Three unsolved problems in applied mathematics

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20/10/11
There are several famous lists of unsolved mathematical problems.

Hilbert’s problems was (is) a list of 23 unsolved problems compiled by David Hilbert in 1900.

In 2000 the Clay Mathematics Institute proposed the Millennium Prize Problems, a list of seven unsolved problems (including one from Hilbert’s list).
While the importance of these problems cannot be disputed, they are often inaccessible, especially to undergraduate students.

Even worse, they often appear far removed from applied mathematics!

We consider three easy to understand but unsolved (and important) problems from applied mathematics.
The existence and smoothness of solutions to the Navier–Stokes equations is one of the Millennium Prize Problems (so they aren’t all bad…).

These equations govern the flow of fluids (liquids or gasses), and a full solution consists of knowing the velocity and pressure at every point in the domain.

For physical realistic results, we expect the solutions to be both smooth and bounded.
In two dimensions it is known that, given a smooth, divergence-free initial velocity field, and in the absence of external forces, smooth solutions exist for all future times.

No such result exists for three dimensions.

This means that, given some suitable initial condition, we do not know if a solution exists for all future times. Mathematically, we cannot claim that a three-dimensional fluid will flow!

See http://www.claymath.org/millennium for further details.
Consider the following differential equation:

\[ \frac{dy}{dx} = 1 \]

Given a suitable condition \( y(0) = y_0 \), can you tell if we can find a closed-form solution to this equation?

Yes! \( y(x) = x + y_0 \).
What about more complicated differential equations?

The Korteweg–de Vries equation for the amplitude $f(\zeta)$ of a solitary wave travelling on the surface of a shallow fluid is

$$-2cf' + 3ff' + \frac{1}{3}f''' = 0$$

Can we solve this exactly? In other words, can we integrate this? Can you tell just by looking?
We can in fact solve this equation as follows\(^1\):

\[-2cf + \frac{3}{2}f^2 + \frac{1}{3}f'' = K, \quad K \in \mathbb{R}\]

\[-2cff' + \frac{3}{2}f'f^2 + \frac{1}{3}f'f'' = Kf'\]

\[-c \frac{\partial}{\partial \zeta} f^2 + \frac{1}{2} \frac{\partial}{\partial \zeta} f^3 + \frac{1}{6} \frac{\partial}{\partial \zeta} [(f')^2] = K \frac{\partial f}{\partial \zeta}\]

\[-cf^2 + \frac{1}{2}f^3 + \frac{1}{6}(f')^2 = Kf + L, \quad L \in \mathbb{R}\]

\[(f')^2 = 6\zeta cf^2 - \frac{1}{3}f^3 + Kf + L\]

\[f = 2c \operatorname{sech}^2 \left( \sqrt{\frac{3c}{2}} \zeta \right)\]

\(^1\)My lecture notes

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In general, there is no (known) condition that can determine whether we can solve a differential equation exactly or not.

Galois theory considers when a “closed form” solution to an equation can be found.
Solution of linear systems of equations

- Linear systems arise in many different areas of mathematics, and are one of the fundamental entities in applied mathematics.
- Linear systems form the basis of many numerical methods, such as finite differences and finite elements.
- If you can solve linear systems, you can solve applied maths problems.
In first year, we learn that the linear system $Ax = b$ can be solved by premultiplying by the inverse of $A$.

Calculating the inverse using Gauss-Jordan elimination requires $O(n^3)$ operations for an $n \times n$ matrix. We are not this patient!

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Using the Coppersmith–Winograd algorithm the problem can be solved in $O(n^{2.376})$ operations\(^2\).

This is currently the fastest method known (these methods are actually for multiplication but can be used to solve linear equations).

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\(^2\)Robinson, *SIAM News*, Volume 38 (9), 2005
Peter, you deserve your own copy. Be sure to look me up next time you're in Bergen —
Nick, 3/6/85

L. N. Trefethen hereby bets Peter Alfeld that by 31 December, 1994, a method will have been found to solve \( Ax = b \) (an \( n \times n \) linear system of \( n \) \( \times 1 \) equations) in \( O(n^{2+\varepsilon}) \) operations for any \( \varepsilon > 0 \). Numerical stability is not required.

The winner gets \$100. — from the loser.

Peter J. Alfeld
Lloyd N. Trefethen

Witnesses:
Per Erik Koch
S. P. Novotný (This is a made-up problem)
Nick Trefethen paid up in February 1996. The bet was renewed for another 10 years.
The bet was also amended so that if either party solved the problem, the other had to pay an additional $100.
The current status of the bet is unknown.
For more information see http://people.maths.ox.ac.uk/trefethen/.
Three important problems in applied maths:

- Existence and smoothness of solutions to the Navier–Stokes equations in three dimensions.
- Determining the integrability of differential equations.
- Solution of linear systems in $O(n^{2+\epsilon})$ operations.

These provide examples of areas of maths that are the subject of current research, but are also accessible enough for undergraduate students.

Discuss!