1. Mathematical conjuring

Mathematics could be described as the science of learning a surprising amount from surprisingly little. From a seemingly innocent collection of equations a good mathematician can extract the most fantastic conclusions that extend beyond all reasonable expectations. At times, some results seem so fanciful that a mathematician’s work can seem reminiscent of a magician performing illusions. The infinite number of primes may be confirmed by a few lines of reasoning, while $\sqrt{2}$ may be shown to be irrational with a similar amount of work. On occasion, results extend beyond mathematics and reach into the real world; it is one such result that provides the topic of this article, in which Borsuk, Ulam and the intermediate value theorem conjure a surprise from the Earth’s equator (and much more).

2. The disappearing act

In order to prove our initial result we first need to recall the intermediate value theorem.

Theorem 1 (Intermediate value). If $f$ is a real-valued continuous function on $[a, b]$ and $u$ lies between $f(a)$ and $f(b)$, then there is some $c \in [a, b]$ such that $f(c) = u$.

We need to make an assumption about the temperature of the Earth along the equator; specifically, we assume that the temperature along the equator varies continuously. That is, if we consider temperature as a function of position along the equator we assume that this function is continuous. The kinetic description of temperature states that the temperature of a collection of particles is related to the average velocity of the particles. While this could give rise to discontinuities on small enough scales, on the scales we are usually interested in temperature can certainly be considered to be continuous. In particular, most measurement devices would show temperature to vary continuously with position. We also assume that the equator is circular. While this is not exact, it is “close enough” for our result to hold.

With these assumptions we can state our first result.

Theorem 2. There always exist antipodal points on the equator with the same temperature.

That is, at any given time we can find two points on opposite sides of the equator that have exactly the same temperature. The proof of this is as follows.

Proof. Describe each point along the equator by an angle $\theta$ in the interval $[0, 2\pi)$, which represents the position of the point from some reference point and “wraps around” after one rotation. Let $f(\theta)$ be a continuous function that represents the temperature along the equator. Define $d(\theta) = f(\theta) - f(\theta + \pi)$, which is the difference in temperature between the point $\theta$ and its antipodal point (Figure 1). If we have some point $\theta$ such that $d(\theta) = 0$ then we are done, so assume that $d(\theta) \neq 0$. Then, $d(\theta + \pi) = f(\theta + \pi) - f(\theta + 2\pi) = f(\theta + \pi) - f(\theta) = -d(\theta)$. Since $f$ is continuous $d$ must also be continuous. By the intermediate value theorem there must be some point $\phi$ between $\theta$ and $\theta + \pi$ such that $d(\phi) = 0$, which means that $f(\phi) = f(\phi + \pi)$. Therefore, there are two antipodal points on the equator with the same temperature.

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1. The word “shazam” was created by conjurers, and it is this connection that lends the title its frivolous rhyme.
2. In addition, the equations governing heat flow show that any discontinuities “smooth out” and neighbouring regions approach the same temperature.
\[ \theta \text{ and } \theta + \pi \text{ such that } d(\phi) = f(\phi) - f(\phi + \pi) = 0, \]
so that \( f(\phi) = f(\phi + \pi) \).

\[ \begin{array}{c}
\text{Figure 1: Antipodal points } \theta \text{ and } \theta + \pi. \\
\end{array} \]

This result also holds if temperature is replaced by any other function that can reasonably be assumed continuous, such as humidity or the distance between the point and Sir Patrick Stewart. We can also replace the equator by any great circle around the Earth (or any closed convex path along the Earth’s surface, adjusting the meaning of “centre”).

3. The reveal

It seems our magicians Borsuk and Ulam have vanished from in front of our very eyes! They now reappear in the grand finale: the result proved above is just a special case of a more-general result known as the Borsuk–Ulam theorem. A proof of this was given by Karol Borsuk (1933), who attributed the statement of the problem to Stanislaw Ulam.

Before we can state this result we need a way to extend the idea of a circle and sphere to higher dimensions. In \( \mathbb{R}^2 \), a circle centred at the origin is just the set of points in \( \mathbb{R}^2 \) that are some fixed distance (the radius) away from the origin, with a similar definition for a sphere in \( \mathbb{R}^3 \). Generalising this idea, we define an \( n \)-sphere of radius \( r \) to be the set of points

\[ S^n = \{ x \in \mathbb{R}^{n+1} : ||x|| = r \}, \]

where \( || \cdot || \) is the standard Euclidean distance. With this definition \( S^1 \) is a circle and \( S^2 \) is a sphere, and we are able to talk about spheres of any dimension \( n \in \mathbb{N} \). We can now state the full Borsuk–Ulam theorem.

**Theorem 3** (Borsuk–Ulam). Every continuous function from an \( n \)-sphere \( S^n \) into \( \mathbb{R}^n \) maps some pair of antipodal points to the same point.

The result from the previous section is just an application of the Borsuk–Ulam theorem for \( n = 1 \); that is, a function from a circle \( S^1 \) into \( \mathbb{R} \) (the proof of the general result is more complicated so we omit it here). For \( n = 2 \) this theorem says that any function from a sphere \( S^2 \) into \( \mathbb{R}^2 \) has a pair of antipodal points that have the same value. This again can be given a profound meteorological interpretation: if we assume that two quantities on the surface of the Earth are continuous, such as temperature and humidity, the Borsuk–Ulam theorem states that there are antipodal points on the Earth’s surface that have the same temperature and humidity. We could also conclude that there are antipodal points on the Earth’s surface that are both the same distance from Sir Patrick Stewart and both the same distance from Sir Ian McKellen. Given any two continuous functions we can always find two such points.

4. Curtain call

Mathematics is filled with many amazing results that at first seem like magic. While some of these are remarkable simply for their theoretical implications, some reach beyond mathematical domains and reveal incredible real-world phenomena. These theorems, like that described above, require a sound understanding of both mathematics and the phenomenon of interest, and show just how powerful applied maths can be. Best of all, by studying maths you have the ability to create similar illusions of your own.

Hayden needs to finish his PhD in applied maths.

References