The area I am interested is the interaction between Topology and Geometry.

In topology, we care about "topological invariants" (something quantity depending on an object (say, manifold) that does not change when you deform the geometric object continuously).

For example, Euler number of a surface is a topological invariant.

\[ X(\infty) = V - E + F = 7 - 15 + 10 = 2 \]

This does not depend on the choice of triangulations and does not change under homeomorphism.

Topological invariants are invariant to classify geometric object.

For example, \( X(\infty) = 2, X(\infty) = 0 \)

\[ X(\infty) = -2 \]

Different Euler numbers \( \Rightarrow \) the geometric surfaces are not homeomorphic to each other.
Topology is flexible, geometry is rigid, but topology provides obstructions to existence of certain geometry.

For example, we learned Gauss–Bonnet formula:

\[ X(M) = \frac{1}{2\pi} \int_M k \ d\text{vol}(M) \]

where \( k \) is the Gaussian curvature on \( M \):

- \( k = 0 \)
- \( k < 0 \)
- \( k > 0 \)

\[ X(\mathbb{S}^2) = 0 \]

The Gauss–Bonnet formula tells us that a torus cannot have positive scalar curvature everywhere.

\[ k < 0 \]

\[ k > 0 \]

(You can use a piece of paper \((k = 0)\) to glue a torus, but not a ball or a 2–torus.)

There is an area called "index theory" created by Atiyah–Singer, studying generalised Gauss–Bonnet theorem, which relates geometry and topology of a geometric object.

Atiyah used some differential operators, known as elliptic operators, on compact manifold and define the index for such kind of operators.
\[ D : C^\infty(M, E) \rightarrow C^\infty(M, E) \] 

index \( D = \dimker D - \dimcoker D \)

This is an invariant measuring how far is \( D \) from being invertible. (Note: if \( D = id \), \( \text{ind} D = 0 \)).

Atiyah-Singer studied this index and relate it to the topology and geometry of \( M \).

For example, think of differentials on the space of forms:

\[ \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \rightarrow \cdots \]

we have the de Rham operator \( d + d^* : \Omega^{ev}(M) \rightarrow \Omega^{odd}(M) \)

\( d + d^* \) can be verified to be elliptic on \( M \).

and \( \text{ind} (d + d^*) = \chi(M) = \frac{1}{2\pi i} \int_M K \)

\[ \uparrow \text{ Gauss-Bonnet} \]

This can be obtained using homology & cohomology of \( M \), a group-valued topological invariants of \( M \).

This suggests that solution space of elliptic differential operator is related to geometry and topology closely.
I am not only interested in nice geometric objects such as surfaces or manifolds. I am also interested in spaces with bad topology. That is where noncommutative geometry comes as part of my research interests.

**Commutative** geometry

Hausdorff space \( M \) (compact) \( \leftrightarrow \) Algebra of continuous functions \( C(M) \) on \( M \).

\( x \) corresponds to maximal ideals.

\( I_x = \{ f \in C(M) | f(x) = 0 \} \).

**Noncommutative** geometry

Space with "bad" topology \( \leftrightarrow \) Noncommutative algebra.

- n point space with trivial topology \( \leftrightarrow \) \( M_n(\mathbb{C}) \)
- \( S^1/\mathbb{Z} \) (quotient space of irrational rotations on \( S^1 \)) \( \leftrightarrow \) noncommutative torus \( A_\theta \) algebra generated by \( u, v \in C(S^1) \) where \( uv = e^{2\pi i \theta} v u \), \( \theta \in \mathbb{R} \) (Note: \( \theta = 0 \Rightarrow A_0 = C(T^2) \)) \( \leftrightarrow \) \( C(S^2) \times \mathbb{Z}_2 \)
It is an interesting project to study topological invariants for noncommutative spaces. For example, can we talk about Euler char, genus, for these noncommutative spaces?

What do I do?

Index theory of elliptic operators on a manifold with group action.

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this is a noncommutative space.

[Relationship between geometry and topology in the context of noncommutative geometry.]

In detail, I am interested in

- How to define index (when dim ker D = \infty).
- Noncommutative geometry involving group action.
- K-theory and (co)homology in NCG.
- Co homological formula for index of D (analogue to Gauss-Bonnet formula).
- Relation to representation theory of groups.
- Noncommutative harmonic analysis, Fourier transform on non-abelian groups.

My research area is related closely to other group members in pure math.