What is Index Theory (I)?

Hang Wang

Mathematical Sciences Center
Tsinghua University

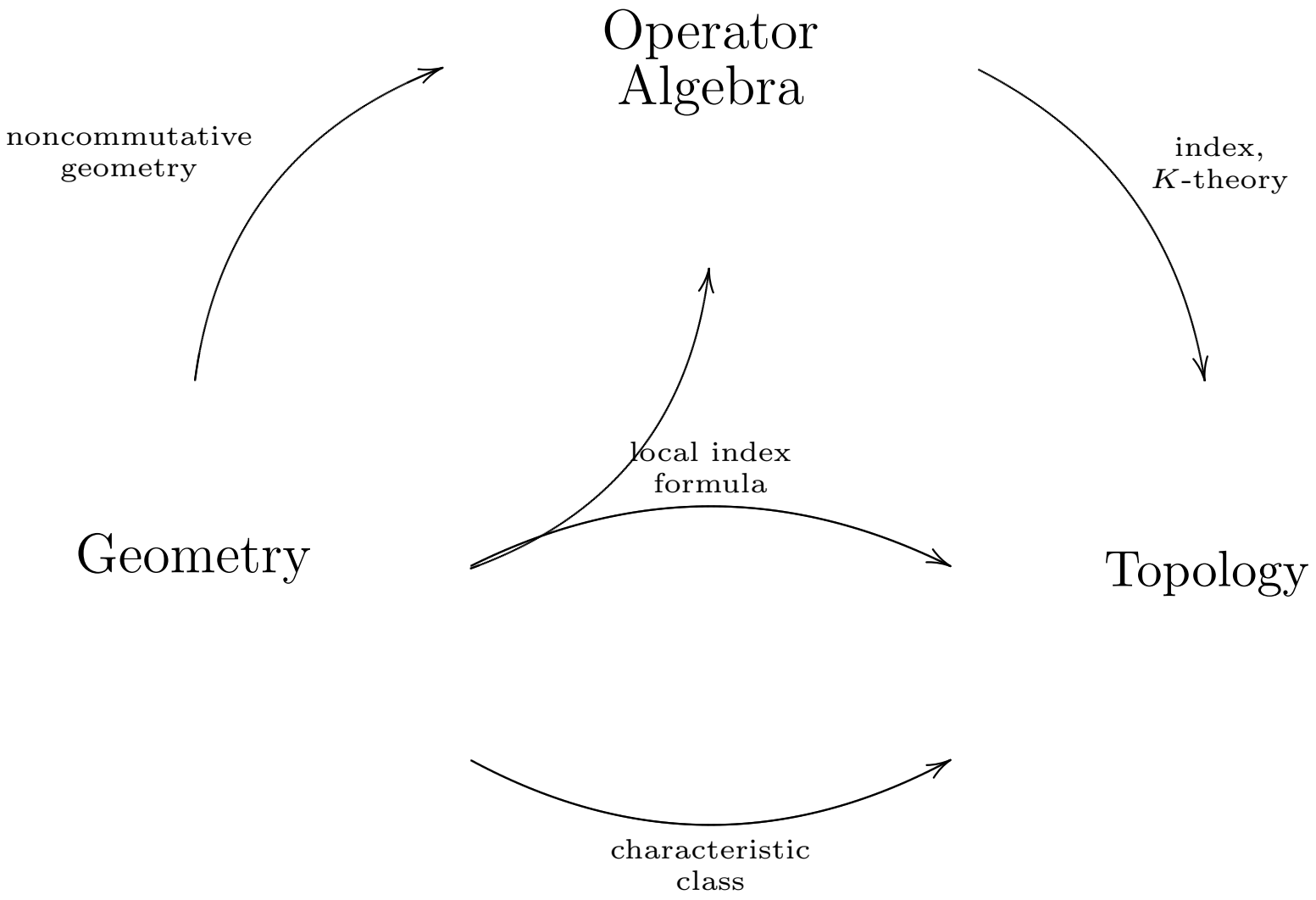
May 17, 2012

Outline

Index theory studies a topological invariant of a type of differential operators on a manifold, as well as the local formula of the invariant in terms of the geometry of the manifold.

- Motivating examples.
- Elliptic operators
- Atiyah-Singer index theorem.
- Dirac operators.
- Topological proof and geometric proof.

Relavent Areas



Fredholm Operator

Let H be a Hilbert space, and let

$$\mathcal{L}(H) = \{T : \sup_{\|x\|=1} \frac{\|Tx\|}{\|x\|} < \infty\}$$

be the set of bounded operators.

Let $\mathcal{K}(H)$ be the ideal of compact operators, that is, the norm limit of finite rank operators on H.

Definition

$F \in \mathcal{L}(H)$ is a Fredholm operator if it is invertible in $\mathcal{L}(H)/\mathcal{K}(H)$, that is, there exists $Q \in \mathcal{L}(H)$ so that

$$FQ - I, QF - I \in \mathcal{K}(H).$$

Fredholm Index

Proposition

A bounded operator F is Fredholm if and only if

- $\ker F$ *and* $\ker F^*$ *are of finite dimensional.*
- F *has closed range.*

Definition

The index of a Fredholm operator F is given by

$$\mathrm{ind} F = \dim \ker F - \dim \ker F^*.$$

Proposition

There exists a homomorphism $\mathrm{ind} : \mathcal{L}(H)/\mathcal{K}(H) \to \mathbb{Z}$. *In particular, if* $K \in \mathcal{K}(H)$, *then*

$$\mathrm{ind} F = \mathrm{ind}(F + K).$$

Toeplitz Operator

Let $L^2(S^1)$be the Hilbert space spanned by $e_k = e^{2\pi ik}$. Let P be the projection onto the Hardy space $H^2 := \text{span}\{e_k | k \geq 0\}$.

Let M_f be the multiplication operator on $L^2(S^1)$ by $f \in C(S^1)$.

Then

$$T_f := PM_fP : H^2 \to H^2$$

is called a Toeplitz operator.

Proposition

The Toeplitz operator T_f is a Fredholm operator if $f \neq 0$, that is, f reduces to $\tilde{f} : S^1 \to S^1$. Furthermore,

$$\text{ind}T_f = -\textit{winding number of } \tilde{f}.$$

Euler Number

Let S be a closed surface with a triangulation, and let V, E, F be the number of vertices, edges and faces respectively.

The Euler number

$$\chi(S) = V - E + F$$

is a topological invariant.

Let X be a compact Riemannian manifold and let $\chi(X)$ be its Euler number, defined by

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \dim H^i(X).$$

Recall that using the exterior derivative $d_i : \Lambda^i X \to \Lambda^{i+1} X$ we define $H^i(X) = \ker d_i / \mathrm{im} d_{i-1}$, the cohomology of X.

Gauss-Bonnet-Chern formula

Proposition

$d : \Lambda^{ev} X \to \Lambda^{od} X$, and then the de Rham operator $d + d^* : \Lambda^{ev} X \to \Lambda^{od} X$ is a Fredholm operator and

$$\mathrm{ind}(d + d^*) = \chi(X).$$

Theorem (Gauss-Bonnet-Chern)

Let R be the Riemannian curvature tensor of X, then

$$\mathrm{ind}(d + d^*) = \chi(X) = \int_X \mathrm{Pf}(R).$$

In particular, if $\dim X = 2$ and k is the Gauss curvature of X, then

$$\chi(X) = \int_X k(x) d\mathrm{vol}.$$

Elliptic Operators

In the study of a differential operator, that is, a system of PDE, on a compact manifold X, a type of operators, called elliptic operators, display similar properties of Fredholm operators.

The symbol of a differential operator

$$F = \sum p_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}}$$

is given by

$$\sum i^{|\alpha|} p_\alpha(x) \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}$$

and the principal symbol $\sigma_F(x, \xi)$ is the component of its highest order in ξ. σ_F defined globally on T^*X.

Definition

A differential operator F on a compact manifold M is elliptic if $\sigma_F(x, \xi)$ is invertible when $\xi \neq 0$.

Being Elliptic

Bounded solution?	$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0$	$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$
Symbol	$i(\xi_1 + \xi_2)$	$i\xi_1 - \xi_2$
Solution Type	$g(x - y)$	$g(x + iy) = g(z)$
Solution Space	infinite dimensional	has dimension 1
Solution Property	differentiable	analytic

Property of Elliptic Operators

The Schwartz kernel $k_F(x, y)$ of an operator F on Hilbert space $L^2(X, E)$ is given by

$$Fu(x) = \int_X k_F(x, y)u(y)dy.$$

F is called smoothing if k_F is smooth.

Theorem

Let P be an elliptic operator of order m on a compact manifold X, then there exists a operator Q, such that $PQ - I$ and $QP - 1$ are smoothing operators.

Remark

By carefully choosing the Hilbert space or normalizing P, P is made bounded, and hence is a Fredholm operator.

Algebraic Topology Model for Elliptic Operators

K-homology group
$K_0(X) = \{(H, \phi, P)\}/$ "appropriate" equivalence relations.

Here, $H = H_0 \oplus H_1$ is a Hilbert space, $\phi : C(X) \to \mathcal{L}(H)$ and $P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix} \in \mathcal{L}(H)$, satisfying $P^2 - 1 \in \mathcal{K}(H)$ and $[a, P] \in \mathcal{K}(H)$ for all $a \in C(X)$.

Remark

Let $P_0 : L^2(X, E_0) \to L^2(X, E_1)$ be a 0-order elliptic operator and $M : C_0(X) \to \mathcal{L}(L^2(X, E = E_0 \oplus E_1))$ be the point-wise multiplication, then $[(L^2(X, E), M, P)] \in K_0(X)$. The index of P_0 induces a homomorphism:

$$\mathrm{ind} : K_0(X) \to \mathbb{Z}.$$

Atiyah-Singer Index Theorem

Let X be a compact manifold and P be an elliptic operator, then the Fredfolm index of P is the index of P.

Theorem (Atiyah-Singer index formula)

$$\mathrm{ind} P = \int_{TX} \mathrm{ch}(\sigma_P)\mathrm{Td}(TX \otimes \mathbb{C}).$$

Here ch *is the Chern character map*

$$\mathrm{ch} : K_c^0(TX) \to H_c^{ev}(TX).$$

The Todd class of a complex vector bundle E is given by

$$\mathrm{Td}(E) := \det{}^{1/2} \frac{\Omega}{1 - e^{-\Omega}},$$

where Ω is the curvature of a connection of E.

Dirac Operator

Proposition

For any elliptic operator P on compact X, there is a Dirac type operator D so that $[P] = [D] \in K_0(X)$.

Dirac type operator is intimately related to the geometry of X and the symbol class of the operator.

The canonical Dirac operator is only defined for spin manifold X with spin bundle S. An operator D is a Dirac operator if

$$\sigma_{D^2}(x, \xi) = \|\xi\|_g^2,$$

where $\|\cdot\|_g$ is the norm of the convector ξ with respect to the Riemannian metric g.

Example

Let D be a order 1 operator on a n-dim torus $M = \mathbb{T}^n$, given by

$$D = \gamma_1 \frac{\partial}{\partial x_1} + \cdots + \gamma_n \frac{\partial}{\partial x_n},$$

where the Pauli matrices $\gamma_1, \cdots, \gamma_n$ are generators of a Clifford algebra, that is

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij}.$$

- When $n = 1$, $D = i\frac{\partial}{\partial x}$;
- When $n = 2$,

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial y} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0 \end{pmatrix}.$$

Properties

Proposition

$$D^2 = \Delta + \frac{\kappa}{4}.$$

Proposition

- *D is an unbounded selfadjoint operator on L^2.*
- *D has discrete eigenvalues tending to infinity, with finite-dimensional eigenspaces, that is, the resolvent operator $(D+i)^{-1}$ is compact.*
- *$[D, f]$ is bounded for $f \in C^\infty(M)$, the set of smooth functions on M.*

($\mathbb{Z}/2$-grading in the even-dimensional case).

Spectral Triple

Definition (Connes)

A spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$, where H is a Hilbert space, $\mathcal{A}$ is an algebra with a representation π on H, and D is a selfadjoint unbounded operator on H such that

- $(D+i)^{-1}$ is compact,
- $[D, a] := Da - aD$ is bounded, $\forall a \in \mathcal{A}$.

Let D be Dirac operator and S be the spin bundle over X, the Dirac spectral triple

$$(C^\infty(X), L^2(X, S), D)$$

recovers Riemannian geometry of X, where $C^\infty(X)$ reflects space information. and D reflects geometry information:

$$d(x, y) = \sup\{|f(x) - f(y)| : \|[D, f]\| \leq 1\}.$$

Index Theorem for Dirac operator

Corollary (Atiyah-Singer index theorem)

The index of the Dirac operator D on compact manifold X, is calculated by

$$\mathrm{ind} D = \int_X \hat{A}(X).$$

The characteristic class $\hat{A}$ is given by

$$\hat{A}(X) = \det{}^{1/2}\left(\frac{R/2}{\sinh R/2}\right)$$

and R is the Riemannian curvature tensor of X.

Proof: Topological Approach

Find index formula for any elliptic operator P over compact X.

Theorem (Kasparov)

$$K_0(X) \longleftarrow KK(C_0(X), C_0(TX)) \times K_0(TX)$$

$$[P] \quad = \quad [\sigma_P] \quad \otimes \quad [\mathcal{D}_{TX}]$$

Proof of AS by Atiyah-Singer, Kasparov).

We map X to a point, and taking Chern character on both sides. The left had side is $\mathrm{ind}P$, the right hand side is $\mathrm{ch}(\sigma_P)\mathrm{ch}(\mathcal{D}_{TX})$, where

$$\mathrm{ch}(\mathcal{D}_{TX}) = \int_{TX} \cdot \mathrm{Todd}(TX \otimes \mathbb{C}).$$

Proof: Geometrical Approach

Find index formula for the Dirac operator D on compact X.

Theorem

The heat operator $e^{-t\tilde{D}^2}, t > 0$ is a smoothing operator, and for all $t > 0$,

$$\mathrm{ind} D = \mathrm{tr}(e^{-tD^*D}) - \mathrm{tr}(e^{-tDD^*}).$$

Proof of AS by Getzler.

$\mathrm{tr}(e^{-tD^*D}) - \mathrm{tr}(e^{-tDD^*})$ is independent of t. As $t \to \infty$,

$$\mathrm{tr}(e^{-tD^*D}) - \mathrm{tr}(e^{-tDD^*}) \to \mathrm{tr} P_{kerD} - \mathrm{tr} P_{\ker D^*};$$

As $t \to 0+$,

$$\mathrm{tr}(e^{-tD^*D}) - \mathrm{tr}(e^{-tDD^*}) \to \int_X \hat{A}(X).$$

Heat Kernel Asymptotics

The Schwartz kernel of $e^{-t\tilde{D}^2}$ is called heat kernel $k_t(x,y)$. Then

$$\mathrm{str}\left(e^{-t\tilde{D}^2}\right) = \int_X k_t(x,x)dx.$$

As $t \to 0+$,

$$k_t(x,y) \sim \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t}[a_0(x,y)+ta_1(x,y)+t^2a_2(x,y)+\cdots],$$

where $a_k(x,x)$ is computed by algebraic expressions involving the metrics, connection coefficients and their derivatives.

For example,

$$a_0(x,x) = 1, a_1(x,x) = \frac{1}{6}\kappa(x),$$

where κ is the scalar curvature.

Variations of Atiyah-Singer index theorem

- Index theorems for families of elliptic operators.
- Index theorem for manifold with boundary, with singularities.
- Index theorem for some noncompact manifold or orbifold.
- L^2-index theorem.
- K-theoretic index theorem.
- Index theorem for "noncommutative spaces".