What is Index Theory (I)?

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Index theory studies a topological invariant of a type of differential operators on a manifold, as well as the local formula of the invariant in terms of the geometry of the manifold.

- Motivating examples.
- Elliptic operators
- Atiyah-Singer index theorem.
- Dirac operators.
- Topological proof and geometric proof.
Relavent Areas

noncommutative geometry

Geometry

local index formula

Operator Algebra

index, $K$-theory

Topology

characteristic class
Fredholm Operator

Let $H$ be a Hilbert space, and let

$$\mathcal{L}(H) = \{ T : \sup\|x\|=1 \frac{\|Tx\|}{\|x\|} < \infty \}$$

be the set of bounded operators.

Let $\mathcal{K}(H)$ be the ideal of compact operators, that is, the norm limit of finite rank operators on $H$.

**Definition**

$F \in \mathcal{L}(H)$ is a Fredholm operator if it is invertible in $\mathcal{L}(H)/\mathcal{K}(H)$, that is, there exists $Q \in \mathcal{L}(H)$ so that

$$FQ - I, QF - I \in \mathcal{K}(H).$$
Proposition

A bounded operator $F$ is Fredholm if and only if

- $\ker F$ and $\ker F^*$ are of finite dimensional.
- $F$ has closed range.

Definition

The index of a Fredholm operator $F$ is given by

$$\text{ind}F = \dim \ker F - \dim \ker F^*.$$  

Proposition

There exists a homomorphism $\text{ind} : \mathcal{L}(H)/\mathcal{K}(H) \to \mathbb{Z}$. In particular, if $K \in \mathcal{K}(H)$, then

$$\text{ind}F = \text{ind}(F + K).$$
Let $L^2(S^1)$ be the Hilbert space spanned by $e_k = e^{2\pi ik}$. Let $P$ be the projection onto the Hardy space $H^2 := \text{span}\{e_k | k \geq 0\}$. Let $M_f$ be the multiplication operator on $L^2(S^1)$ by $f \in C(S^1)$. Then

$$T_f := PM_fP : H^2 \to H^2$$

is called a Toeplitz operator.

**Proposition**

The Toeplitz operator $T_f$ is a Fredholm operator if $f \neq 0$, that is, $f$ reduces to $\tilde{f} : S^1 \to S^1$. Furthermore,

$$\text{ind}T_f = -\text{winding number of } \tilde{f}.$$
Euler Number

Let $S$ be a closed surface with a triangulation, and let $V, E, F$ be the number of vertices, edges and faces respectively.

The Euler number

$$\chi(S) = V - E + F$$

is a topological invariant.

Let $X$ be a compact Riemannian manifold and let $\chi(X)$ be its Euler number, defined by

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \dim H^i(X).$$

Recall that using the exterior derivative $d_i : \Lambda^i X \to \Lambda^{i+1} X$ we define $H^i(X) = \ker d_i / \text{im} d_{i-1}$, the cohomology of $X$. 
Gauss-Bonnet-Chern formula

**Proposition**

d : $\Lambda^{ev}X \rightarrow \Lambda^{od}X$, and then the de Rham operator

d + d^* : $\Lambda^{ev}X \rightarrow \Lambda^{od}X$ is a Fredholm operator and

$$\text{ind}(d + d^*) = \chi(X).$$

**Theorem (Gauss-Bonnet-Chern)**

Let $R$ be the Riemannian curvature tensor of $X$, then

$$\text{ind}(d + d^*) = \chi(X) = \int_X \text{Pf}(R).$$

In particular, if $\dim X = 2$ and $k$ is the Gauss curvature of $X$, then

$$\chi(X) = \int_X k(x) dvol.$$
Elliptic Operators

In the study of a differential operator, that is, a system of PDE, on a compact manifold $X$, a type of operators, called elliptic operators, display similar properties of Fredholm operators.

The symbol of a differential operator

$$F = \sum p_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}}$$

is given by

$$\sum i^{|\alpha|} p_\alpha(x) \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}$$

and the principal symbol $\sigma_F(x, \xi)$ is the component of its highest order in $\xi$. $\sigma_F$ defined globally on $T^*X$.

**Definition**

A differential operator $F$ on a compact manifold $M$ is elliptic if $\sigma_F(x, \xi)$ is invertible when $\xi \neq 0$. 
<table>
<thead>
<tr>
<th>Bounded solution?</th>
<th>$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0$</th>
<th>$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symbol</td>
<td>$i(\xi_1 + \xi_2)$</td>
<td>$i\xi_1 - \xi_2$</td>
</tr>
<tr>
<td>Solution Type</td>
<td>$g(x - y)$</td>
<td>$g(x + iy) = g(z)$</td>
</tr>
<tr>
<td>Solution Space</td>
<td>infinite dimensional</td>
<td>has dimension 1</td>
</tr>
<tr>
<td>Solution Property</td>
<td>differentiable</td>
<td>analytic</td>
</tr>
</tbody>
</table>
The Schwartz kernel $k_F(x, y)$ of an operator $F$ on Hilbert space $L^2(X, E)$ is given by

$$F u(x) = \int_X k_F(x, y) u(y) dy.$$ 

$F$ is called smoothing if $k_F$ is smooth.

**Theorem**

Let $P$ be an elliptic operator of order $m$ on a compact manifold $X$, then there exists a operator $Q$, such that $PQ - I$ and $QP - 1$ are smoothing operators.

**Remark**

By carefully choosing the Hilbert space or normalizing $P$, $P$ is made bounded, and hence is a Fredholm operator.
**Algebraic Topology Model for Elliptic Operators**

**K-homology group**

\[ K_0(X) = \{(H, \phi, P)\}/ \text{“appropriate” equivalence relations.} \]

Here, \( H = H_0 \oplus H_1 \) is a Hilbert space, \( \phi : C(X) \to \mathcal{L}(H) \) and \( P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix} \in \mathcal{L}(H) \), satisfying \( P^2 - 1 \in \mathcal{K}(H) \) and 

\[ [a, P] \in \mathcal{K}(H) \text{ for all } a \in C(X). \]

**Remark**

Let \( P_0 : L^2(X, E_0) \to L^2(X, E_1) \) be a 0-order elliptic operator and \( M : C_0(X) \to \mathcal{L}(L^2(X, E = E_0 \oplus E_1)) \) be the point-wise multiplication, then \([ (L^2(X, E), M, P) ] \in K_0(X). \) The index of \( P_0 \) induces a homomorphism:

\[ \text{ind} : K_0(X) \to \mathbb{Z}. \]
Atiyah-Singer Index Theorem

Let $X$ be a compact manifold and $P$ be an elliptic operator, then the Fredholm index of $P$ is the index of $P$.

**Theorem (Atiyah-Singer index formula)**

$$\text{ind} P = \int_{TX} \text{ch}(\sigma_P) \text{Td}(TX \otimes \mathbb{C}).$$

Here $\text{ch}$ is the *Chern character* map

$$\text{ch} : K^0_c(TX) \rightarrow H^*_c(TX).$$

The *Todd class* of a complex vector bundle $E$ is given by

$$\text{Td}(E) := \det^{1/2} \frac{\Omega}{1 - e^{-\Omega}},$$

where $\Omega$ is the curvature of a connection of $E$. 
Proposition

For any elliptic operator $P$ on compact $X$, there is a Dirac type operator $D$ so that $[P] = [D] \in K_0(X)$.

Dirac type operator is intimately related to the geometry of $X$ and the symbol class of the operator.

The canonical Dirac operator is only defined for spin manifold $X$ with spin bundle $S$. An operator $D$ is a Dirac operator if

$$\sigma_{D^2}(x, \xi) = \|\xi\|^2_g,$$

where $\| \cdot \|_g$ is the norm of the convector $\xi$ with respect to the Riemannian metric $g$. 
Example

Let $D$ be a order 1 operator on a $n$-dim torus $M = \mathbb{T}^n$, given by

$$D = \gamma_1 \frac{\partial}{\partial x_1} + \cdots + \gamma_n \frac{\partial}{\partial x_n},$$

where the Pauli matrices $\gamma_1, \cdots, \gamma_n$ are generators of a Clifford algebra, that is

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij}.$$

- When $n = 1$, $D = i \frac{\partial}{\partial x}$;
- When $n = 2$,

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial y} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} & 0 \end{pmatrix}.$$
Properties

Proposition

\[ D^2 = \Delta + \frac{\kappa}{4}. \]

Proposition

\begin{itemize}
  \item \textit{D is an unbounded selfadjoint operator on }\textit{L}^2.\
  \item \textit{D has discrete eigenvalues tending to infinity, with finite-dimensional eigenspaces, that is, the resolvent operator }\textit{(D + i)^{-1} is compact.}\n  \item \textit{[D, f] is bounded for }f \in C^\infty(M), \textit{the set of smooth functions on }M.\n\end{itemize}

\textit{(Z/2-grading in the even-dimensional case).}
Spectral Triple

Definition (Connes)

A spectral triple is a triple \((\mathcal{A}, \mathcal{H}, D)\), where \(H\) is a Hilbert space, \(\mathcal{A}\) is an algebra with a representation \(\pi\) on \(H\), and \(D\) is a selfadjoint unbounded operator on \(H\) such that

- \((D + i)^{-1}\) is compact,
- \([D, a] := Da - aD\) is bounded, \(\forall a \in \mathcal{A}\).

Let \(D\) be Dirac operator and \(S\) be the spin bundle over \(X\), the Dirac spectral triple

\[(C^\infty(X), L^2(X, S), D)\]

recovers Riemannian geometry of \(X\), where \(C^\infty(X)\) reflects space information, and \(D\) reflects geometry information:

\[d(x, y) = \sup\{|f(x) - f(y)| : \|[D, f]\| \leq 1\}\.\]
Index Theorem for Dirac operator

Corollary (Atiyah-Singer index theorem)

The index of the Dirac operator $D$ on compact manifold $X$, is calculated by

$$\text{ind} D = \int_X \hat{A}(X).$$

The characteristic class $\hat{A}$ is given by

$$\hat{A}(X) = \det^{1/2} \left( \frac{R/2}{\sinh R/2} \right)$$

and $R$ is the Riemannian curvature tensor of $X$. 
Proof: Topological Approach

Find index formula for any elliptic operator $P$ over compact $X$.

**Theorem (Kasparov)**

$$K_0(X) \leftarrow KK(C_0(X), C_0(TX)) \times K_0(TX)$$

$$[P] \quad \bigotimes \quad [\sigma_P] \quad \bigotimes \quad [D_{TX}]$$

**Proof of AS by Atiyah-Singer, Kasparov.**

We map $X$ to a point, and taking Chern character on both sides. The left hand side is $\text{ind} P$, the right hand side is $\text{ch}(\sigma_P)\text{ch}(D_{TX})$, where

$$\text{ch}(D_{TX}) = \int_{TX} \cdot \text{Todd}(TX \otimes \mathbb{C}).$$
Proof: Geometrical Approach

Find index formula for the Dirac operator $D$ on compact $X$.

Theorem

The heat operator $e^{-t\tilde{D}^2}$, $t > 0$ is a smoothing operator, and for all $t > 0$,

$$\text{ind}D = \text{tr}(e^{-tD^*D}) - \text{tr}(e^{-tDD^*}).$$

Proof of AS by Getzler.

$\text{tr}(e^{-tD^*D}) - \text{tr}(e^{-tDD^*})$ is independent of $t$. As $t \to \infty$,

$$\text{tr}(e^{-tD^*D}) - \text{tr}(e^{-tDD^*}) \to \text{tr}P_{kerD} - \text{tr}P_{kerD^*};$$

As $t \to 0+$,

$$\text{tr}(e^{-tD^*D}) - \text{tr}(e^{-tDD^*}) \to \int_X \hat{A}(X).$$
The Schwartz kernel of $e^{-t\tilde{D}^2}$ is called heat kernel $k_t(x, y)$. Then

$$\text{str} \left( e^{-t\tilde{D}^2} \right) = \int_X k_t(x, x) dx.$$ 

As $t \to 0+$,

$$k_t(x, y) \sim \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t} [a_0(x, y) + ta_1(x, y) + t^2 a_2(x, y) + \cdots],$$

where $a_k(x, x)$ is computed by algebraic expressions involving the metrics, connection coefficients and their derivatives.

For example,

$$a_0(x, x) = 1, a_1(x, x) = \frac{1}{6} \kappa(x),$$

where $\kappa$ is the scalar curvature.
Variations of Atiyah-Singer index theorem

- Index theorems for families of elliptic operators.
- Index theorem for manifold with boundary, with singularities.
- Index theorem for some noncompact manifold or orbifold.
- $L^2$-index theorem.
- $K$-theoretic index theorem.
- Index theorem for “noncommutative spaces”.