\[ T_1 e^{ik(x \cos \psi_1 + y \sin \psi_1)} \]

\[ \phi_{\text{amb}} = e^{ik(x \cos \psi + y \sin \psi)} \]

Incident amplitudes

\[ c_m^{(p)} = e^{i2kd \cos \psi} c_m \quad \Rightarrow \quad Q = 2k \cos \psi \]

Transmitted field (similar for reflected)

\[ \phi = \sum_{n=-\infty}^{\infty} T_n e^{ik(x \cos \psi_n + y \sin \psi_n)} \]

where

\[ \cos \psi_n = \cos \psi_0 + n\pi / kd \quad \Rightarrow \quad \psi_0 = \psi \]
Forced problem

\[ T_0 e^{i k(x \cos \psi_0 + y \sin \psi_0)} = \phi_{\text{amb}} = e^{i k(x \cos \psi + y \sin \psi)} \]

Incident amplitudes

\[ c_m^{(p)} = e^{i 2 k d p \cos \psi} c_m \Rightarrow Q = 2 k \cos \psi \]

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Incident amplitudes

\[ c_m^{(p)} = e^{i2kdp \cos \psi} c_m \implies Q = 2k \cos \psi \]

Transmitted field (similar for reflected)

\[ \phi = \sum_{n=-\infty}^{\infty} T_n e^{ik(x \cos \psi_n + y \sin \psi_n)} \]

\[ \psi_n \in \mathbb{R} \quad \text{travelling waves}; \quad \psi_n \in \mathbb{C} \quad \text{decaying waves} \]
Unforced problem (e.g. Porter & Evans, 1999)

Rayleigh-Bloch wave modes

- Incident amplitudes \( c_m = 0 \)
- Seek eigenvalues \( e^{i Q^{RB} d} \) and associated eigenfunctions
- RB modes bound to array
- (In general) solutions exist for \( kd < Q^{RB} d < \pi \)
- Incident wave cannot excite them

Peter & Meylan (JFM, 2007)
Semi-infinite problem (e.g. Peter & Meylan, 2007)

Solution method

Write

\[ b_m^{(p)} = e^{i2kdp \cos \psi} b_m^\psi \]
\[ + \alpha e^{iQ^{RB} dp} b_m^{RB} \]
\[ + \tilde{b}_m^{(p)} \]

where

- \( b_m^\psi \) are amplitudes for forced problem with incident angle \( \psi \).
- \( b_m^{RB} \) are amplitudes for unforced problem and \( \alpha \) is a constant.
- \( \tilde{b}_m^{(p)} \rightarrow 0 \) as \( p \rightarrow \infty \).
Semi-infinite problem (e.g. Peter & Meylan, 2007)

Solution method

Write

\[ b_m^{(p)} = e^{i2kdp \cos \psi} b_{\psi}^m + \alpha e^{iQ_{RB} dp} b_{RB}^m + \tilde{b}_{m}^{(p)} \]

where

- \( b_{\psi}^m \) are amplitudes for forced problem with incident angle \( \psi \).
- \( b_{RB}^m \) are amplitudes for unforced problem and \( \alpha \) is a constant.
- \( \tilde{b}_{m}^{(p)} \rightarrow 0 \) as \( p \rightarrow \infty \).

Scattered field \( \phi_{sca} \)

Linton et al. (2007)
Solution method

\[ b_m^{(p)} = e^{i2kdp \cos \psi} b_m^\psi \]

\[ + \alpha_+ e^{iQ^{RB} dp} b_m^{RB} + \alpha_- e^{-iQ^{RB} dp} (-1)^m b_m^{RB} \]

\[ + \tilde{b}_m^{(p)+} + \tilde{b}_m^{(p)-} \]

where

\[ \tilde{b}_m^{(p)+} \rightarrow 0 \text{ as } p \rightarrow N. \]

\[ \tilde{b}_m^{(p)-} \rightarrow 0 \text{ as } p \rightarrow 1. \]
Resonance requires

1. **Strong excitation** of Rayleigh-Bloch mode.
2. **Strong reflection** of Rayleigh-Bloch mode at ends.
3. **Coherence** of Rayleigh-Bloch modes.
A different approach

Motivation

- Plane wave with amplitude $A$ and at angle $\chi$

$$\varphi(x, y : \chi) \equiv A e^{ik(x \cos \chi + y \sin \chi)}$$

is a solution of $\varphi_{xx} + \varphi_{yy} + k^2 \varphi = 0$

- Where are the plane waves in the direct formulation?

$$J_n(kr)e^{in\theta} = \frac{(-i)^n}{2\pi} \int_{-\pi}^{\pi} e^{inx} \varphi(\chi) \, d\chi$$

$$H_n(kr)e^{in\theta} = \frac{(-i)^n}{\pi} \int_{\gamma+\theta}^{\gamma} e^{inx} \varphi(\chi) \, d\chi$$
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\[ H_n(kr) e^{in\theta} = \frac{(-i)^n}{\pi} \int_{\gamma + \theta}^{-\pi} e^{in\chi} \phi(\chi) \, d\chi \]
\[ \chi \in C \]
A different approach

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- Plane wave with amplitude $A$ and at angle $\chi$

\[ \varphi(x, y : \chi) \equiv A e^{ik(x \cos \chi + y \sin \chi)} \]

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\]

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H_n(kr)e^{in\theta} = \frac{(-i)^n}{\pi} \int_{\gamma + \theta} \rho e^{inx} \varphi(\chi) \, d\chi
\]

$\chi \in \mathbb{C}$
A different approach

Motivation

- Plane wave with amplitude $A$ and at angle $\chi$

$$\varphi(x, y : \chi) \equiv A e^{i k (x \cos \chi + y \sin \chi)}$$

is a solution of $\varphi_{xx} + \varphi_{yy} + k^2 \varphi = 0$

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H_n(kr)e^{in\theta} = \frac{(-i)^n}{\pi} \int_{\gamma+\theta}^{\gamma} e^{inx} \varphi(\chi) \, d\chi
\]
A different approach

Hankel functions

\[ H_n(kr)e^{in\theta} = \frac{(-i)^n}{\pi} \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{i\infty} + \int_{\frac{-\pi}{2} + i\infty}^{\pi} \right\} e^{in\chi} \varphi(\chi) \, d\chi \]

if \( x \geq 0 \)

i.e. right travelling/decaying waves (real/complex branches)

\[ H_n(kr)e^{in\theta} = \frac{(-i)^n}{\pi} \left\{ \int_{\frac{\pi}{2}}^{3\pi/2} + \int_{3\pi/2}^{\infty} + \int_{3\pi/2}^{\pi/2 + i\infty} \right\} e^{in\chi} \varphi(\chi) \, d\chi \]

if \( x \leq 0 \)

i.e. left travelling/decaying waves (real/complex branches)
Decompose geometry

\[ d \]
Single cylinder

\[ \phi_{\text{inc}}^- = \int_{-\frac{\pi}{2} + i\infty}^{\frac{\pi}{2} - i\infty} c_-(\chi)\varphi(\chi) \, d\chi \]
\[ \phi_{\text{sca}}^- = \int_{\frac{\pi}{2} + i\infty}^{\frac{3\pi}{2} - i\infty} b_-(\chi)\varphi(\chi) \, d\chi \]
\[ \phi_{\text{sca}}^+ = \int_{-\frac{\pi}{2} + i\infty}^{\frac{\pi}{2} - i\infty} b_+(\chi)\varphi(\chi) \, d\chi \]
\[ \phi_{\text{inc}}^+ = \int_{\frac{\pi}{2} + i\infty}^{\frac{3\pi}{2} - i\infty} c_+(\chi)\varphi(\chi) \, d\chi \]

- \( c_\pm(\chi) \) are incident amplitude functions
- \( b_\pm(\chi) \) are scattered amplitude functions
Single slab

- Introduce scattering kernels $R$ and $T$:

  $$b_-(\chi) = \int_{-\frac{\pi}{2} + i\infty}^{\frac{\pi}{2} - i\infty} R(\chi|\psi)c_-(\psi) \, d\psi + \int_{\frac{\pi}{2} + i\infty}^{\frac{3\pi}{2} - i\infty} T(\chi|\psi)c_+(\psi) \, d\psi$$

  $$b_+(\chi) = \int_{-\frac{\pi}{2} + i\infty}^{\frac{\pi}{2} - i\infty} T(\chi|\psi)c_-(\psi) \, d\psi + \int_{\frac{\pi}{2} + i\infty}^{\frac{3\pi}{2} - i\infty} R(\chi|\psi)c_+(\psi) \, d\psi$$

- Truncate complex branches and sample (discretise) $\chi$-space. Gives array expressions:

  $$b_- = Rc_- + Tc_+ \quad \text{and} \quad b_+ = Tc_- + Rc_+$$

- Associated transfer matrix (left-to-right map)

  $$P = \begin{pmatrix} T - RT^{-1}R & RT^{-1} \\ -T^{-1}R & T^{-1} \end{pmatrix}$$

  $\rightarrow$ calculating transfer matrix $\equiv$ solving problem.
Spectrum of transfer matrix

Rayleigh-Bloch mode

Eigenvalues

|Eigenfunction|

\[ e^{iQ^{RB}d} \]

\[ \text{Re}(\chi)/\pi - \text{Im}(\chi) \]

---

rightward

---

leftward
Spectrum of transfer matrix

Forced mode

| Eigenvalues | | Eigenfunction |
|-------------|-----------------|
| $e^{ikd\cos(0)}$ | | rightward, leftward |

Graphs showing eigenvalues and eigenfunctions.
Summary of new approach

**Strengths**
- Solving for infinite, semi-infinite and finite arrays requires linear algebra only.
- Straightforward to incorporate disorder

\[ P_{1,N} = P_N P_{N-1} \ldots P_1. \]

- Bounded system size (independent of num. of cylinders).

**Weaknesses**
- Relies on discretisation and truncation.
- Involves exponentially growing/decaying terms.
Wave localisation

- Appears in all (?) branches on wave science.
- Disorder can localise waves in space without dissipation.
- Manifest as exponential attenuation of wave amplitude.
- Due to interference between scattered waves.
- Attenuation rate depends on frequency and medium properties.

\[ \log_{10} |\text{amplitude}| \]

Distance
Positional disorder damps resonance

Introduce positional disorder

Perturbations chosen randomly.

$|\epsilon_n| \in \mathcal{U}(0, \epsilon)$ for prescribed $\epsilon$.

Max Load

$\psi = 0$

$N = 100$

$d/a = 4$

$\epsilon = 0; \epsilon = 0.1; \epsilon = 0.2; \epsilon = 0.4$
**Are Rayleigh-Bloch waves localised?**

**Question**
- Does positional disorder localise Rayleigh-Bloch waves?
  → Is this the process that damps the resonant loads?

**Idea**
- Excite motions with Rayleigh-Bloch waves rather than plane incident wave.
- Assume Rayleigh-Bloch wave modes dominate.
  → Positional disorder expected to perturb RB modes.
- Extract Rayleigh-Bloch wavenumber $Q_{RB}(\epsilon)$.
  → $Q_{RB}(\epsilon) \in \mathbb{C}$ implies localisation.

**Problem**
- Can’t excite rightward/leftward travelling RB wave only.
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Are Rayleigh-Bloch waves localised?

Method

1. Solve 2 problems:
   1. With forward RB incident wave from left of array.
   2. With backward RB incident wave from right of array.

2. Use ansatz for amplitude spectra

\[
\begin{pmatrix}
  c_n \\
  b_n
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  c_n \\
  b_n
\end{pmatrix}
= \alpha_+ e^{iQ_{RB} n} v_{RB}^+ + \alpha_- e^{iQ_{RB} (N+1-n)} v_{RB}^- ,
\]

through the array \( n = 0, \ldots, N \).

3. Combine solutions to separate rightward and leftward modes.
Example results: no disorder

\( kd = 2.7814; \; d/a = 4; \; N = 100; \; \epsilon = 0 \)

\[ \pm Q^{RB} d \approx \pm 3.104 \pm 0.001i \quad (Q^{RB} d = 3.11003) \]
Example results: with disorder

\[ kd = 2.7814; \ d/a = 4; \ N = 100; \ \epsilon = 0.2 \]

\[ Q^{\text{RB}} d \approx 3.012 + 0.019i \]

\[ -Q^{\text{RB}} d \approx -2.699 - 0.022i \]
Example results: ensemble runs

$\varepsilon = 0.2$; ensemble size 500

$\text{Re}(Q^{RB}d) \approx \pm 2.73$

$\text{Im}(Q^{RB}d) \approx \pm 0.017$
Localisation of Rayleigh-Bloch waves…
Evidence of this.

…and stability of resonant loads on arrays of bottom mounted cylinders with respect to positional disorder
Disorder damps/eliminates resonance.

Still to do
Make the link undeniable.
University of Adelaide
Sunday 6th – Wednesday 9th December 2015