More than just a pretty fractal

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Fractals (and images of fractals) are some of the most famous mathematical objects.

Despite this, often little is known of their origin, uses or even definition!

We look at their history, their importance in mathematics and their application to real-world problems.
Introduction (or rediscovery) of many of the basic elements of calculus, such as the product rule, chain rule and Taylor series followed.
This work lacked rigour.

The idea of a function was not introduced by Euler until the 18th century.

Around 1820 Cauchy began to formalise the ideas developed by calculus.

This field became known as mathematical analysis.

Big players such as Riemann, Poincarè, Poisson, Liouville and Fourier.
- Instead of simply systematising the results of calculus something new appeared: “monsters”.

- These monsters were counter-intuitive objects, such as nowhere continuous functions.

- Another example is the Weierstrass function (1872), given by

\[
f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)
\]

for \(0 < a < 1\), \(b\) a positive integer and \(ab > 1 + \frac{3}{2} \pi\).
Weierstrass function

This function is continuous everywhere but differentiable nowhere.
Weierstrass function

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“Logic sometimes makes monsters. For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some purpose... Indeed, from the point of view of logic, these strange functions are the most general; on the other hand those which one meets without searching for them, and which follow simple laws appear as a particular case which does not amount to more than a small corner.”

— Poincarè
Definition

- It wasn’t until the 1950s that fractals began to gain acceptance.
- This began with the work of Benoit B. Mandelbrot.
- At the time, Mandelbrot was working for IBM and using computers to explore mathematics.
In his book *The Fractal Geometry of Nature* Mandelbrot argued for the acceptance of fractals and popularised the use of fractal dimension.

This provided a way to measure how “rough” an object was. “Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.”

— Mandelbrot
Mandelbrot proposed the following:

**Definition**

A fractal is an object with a fractal dimension greater than its topological dimension.

- Modern definitions are more general and require in addition that a shape have fine detail at any scale, cannot be described by Euclidean geometry, be self-similar and have a recursive definition.
Fractal dimension is a complicated notion but can be understood through an example.

Space filling curves were one of the original monsters and are also fractals.

A curve has a topological dimension one.

A space filling curve in the plane has a fractal dimension of two. The curve *is* whole plane.
A game

Let's play a simple game.

1. Begin with the three vertices of a triangle.
2. Randomly select a point inside the triangle.
3. Randomly select a vertex of the triangle and generate a new point halfway between the vertex and the interior point.
4. Repeat step 3 some number of times.
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More than just a pretty fractal
The Sierpiński triangle! Fractal dimension $\log(3)/\log(2) \approx 1.58$
One method for finding the zeros of a non-linear function is to use Newton’s method.

To find a zero of a function \( f(x) \) we iterate

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

We can also use this method for functions of complex variables.

Consider finding the zeros of \( f(z) = z^3 - 1 \), where \( z \in \mathbb{C} \). We are really finding the cube roots of unity, so there are three solutions.

Given some initial guess \( z_0 \), which root does this converge to? How many iterations does it take for the method to converge?
A Newton fractal, coloured by the number of iterations
The Cantor set, sometimes denoted $C$, is a famous fractal.

It may be constructed by repeatedly removing the middle third of lines an infinite number of times.

Although the Cantor set has zero length, it contains an infinite number of points!

It has fractal dimension $\log(2)/\log(3) \approx 0.63$. 
The Cantor set may be constructed in a different way. Define the functions $w_1(x) = x/3$ and $w_2(x) = x/3 + 2/3$. Let $W(A) = w_1(A) \cup w_2(A)$ where $A$ is an interval in $\mathbb{R}$. The map $W$ is called a Hutchinson operator. Are there any solutions to $W(A) = A$? Yes! The Cantor Set! $W(C) = C$!
Fractals and cancer

- Fractals could provide a way to detect cancers without the need for sophisticated imaging devices.
- Baish & Rakesh (*Fractals and Cancer*, 2000) studied the fractal dimension of blood vessels in health tissue and tumours.
- The following fractal dimensions were observed:

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<th>Fractal dimension</th>
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<td>Normal</td>
<td>$1.89 \pm 0.04$</td>
</tr>
<tr>
<td>Tumour</td>
<td>$1.70 \pm 0.03$</td>
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A — Normal veins
B — Normal capillaries
C — Veins in tumours
From Baish & Rakesh (2000)
Fractals have also been used to study the distribution of trees in forests.

West, Enquist and Brown (2008) studied how the size of the branches on the largest tree in a forest compared to the sizes of all the trees relative to each other.

In other words, the forest exhibited scaling just like a fractal!

They found that the forest is just like a scaled version of a single tree!
Conclusions

- Fractals are not only nice to look at, they are also important mathematical objects.
- Fractals are extremely useful tools for understanding many elements of nature.