PARABOLIC EXHAUSTIONS AND ANALYTIC COVERINGS

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Abstract. Let τ be a parabolic exhaustion on a Stein manifold X such that τ is strictly plurisubharmonic at its zeros. The metric defined by τ on the complement of its degeneracy locus D is shown to be flat if τ is real-analytic or if "most" leaves of the associated Monge-Ampère foliation $\mathcal{F}$ abut the zeros of τ. Then, by an analysis of the singularities of τ, we show that the tangent bundle of $X \setminus D$ extends to a flat hermitian bundle on X with a holomorphic section s such that $\tau = \|s\|^2$, and that $\mathcal{F}$ extends to a singular holomorphic foliation of X. Also, τ is the length-squared of an analytic covering of X onto a ball if and only if the monodromy of the τ-connection is trivial. We obtain a characterization of affine algebraic manifolds as those X possessing τ with finite monodromy and affine leaves.

1. Introduction.

Let X be an n-dimensional Stein manifold and $f : X \to B_R^n$ be a proper holomorphic map, i.e., a finite branched analytic covering, onto the ball of radius $\sqrt{R}$ in $\mathbb{C}^n$, $R \in (0, +\infty]$. Such a map gives rise to a smooth exhaustion $\tau = |f|^2 : X \to [0, R)$, which is said to be *parabolic* because its logarithm ρ is plurisubharmonic and satisfies the so-called Monge-Ampère equation $(\partial\bar{\partial}\rho)^n = 0$ on $X' = X \setminus \tau^{-1}(0)$. This means that the complex Hessian of ρ has vanishing determinant in any local coordinates. The Monge-Ampère equation is a natural generalization of Laplace's equation to higher dimensions and has found many applications in complex analysis.

Unbounded parabolic exhaustions have been used in Nevanlinna theory, in particular for algebraic submanifolds of $\mathbb{C}^N$, where f can be taken to be a generic linear projection of X onto a linear subspace of the same dimension. Cornalba, Griffiths and others have asked if possessing a parabolic exhaustion with some special properties characterizes affine algebraic manifolds. It would also be of interest to know when a parabolic exhaustion arises from an analytic covering. This work is a study of parabolic exhaustions and their singularities. In particular, the two above questions will be addressed.

A parabolic exhaustion τ gives rise to a rich geometric structure on X. The Levi form $i\partial\bar{\partial}\tau$ defines a Hermitian metric on the complement of the degeneracy locus $D =$

$\{(\partial\bar\partial\tau)^n = 0\}$, and there is a smooth foliation $\mathcal{F}$ by Riemann surfaces on the complement in X' of the singularity set $E = \{(\partial\bar\partial\rho)^{n-1} = 0\} \subset D$, such that ρ is harmonic along every leaf. Foliations that arise in this fashion from the Monge-Ampère equation have been studied since the late seventies by Bedford, Burns, Duchamp, Kalka, Lempert, Patrizio, Stoll, Wong and others.

If τ is strictly plurisubharmonic everywhere, i.e., $D = \varnothing$, then Stoll has shown that up to a biholomorphism, $X = B_R^n$ and $\tau = |\cdot|^2$. In general, τ must be strictly plurisubharmonic somewhere on X; we assume that this holds at the zeros of τ. (Note that if τ is given by a covering f, then this can always be achieved by postcomposing f with a generic automorphism of B_R^n.) Under this hypothesis, we prove the following structure theorem, which is the main result of the paper. Without the hypothesis, the theorem fails.

Main Theorem. *Let τ be a parabolic exhaustion on a Stein manifold X of dimension n such that τ is strictly plurisubharmonic at its zeros. Then the following conditions are equivalent.*

- (i) *The τ-metric on $X \setminus D$ is flat.*
- (ii) *τ is real-analytic.*
- (iii) *ρ is unbounded below on every leaf of $\mathcal{F}$.*
- (iv) *Locally, $\tau = |f|^2$, where f is a holomorphic map into $\mathbb{C}^n$.*
- (v) *$\mathcal{F}$ extends to a singular holomorphic foliation of X'.*
- (vi) *There exists a holomorphic section s of a flat hermitian vector bundle of rank n over X such that $\tau = \|s\|^2$.*

Moreover, there is an analytic covering $f : X \to B_R^n$ such that $\tau = |f|^2$ if and only if these conditions are satisfied and the monodromy of the τ-connection is trivial.

I do not know if conditions (i)–(vi) always hold for a parabolic exhaustion as in the theorem.

As an application, we give the following solution to the problem of characterizing affine algebraic manifolds by means of parabolic exhaustions.

Algebraicity Theorem. *A Stein manifold X is affine algebraic if and only if X has a parabolic exhaustion τ such that*

- (a) *τ is strict at its zeros,*
- (b) *the leaves of the Monge-Ampère foliation $\mathcal{F}$ of $X' \setminus E$ are affine curves, and*
- (c) *the τ-connection has finite monodromy.*

We shall see that (a) and (b) actually imply that the τ-metric is flat, so (c) makes sense. It would be interesting to have a simple condition on τ implying (b) or merely that the leaves have finitely generated homology.

The paper is organized as follows. In sections 2 and 3 we summarize the necessary background material on singular holomorphic foliations and the geometry of strictly parabolic functions.

In section 4 we set the stage for the proof of the main theorem.

In section 5 we give an easy and complete analysis of parabolic exhaustions on Riemann surfaces. This should motivate and clarify our approach to the general case. We also give two one-dimensional examples of parabolic exhaustions which do not arise from analytic coverings.

In section 6 we show that vanishing of the curvature of the τ-metric propagates along leaves of $\mathcal{F}$. Applying Stoll's theorem on connected components of $\{\tau < r\}$ with r sufficiently small, we see that near every zero of τ the τ-metric is flat. Therefore, the τ-metric is flat if ρ is unbounded below on "most" leaves of $\mathcal{F}$.

In section 7, we assume that the τ-metric is flat on $X \setminus D$. The gradient of τ in the τ-metric is a holomorphic vector field on $X \setminus D$ and $\|\operatorname{grad}\tau\|^2 = \tau$ (this is in fact equivalent to the Monge-Ampère equation). The idea is to extend the flat bundle $T(X \setminus D)$ to a unitary local system $\mathcal{L}$ on all of X and obtain from $\operatorname{grad}\tau$ a holomorphic section s of the flat bundle $\mathcal{L} \otimes \mathcal{O}$ such that $\tau = \|s\|^2$.

This is possible because the singularities that might interfere are not too severe. Namely, using a theorem of Simha (essentially dating back to Hartogs) on analyticity of certain singularity sets, we show that D is an analytic hypersurface (unless it is empty) and that E is analytic of codimension at least two. Furthermore, D is seen to be generically transverse to $\mathcal{F}$. Knowing this, we can prove that the flat canonical connection of the τ-metric extends to a regular meromorphic connection on all of X with no monodromy locally at points of D.

Thus, $\tau = \|s\|^2$ where s is a holomorphic section of a flat hermitian vector bundle of rank n on X. In particular, τ is real-analytic. We see that τ comes from an analytic covering of B^n_R if and only if the monodromy representation of the fundamental group $\pi_1(X)$ into the unitary group $U(n)$, given by the τ-connection, is trivial.

In section 8, we complete the proof of the main theorem. We show that the τ-metric is flat if τ is real-analytic, because then D is pluripolar and cannot disconnect X.

We also prove that $\mathcal{F}$ extends to a singular holomorphic foliation on all of X', assuming condition (vi) in the Main Theorem. To get an atlas for such a foliation, we observe that, regardless of monodromy, locally $\tau = |f|^2$, where f is an analytic covering of an open subset of $\mathbb{C}^n$. We postcompose these with the projection onto $\mathbb{P}^{n-1}$ to obtain local maps whose fibres in $X' \setminus E$ are just plates of the original foliation. These maps do form the required atlas; the non-trivial part of the proof uses a theorem of Bohnhorst to show that they are simple.

Having proved the main theorem, we show that it fails if we do not require τ to be strict at its zeros.

Section 9 contains some further remarks on the Monge-Ampère foliation.

In section 10, we prove the Algebraicity Theorem. If the τ-metric is flat with trivial monodromy and $R = +\infty$, then $\tau = |f|^2$ where $f : X \to \mathbb{C}^n$ is an analytic covering. By the *Fortsetzungssatz* of Grauert and Remmert, if the critical locus of f is algebraic, then X is affine algebraic. We prove that this holds if and only if the leaves of the foliation $\mathcal{F}$ are affine curves. It actually suffices to assume that the monodromy is finite.

Finally, in section 11, we briefly discuss what happens to our theory if we omit the assumption that τ be proper.

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2. Holomorphic foliations.

This section contains some necessary background material on holomorphic foliations. The following definitions are due to Holmann [Ho1; Ho2].

A local holomorphic foliation on a complex manifold X is a simple open holomorphic map $\phi : U \to V$, where $U \subset X$ is open and V is a complex manifold. By definition, ϕ is *simple* if there is a basis $\mathcal{B}$ for the topology of U such that $\phi|W$ has connected fibres for all $W \in \mathcal{B}$. Two local holomorphic foliations $\phi_i : U_i \to V_i$, $i = 1, 2$, are *compatible* if for every $x \in U_1 \cap U_2$ there is a neighbourhood $W \subset U_1 \cap U_2$ of x and a biholomorphism $h : \phi_1(W) \to \phi_2(W)$ such that $h \circ \phi_1|W = \phi_2|W$. *A holomorphic foliation* $\mathcal{F}$ of X is a maximal atlas of mutually compatible local holomorphic foliations on X. These local holomorphic foliations are called $\mathcal{F}$-*charts* and their fibres are sometimes called *plates* of $\mathcal{F}$.

Let $\mathcal{F}$ be a holomorphic foliation of a complex manifold X. Take as a basis for a new topology on X the fibres of all the $\mathcal{F}$-charts. The *leaves* of $\mathcal{F}$ are the connected components of X in this topology. Each leaf is second countable and has an induced complex structure. They all have the same pure dimension, which is called the *dimension* of $\mathcal{F}$.

The following useful lemma says, roughly speaking, that a holomorphic foliation looks the same near each point of any of its leaves.

2.1. Lemma (Holmann [Ho1]**).** *Let $\mathcal{F}$ be a holomorphic foliation of a complex manifold X and $p_1, p_2 \in X$ belong to the same leaf of $\mathcal{F}$. Then there exist $\mathcal{F}$-charts $\phi_i : U_i \to V_i$, $i = 1, 2$, with $p_i \in U_i$, and a biholomorphism $h : V_1 \to V_2$ such that $\phi_1^{-1}(x)$ and $\phi_2^{-1}(h(x))$ belong to the same leaf for every $x \in V_1$.*

As an immediate consequence we have:

2.2. Corollary. *The closure of a leaf in a holomorphic foliation is saturated.*

A set is *saturated* if it is a union of leaves. *The saturation* of a set is the union of all leaves intersecting it.

Lemma 2.1 is also used to prove the following facts.

2.3. Lemma (Holmann [Ho1]**).** *Let $\phi : U \to V$ be an $\mathcal{F}$-chart and say that $x, y \in V$ are equivalent if $\phi^{-1}(x)$ and $\phi^{-1}(y)$ belong to the same leaf of $\mathcal{F}$.*

(1) This defines an open equivalence relation on V. Hence, the saturation of an open set in X is open.

(2) An equivalence class is at most countable and a closed equivalence class is discrete.

2.4. Corollary (Holmann [Ho1]**).** *A leaf of a holomorphic foliation of a complex manifold X is a (locally) analytic subvariety of X if and only if it is (locally) closed in X.*

A holomorphic foliation $\mathcal{F}$ of a complex manifold X is *smooth* if all the $\mathcal{F}$-charts are submersions. Then the leaves of $\mathcal{F}$ are smooth. By the theorem of Frobenius, a smooth holomorphic foliation of X is nothing but a subbundle of the holomorphic tangent bundle TX which is involutive, i.e., closed under the bracket. This subbundle consists of all tangent vectors to the leaves of $\mathcal{F}$.

To any holomorphic foliation $\mathcal{F}$ of X there is associated an involutive coherent analytic subsheaf $T\mathcal{F}$ of TX of tangent vectors to the leaves of $\mathcal{F}$. For any $\mathcal{F}$-chart $\phi : U \to V$, $T\mathcal{F}|U$ is the kernel of the tangent map $\phi_* : TU \to TV$. Baum and Bott [BauBo; Bau] have given an alternative definition of a holomorphic foliation of a complex manifold X as an involutive coherent analytic subsheaf of TX. According to this definition, if $\mathcal{F}$ is a holomorphic foliation of X and $f : Y \to X$ is any holomorphic map, then the pullback $f^*\mathcal{F}$ is a holomorphic foliation of Y. Holmann's definition is stronger and does not enjoy this property without restrictions on the map f.

A third definition is due to Gómez-Mont [Gó-M]. He defines a holomorphic foliation to be a smooth holomorphic foliation of the complement of a closed analytic subvariety of codimension at least two.

The singularity set E of a holomorphic foliation $\mathcal{F}$ of a complex manifold X is the set of critical points of the $\mathcal{F}$-charts. This is a closed analytic subvariety of X, which may also be described as the set of points at which the quotient sheaf $TX/T\mathcal{F}$ is not locally free. The restriction of $\mathcal{F}$ to $X \setminus E$ is a smooth holomorphic foliation. We shall distinguish between leaves of $\mathcal{F}$ and leaves of the restriction by referring to the latter as *smooth leaves.*

2.5. Proposition. *Let $\mathcal{F}$ be a holomorphic foliation of a complex manifold X with singularity set E. If M is a leaf of $\mathcal{F}$, then $E \cap M$ is a thin analytic subset of M. In particular, $M \not\subset E$. Also, $M \setminus E$ is a union of smooth leaves, each of which is the complement of E in an irreducible component of M.*

This is is a direct consequence of the following characterization of simple open maps.

2.6. Theorem (Bohnhorst [Boh]**).** *Let X, Y be complex manifolds and $\phi : X \to Y$ be a holomorphic map. Suppose ϕ is open, so all fibres of ϕ have pure dimension $k = \dim X - \dim Y$. Let E be the set of critical points of ϕ. Then ϕ is simple if and only if*

$$\dim E \cap \phi^{-1}(y) < k \quad \textit{for all } y \in Y.$$

Finally, we shall prove an analogue of Corollary 2.2 for irreducible components of leaves.

2.7. Lemma. *Let $\mathcal{F}$ be a holomorphic foliation of a complex manifold X. Then the closure of an irreducible component of a leaf of $\mathcal{F}$ is a union of irreducible components of leaves.*

Proof. If p is a point outside the singularity set E of $\mathcal{F}$, let C_p denote the irreducible component through p of the leaf through p, and M_p denote the smooth leaf through p.

Let M be a smooth leaf contained in an irreducible component C of a leaf of $\mathcal{F}$ and let $p \in \overline{C} \setminus C$. If $p \notin E$, then $M_p \subset \overline{M}$ by Lemma 2.2, and $\overline{M} = \overline{C}$, so $C_p \subset \overline{C}$. Suppose $p \in E$. We need to show that there is $q \notin E$ with $p \in C_q$ and $q \in \overline{C}$, so $C_q \subset \overline{C}$.

Let $\phi : U \to V$ be an $\mathcal{F}$-chart at p and denote the fibre $\phi^{-1}(\phi(p))$ by F. There is a sequence (y_i) in V converging to $\phi(p)$ and irreducible components $C_i \subset C$ of $\phi^{-1}(y_i)$, whose union A is a closed analytic subset of $U \setminus F$, such that $p \in \overline{A}$. Now $\overline{A}$ is clearly not analytic in any neighbourhood of p, so by the Remmert-Stein theorem [ReSt], $\overline{A}$ contains an irreducible component C' of F with $p \in C'$. Since $E \cap F$ is thin in F, C' must contain a point q as above. $\square$

3. The geometry of strictly parabolic functions.

In this section we will summarize the basic results on the Monge-Ampère foliation induced by a strictly parabolic function, as well as the various associated metrics and their curvatures. The references [BeBu], [BeKa], [Bu1], [Bu2], [DuKa1], [DuKa2] and [Wo] contain all the facts that we state without proof.

Let X be an n-dimensional complex manifold and ρ be a smooth plurisubharmonic function on X satisfying the homogeneous Monge-Ampère equation

$$(\partial\bar\partial\rho)^n = 0$$

with the non-degeneracy condition

$$\operatorname{rank} \partial\bar\partial\rho = n - 1.$$

Then the annihilator of $\partial\bar\partial\rho$ is a smooth complex line subbundle of the holomorphic tangent bundle TX, integrable as a subbundle of the real tangent bundle, and hence gives rise to a smooth (but not necessarily holomorphic) foliation $\mathcal{F}$ of X by Riemann surfaces. Along every leaf, ρ is harmonic and the (1,0)-derivative $\partial\rho$ is holomorphic. Also, if a smooth function f on X is holomorphic along the leaves, then the interior of the zero locus of $\bar\partial f$ is saturated.

Let $N = TX/T\mathcal{F}$ be the normal bundle of $\mathcal{F}$ and $N^{\mathbb{C}} = N \oplus \overline{N}$ be the complexification of the real normal bundle. Restricted to a leaf, $N^{\mathbb{C}}$ is flat: we have a natural notion of parallel translation of a normal vector along a leaf. This determines a flat partial connection on $N^{\mathbb{C}}$, the Bott connection

$$\nabla : N^{\mathbb{C}} \to N^{\mathbb{C}} \otimes T^\vee\mathcal{F}, \quad \nabla_v(w) = \text{projection of } [v, \text{lifting of } w].$$

The extent to which ∇ fails to respect the complex structure of X is measured by the anti-holomorphic torsion tensor $\overline{N} \to N \otimes T^\vee \mathcal{F}$, which vanishes if and only if the foliation $\mathcal{F}$ is holomorphic.

The normal bundle N carries a hermitian metric, called *the ρ-metric*, defined by the $(1,1)$-form $i\partial\bar\partial\rho$, which is non-positively curved along leaves. The foliation $\mathcal{F}$ is holomorphic if and only if the curvature (or just the Ricci curvature) vanishes along every leaf, and then the metric connection $N \to N \otimes T^\vee X$ restricts to the Bott connection.

Let us now consider the case when $\tau = e^\rho$ is strictly plurisubharmonic and thus defines a metric on X.

A parabolic function on an n-dimensional complex manifold X is a smooth non-negative function τ on X whose logarithm ρ is plurisubharmonic and satisfies the homogeneous Monge-Ampère equation $(\partial\bar\partial\rho)^n = 0$ on $X' = X \setminus \tau^{-1}(0)$. Then ρ and τ are plurisubharmonic on all of X. We say that τ is *strictly parabolic* on X if it is strictly plurisubharmonic at every point of X. The closed subset D of X where τ is not strictly plurisubharmonic is called *the degeneracy locus* of τ.

Example. If $f_1, \dots, f_n$ are holomorphic functions on X, then $\tau = |f_1|^2 + \cdots + |f_n|^2$ is a parabolic function on X. The degeneracy locus of τ is the set of points where the Jacobian determinant $\det(\partial f_i/\partial z_j)$ vanishes, i.e., where the map $(f_1, \dots, f_n) : X \to \mathbb{C}^n$ fails to be a local biholomorphism. More generally, if s is a holomorphic section of a flat hermitian vector bundle of rank n on X, then $\|s\|^2$ is a parabolic function on X.

Now let τ be a positive strictly parabolic function on X. The Monge-Ampère condition is equivalent to the equation

$$(\partial\bar\partial\tau)^n = (\partial\bar\partial e^\rho)^n = \tau^n(\partial\bar\partial\rho + \partial\rho\wedge\bar\partial\rho)^n = n\tau^n\partial\rho\wedge\bar\partial\rho\wedge(\partial\bar\partial\rho)^{n-1}, \tag{3.1}$$

so $\operatorname{rank}\partial\bar\partial\rho = n-1$ and we have an associated foliation $\mathcal{F}$ of X with all the properties described above.

The $(1,1)$-form $i\partial\bar\partial\tau$ defines a Kähler metric on X, called *the τ-metric*, with norm $\|\cdot\|$. The leaves of $\mathcal{F}$ are totally geodesic in this metric, i.e., the second fundamental form of $T\mathcal{F}$ in TX vanishes along every leaf. This is the central fact relating the metric and the foliation.

Since the following proposition is not explicitly contained in any of our references, we shall, for completeness, give a detailed proof of it.

3.1. Proposition. *The quotient metric in N induced by the τ-metric in TX is equal to the ρ-metric multiplied by τ.*

Proof. Let $z_1, \dots, z_n$ be local coordinates on X such that $\{z_2, \dots, z_n = 0\}$ is a plate M of $\mathcal{F}$ with coordinate z_1. With respect to the frame $\partial/\partial z_2, \dots, \partial/\partial z_n$ for $N|M$, the ρ-metric is given by the matrix $H_\rho = (\rho_{i\bar j})_{i,j=2}^n$. (Here, and frequently in what follows, subscripts are used to denote partial derivatives.)

The dual bundle $N^\vee$ of N is a subbundle of the cotangent bundle $T^\vee X$, so the dual of the τ-metric defines a metric in $N^\vee$ in the obvious way. By definition, the quotient

τ-metric in N is the dual of this metric, so with respect to the above frame, it is given by the matrix

$$^{t}[(\tau^{i\bar{j}})_{i,j=2}^{n}]^{-1}.$$

Here, $(\tau^{i\bar{j}})_{i,j=1}^{n}$ denotes the transposed inverse of the Hessian $H_\tau = (\tau_{i\bar{j}})_{i,j=1}^{n}$ of τ. Similarly, $(\rho^{i\bar{j}})_{i,j=2}^{n}$ will denote the transposed inverse of H_ρ.

On M, we have $\rho_{1\bar{j}} = 0$ for $j = 1, \ldots, n$, so

$$\det H_\tau = \det(\tau(\rho_{i\bar{j}} + \rho_i\rho_{\bar{j}})) = \tau^n \begin{vmatrix} \rho_1\rho_{\bar{1}} & \rho_1\rho_{\bar{2}} & \cdots & \rho_1\rho_{\bar{n}} \\ \rho_2\rho_{\bar{1}} & \rho_{2\bar{2}} + \rho_2\rho_{\bar{2}} & \cdots & \rho_{2\bar{n}} + \rho_2\rho_{\bar{n}} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_n\rho_{\bar{1}} & \rho_{n\bar{2}} + \rho_n\rho_{\bar{2}} & \cdots & \rho_{n\bar{n}} + \rho_n\rho_{\bar{n}} \end{vmatrix}$$

$$= \tau^n \rho_1\rho_{\bar{1}} \begin{vmatrix} 1 & \rho_{\bar{2}} & \cdots & \rho_{\bar{n}} \\ \rho_2 & \rho_{2\bar{2}} + \rho_2\rho_{\bar{2}} & \cdots & \rho_{2\bar{n}} + \rho_2\rho_{\bar{n}} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_n & \rho_{n\bar{2}} + \rho_n\rho_{\bar{2}} & \cdots & \rho_{n\bar{n}} + \rho_n\rho_{\bar{n}} \end{vmatrix}.$$

Subtracting the first column multiplied by $\rho_{\bar{k}}$ from the k-th column for $k = 2, \ldots, n$, we obtain

$$\det H_\tau = \tau^n |\rho_1|^2 \begin{vmatrix} 1 & 0 & \cdots & 0 \\ \rho_2 & \rho_{2\bar{2}} & \cdots & \rho_{2\bar{n}} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_n & \rho_{n\bar{2}} & \cdots & \rho_{n\bar{n}} \end{vmatrix} = \tau^n |\rho_1|^2 \det H_\rho. \tag{3.2}$$

Now let c_{ij}^{τ} be the cofactor obtained by deleting the i-th row and j-th column of H_τ and taking the determinant with sign $(-1)^{i+j}$. Let c_{ij}^{ρ} be the analogous cofactor for H_ρ. Then a similar computation gives

$$c_{ij}^{\tau} = \tau^{n-1}|\rho_1|^2 c_{ij}^{\rho},$$

so

$$\tau^{i\bar{j}} = \frac{c_{ij}^{\tau}}{\det H_\tau} = \frac{1}{\tau}\frac{c_{ij}^{\rho}}{\det H_\rho} = \frac{1}{\tau}\rho^{i\bar{j}}$$

for $i, j = 2, \ldots, n$. Hence,

$$^{t}[(\tau^{i\bar{j}})_{i,j=2}^{n}]^{-1} = {}^{t}[\frac{1}{\tau}(\rho^{i\bar{j}})]^{-1} = \tau\, {}^{t}[{}^{t}H_\rho^{-1}]^{-1} = \tau H_\rho. \quad \square$$

The proposition implies that the two metrics in N have the same non-positive curvature along leaves. The τ-metric in $T\mathfrak{F}$ is flat along leaves. Since the leaves are totally geodesic, the τ-metric in TX is non-positively curved along leaves, and the foliation $\mathcal{F}$ is holomorphic if and only if the curvature vanishes along leaves.

In local coordinates, the complex gradient $\operatorname{grad}\tau$ of τ with respect to the τ-metric is given by the formula

$$\operatorname{grad}\tau = \sum_{i,j=1}^{n} \tau^{i\bar{j}}\tau_{\bar{j}}\frac{\partial}{\partial z_i}.$$

The Monge-Ampère condition is equivalent to the equation

$$\|\operatorname{grad}\tau\|^2 = (\operatorname{grad}\tau)(\tau) = \sum_{i,j}\tau_i\tau^{i\bar{j}}\tau_{\bar{j}} = \tau.$$

The gradient is tangent to the foliation $\mathcal{F}$ and holomorphic along every leaf. Furthermore, $\mathcal{F}$ is holomorphic if and only if $\operatorname{grad}\tau$ is holomorphic on X.

4. Parabolic exhaustions.

A parabolic exhaustion on an n-dimensional complex manifold X is a parabolic function $\tau : X \to [0, R)$, $0 < R \leq +\infty$, which is also an exhaustion. This means that the sublevel sets $\{\tau < c\}$ are relatively compact in X for every $c < R$.

Examples. (1) Let $f : X \to B^n_R$ be a proper holomorphic map, where $B^n_R = \{z \in \mathbb{C}^n : |z|^2 < R\}$ is the ball of radius $\sqrt{R}$ in $\mathbb{C}^n$. Then $|f|^2$ is a parabolic exhaustion on X. (Here, $|\cdot|$ denotes the euclidean norm.)

(2) If τ is a parabolic exhaustion on X and $f : Y \to X$ is a proper holomorphic map, then $\tau \circ f$ is a parabolic exhaustion on Y.

(3) If τ, σ are parabolic exhaustions on X, then the product $\tau\sigma$ is also a parabolic exhaustion on X.

(4) If τ, σ are parabolic exhaustions on X, Y respectively, then the function $(x, y) \mapsto \tau(x) + \sigma(y)$ is a parabolic exhaustion on the product $X \times Y$.

In what follows, we let X be an n-dimensional Stein manifold and $\tau : X \to [0, R)$ be a parabolic exhaustion, where $0 < R \leq +\infty$. These are our basic objects of interest.

We shall need the following minimum principle for the Monge-Ampère operator.

4.1. The Monge-Ampère Minimum Principle [BeTa; Sa]**.** *Let Ω be a relatively compact open set in an n-dimensional complex manifold and u, v be smooth plurisubharmonic functions on Ω, continuous on $\overline{\Omega}$, such that $(i\partial\bar{\partial}u)^n \leq (i\partial\bar{\partial}v)^n$ on Ω. If $u \geq v$ on $\partial\Omega$, then $u \geq v$ on Ω.*

By the minimum principle, $\rho = \log\tau$ must be unbounded below on X, so the zero set $\tau^{-1}(0)$ is not empty and τ is surjective. Also, τ must be strictly plurisubharmonic somewhere, i.e., $D \neq X$.

We obtain a Monge-Ampère foliation $\mathcal{F}$ on the complement in $X' = X \setminus \tau^{-1}(0)$ of the set $E = \{(\partial\bar{\partial}\rho)^{n-1} = 0\}$, which we call *the singularity set* of τ or $\mathcal{F}$. By formula (3.1), E is contained in the degeneracy locus $D = \{(\partial\bar{\partial}\tau)^n = 0\}$.

If $\mathcal{F}$ is holomorphic on a neighbourhood of a point $p \in X' \setminus E$, then by *an $\mathcal{F}$-chart* or *$\mathcal{F}$-coordinates* at p we shall mean a coordinate chart $(U, z_1, \dots, z_n)$ in $X' \setminus E$ centred at p with $U = \{|z_k| < 1, k = 1, \dots, n\}$, such that the leaves of $\mathcal{F}$ intersect U in plates of the form $\{(z_2, \dots, z_n) \text{ constant}\}$. Then z_1 is a coordinate on every plate. (This definition differs slightly from that of section 2.)

The following theorem completely classifies strictly parabolic exhaustions. For a proof see [St] or [Bu1].

4.2. Theorem (Stoll). *Let X be an n-dimensional complex manifold and $\tau : X \to [0, R)$ be a strictly parabolic exhaustion, $0 < R \leq +\infty$. Then there exists a biholomorphism $f : X \to B^n_R$ such that $\tau = |f|^2$.*

In this paper we will prove a structure theorem for parabolic exhaustions which are only assumed to be strictly plurisubharmonic at their zeros, but not necessarily everywhere. We shall need the following simple consequence of Stoll's theorem.

4.3. Proposition. *Assume τ is strictly plurisubharmonic at its zeros. Then the zero set of τ is finite and in suitable coordinates near each zero, the foliation $\mathcal{F}$ looks like a pencil of concentric discs. Furthermore, the τ-metric is flat on a neighbourhood of the zero set of τ.*

Proof. If $r > 0$ is sufficiently small, then τ restricted to any connected component of the sublevel set $\{\tau < r\}$ is a strictly parabolic exhaustion. Now apply Stoll's theorem to τ on each connected component. $\square$

5. The case of Riemann surfaces.

In this section, we present a complete analysis of parabolic exhaustions in the one-dimensional case. This serves to motivate and clarify our approach to the general case.

Let τ be a non-constant parabolic function on a Riemann surface X. For example, τ could be a parabolic exhaustion of X. Then ρ is harmonic on X' and hence locally of the form $\log|f|^2$ with f holomorphic. By the following lemma, the zero set $\tau^{-1}(0)$ is discrete.

5.1. Lemma. *Let U be an open subset of $\mathbb{C}$ and $v \not\equiv -\infty$ be a subharmonic function on U such that e^v is C^1. Then the polar set $v^{-1}(-\infty)$ is a discrete subset of U.*

Proof. Since $2v$ is subharmonic on U, by a result of Bombieri [Bom] the set where e^{-2v} is not locally integrable is discrete. Therefore it suffices to show that if $v(p) = -\infty$, then e^{-2v} is not locally integrable at p. Since e^v is C^1, there is a constant $c > 0$ such that $e^{v(z)} \leq c|z - p|$ near p. Hence, $e^{-2v(z)} \geq \frac{1}{c^2}|z - p|^{-2}$, so e^{-2v} is not locally integrable at p. $\square$

In a coordinate z centred at a zero of τ, we have

$$\rho(z) = c \log|z|^2 + \log|f|^2$$

with $0 \leq c < 1$ and f holomorphic where $z \neq 0$. Since τ is smooth, $c = 0$ and f extends holomorphically across the zero.

Thus we obtain an open cover (U_i) of X and holomorphic functions f_i on U_i such that $\tau = |f_i|^2$ on U_i. The quotients f_i/f_j give a cocycle μ in $H^1(X, \mathbb{U})$, where $\mathbb{U}$ denotes the unit circle, which splits if and only if $\tau = |f|^2$ for some holomorphic function f on X. The functions f_i represent a holomorphic section s of the flat line bundle given by μ, and $\|s\|^2 = \tau$ in a suitable hermitian metric.

This approach clearly does not lend itself to direct generalization to higher dimensions. Let us therefore give a different interpretation of the cocycle μ. With respect to a local coordinate z on X the τ-metric is given by the function $\tau_{z\bar{z}}$, and the metric connection by the $(1,0)$-form

$$\theta = \partial \log \tau_{z\bar{z}} = (\rho_z + \frac{\rho_{zz}}{\rho_z})dz.$$

Since ρ_z is holomorphic and not identically zero, θ is meromorphic and its poles are simple with integral residues. Therefore, the τ-metric is flat and its connection extends to a regular meromorphic connection on X with no local monodromy at points of D, which is discrete.

We get an open cover (U_i) of X and meromorphic vector fields e_i on U_i which are unitary on $U_i \setminus D$. The quotients e_i/e_j give a cocycle in $H^1(X, \mathbb{U})$ representing the monodromy of the flat connection. This cocycle is the inverse of the cocycle μ for the following reason: If f is a holomorphic function on an open subset U of X and v is a meromorphic vector field on U such that $\operatorname{grad} \tau = fv$, then

$$\tau = \| \operatorname{grad} \tau \|^2 = |f|^2 \|v\|^2,$$

so $\tau = |f|^2$ if and only if v is unitary. Therefore, $\tau = |f|^2$ for some holomorphic function f on X if and only if the τ-connection has trivial monodromy.

It is this latter approach that we shall try to extend to higher dimensions. Note that we have proved the Main Theorem 8.1 in the one-dimensional case, actually under a substantially weakened hypothesis. We have in fact shown that conditions (i)–(vi) always hold. Therefore, in the following, we may assume that $n = \dim X \geq 2$ whenever this is convenient.

We conclude this section by presenting two examples of parabolic exhaustions on Riemann surfaces which do not arise from analytic coverings.

Examples. (1) On $\mathbb{C}^\times$ consider the parabolic exhaustion

$$\tau(z) = \frac{|z-1|^2}{|z|^{2\epsilon}}$$

with $0 < \epsilon < 1$. Then $\tau^{-1}(0) = \{1\}$ and $D = \{\epsilon/(\epsilon - 1)\}$. It is not hard to check that the monodromy of the connection is generated by $\exp(2\pi i \epsilon) \in \mathbb{U}$, so it is non-trivial and not even finite unless ϵ is rational.

(2) Let $X = \mathbb{C} \setminus \{1, 2, 3, \dots\}$. The series

$$\sigma(z) = -\sum_{n=1}^{+\infty} 2^{-n} \log|z - n|$$

converges locally uniformly on X and thus defines a harmonic function on X. We obtain a parabolic exhaustion τ on X, which is strict at its only zero, by setting $\tau(z) = |z|^2 e^{\sigma(z)}$. However, X is not a covering of $\mathbb{C}$: If $f : X \to \mathbb{C}$ were a proper holomorphic function, then f would extend to a holomorphic function $\mathbb{P} \to \mathbb{P}$ with poles at the points $1, 2, \dots, \infty$ because at each of these points f approaches ∞, but this is absurd.

6. Propagation of flatness along leaves.

We consider a parabolic exhaustion τ which is strictly plurisubharmonic at its zeros and continue to use the notation established in section 4. In this section, we show that flatness of the τ-metric propagates along the leaves of the Monge-Ampère foliation $\mathfrak{F}$. This yields criteria for the τ-metric to be flat on $X \setminus D$.

6.1. Proposition. *Let F be the subset of $X \setminus D$ where the curvature of the τ-metric vanishes. Then the interior F° of F is saturated.*

Proof. Let M be a leaf intersecting F°. Then $M \cap F^\circ$ is open in M and non-empty. Since $M \not\subset D$, $M \cap D$ is discrete, being the set of critical points of the harmonic function $\rho|M$, and $M \setminus D$ is connected. Hence it suffices to show that $M \cap F^\circ$ is closed in $M \setminus D$.

Let $p \in M \setminus D$ be a limit point of $M \cap F^\circ$ and (V, w) be a coordinate chart at p with $V \cap D = \varnothing$ in which $\mathfrak{F}$ is smoothly trivial. By shrinking V, we may assume that the τ-metric is flat on an open set W intersecting every plate in V. Now each component of $\operatorname{grad}\tau$ in terms of the frame $\partial/\partial w_1, \dots, \partial/\partial w_n$ is holomorphic along leaves and holomorphic on W, and hence holomorphic on V. Therefore, $\operatorname{grad}\tau$ is holomorphic on V, so $\mathfrak{F}$ is holomorphic on V, and we can choose an $\mathfrak{F}$-chart (U, z) at p with $U \cap D = \varnothing$, such that the τ-metric is flat on an open set intersecting every plate in U.

With respect to the frame $\partial/\partial z_1, \dots, \partial/\partial z_n$ for TU, the metric connection is given by the $n \times n$ matrix $\theta = \partial h \cdot h^{-1}$, where the matrix $h = (\tau_{i\bar{j}})$ represents the metric, and the curvature operator is given by the matrix $\Theta = \bar{\partial}\theta$ of $(1,1)$-forms on U. The matrix Θh is skew-hermitian.

Consider the dz_1-term, say $f dz_1$, in $\theta_{\alpha\beta}$. Since $\mathfrak{F}$ is holomorphic on U, Θ vanishes along leaves, so the term $f_{\bar{1}} d\bar{z}_1 \wedge dz_1$ in $\Theta_{\alpha\beta}$ must vanish. Hence, $f_{\bar{j}\bar{1}} = 0$ for any j, so $f_{\bar{j}}$ is holomorphic along leaves. Now Θ, and hence $f_{\bar{j}}$, vanishes on an open set in U intersecting every plate, so $f_{\bar{j}} = 0$. Therefore, all terms in Θ containing dz_1 vanish. The same holds for $\Theta h = -h\Theta^*$, and hence for Θ^*, so all terms in Θ containing $d\bar{z}_1$ vanish. Therefore, if $g dz_k$, $k > 1$, is a term in $\theta_{\alpha\beta}$, then the term $g_{\bar{1}} d\bar{z}_1 \wedge dz_k$ in $\Theta_{\alpha\beta}$ vanishes. By an argument analogous to the one above, we have $g_{\bar{j}} = 0$ for all j.

This shows that $\Theta = 0$ on U, so $p \in M \cap F^\circ$. $\square$

6.2. Corollary. *Let $U \subset X' \setminus E$ be the union of those leaves M of $\mathcal{F}$ for which $\rho|M$ is unbounded below. If U is dense in $X \setminus D$, then the τ-metric is flat.*

The set U is in fact open by Lemma 2.1.

Proof. By Proposition 4.3, the τ-metric is flat on a neighbourhood of $\tau^{-1}(0)$, so by Proposition 6.1, the τ-metric is flat on U. $\square$

This result raises the question of when a leaf $M \not\subset D$ abuts the zero set of τ. A sufficient condition, somewhat opaque, follows.

6.3. Proposition. *Let $M \not\subset D$ be a leaf of $\mathcal{F}$. Suppose every $p \in \overline{M} \cap E$ has coordinate neighbourhoods $U \Subset U'$ such that $\overline{V} \setminus (V \cup D)$ has no limit points in $\rho^{-1}(c) \cap D$, where $V = M \cap U$ and $c = \rho(p)$. Then $\rho|M$ is unbounded below.*

Proof. Suppose $\rho|M$ is bounded below, so $\rho|\overline{M}$ has a minimum at a point p. First assume that $p \in E$. The function $\lambda = \log\det(\tau_{i\bar{\jmath}})$ on U' is subharmonic along leaves. We shall compare λ to the harmonic function ρ on V.

Let $Y = \overline{V} \setminus (V \cup D) \neq \varnothing$, $a = \inf \rho(Y)$, $b = \sup \lambda(Y)$. Then $t(\rho - a) \geq \lambda - b$ on Y for all $t \geq 0$, so

$$\liminf_{x \to \infty \text{ in } V} t(\rho - a) - (\lambda - b) \geq 0$$

(recall that $\lambda = -\infty$ on D). Hence, $t(\rho - a) \geq \lambda - b$ on V, so $\rho \geq a$ on $V \setminus D$ and thus on V (because $M \cap D$ is discrete since $M \not\subset D$). Therefore, $a = c$, and $\rho(q) = c$ for some $q \in \overline{Y}$. By assumption, $q \notin D$.

If M' is the leaf through q, then $M' \subset \overline{M}$ by Corollary 2.2, so $\rho|M'$ has a minimum at q. Since $\rho|M'$ is harmonic, it must be constant, so $M' \subset \rho^{-1}(c)$, which is absurd since $\rho^{-1}(c)$ is strictly pseudoconvex off D.

Now suppose that $\rho|\overline{M}$ has no minima in E, so in particular $p \notin E$. Let $M_1 \subset \overline{M}$ be the leaf through p. Then $\rho|M_1$ is constant, so M_1 is relatively compact in X and $\overline{M_1} \cap E = \varnothing$. We shall use an observation attributed to Fornæss in [BeKa] to derive a contradiction from this. Namely, let u be a strictly plurisubharmonic function on X. Then u has a maximum at some point $q \in \overline{M}_1$. Let $M_2 \subset \overline{M_1}$ be the leaf through q. Then $u|M_2$ has a maximum at q, so $u|M_2$ is constant, which is absurd. $\square$

6.4. Corollary. *If $M \not\subset D$ is a leaf of $\mathcal{F}$ such that $\overline{M} \cap E = \varnothing$, then $\rho|M$ is unbounded below.*

7. Singularities of a parabolic exhaustion with a flat metric.

We continue to consider an n-dimensional Stein manifold X with a parabolic exhaustion $\tau : X \to [0, R)$ which is strictly plurisubharmonic at its zeros. In this section we shall assume that the τ-metric on $X \setminus D$ is flat. This has strong consequences for the singularities of τ. It also implies that the foliation $\mathcal{F}$ is holomorphic on $X' \setminus D$.

We will need the following theorem, which was first proved in a special case by Hartogs. For a proof see [Si]. For a somewhat weaker version of the theorem see [Na].

7.1. Hartogs' Singularity Theorem. *Let A be a closed subset of an open set U in $\mathbb{C}^{m+1}$ such that*

(a) *$U \setminus A$ is pseudoconvex at every point of A and*
(b) *the projection $\mathbb{C}^{m+1} \to \mathbb{C}^m$ is discrete on A.*

Then A is an analytic subset of U. Furthermore, by (a)*, A must be a hypersurface unless it is empty.*

7.2. Proposition. *Let U be an open set in $\mathbb{C}^m$ and $v : U \to [-\infty, +\infty)$ be a function such that e^v is smooth and v is pluriharmonic on $U \setminus Z$, where $Z = v^{-1}(-\infty)$. Then Z is an analytic hypersurface in U, unless it is empty or all of U.*

Proof. Suppose $Z \neq \varnothing, U$. Now $-v \to +\infty$ at Z, so $U \setminus Z$ is pseudoconvex at every point of Z. Also, v extends to a plurisubharmonic function on all of U. In particular, Z is pluripolar, so for every $p \in Z$ there is a complex line ℓ such that $p \in \ell \not\subset Z$. If $\ell \not\subset Z$ is a complex line in U, then $\ell \cap Z = (v|\ell)^{-1}(-\infty)$ is discrete by Lemma 5.1, and the same holds for parallel lines sufficiently close to ℓ. Therefore, by Hartogs' singularity theorem 7.1, Z is an analytic hypersurface in U. $\square$

7.3. Proposition. *The degeneracy locus D is an analytic hypersurface in X unless it is empty.*

Proof. Let $p \in \partial D$ and $v = \log\det(\tau_{i\bar{j}})$ with respect to local coordinates on a neighbourhood U of p. The Ricci curvature of the τ-metric is represented by the form $-i\partial\bar{\partial}v$, so by flatness v is pluriharmonic on $U \setminus D$. Now Proposition 7.2 applied to v concludes the proof. $\square$

Since D is in particular nowhere dense, we have the following corollary.

7.4. Corollary. *The foliation $\mathcal{F}$ is holomorphic.*

7.5. Lemma. *Let A be a non-discrete closed analytic subset of X' such that ρ restricted to the smooth locus of A is Monge-Ampère homogeneous. Then ρ is unbounded below on A.*

Proof. Suppose $\rho|A$ is bounded below. Since A is closed, $\rho|A$ has a minimum at some point $x \in A$. Also, since X is Stein and therefore has no non-discrete compact subvarieties, $\rho|A$ cannot be constant. Hence, there is $c \in \rho(A)$ with $c > \rho(x)$.

If A is smooth, we can now apply the Monge-Ampère minimum principle 4.1 to ρ and the constant c on the sublevel set $\Omega = \{\rho < c\} \cap A$, which is relatively compact in A. We find that $\rho \geq c$ on Ω, which is absurd.

If A is not smooth, we take a resolution of singularities $\psi : Y \to A$ and do the same for $\rho \circ \psi$, which is smooth, proper and Monge-Ampère homogeneous on Y. $\square$

7.6. Proposition. *The singularity set E is an analytic subvariety of X of codimension at least two.*

Proof. By formula (3.2), in any $\mathcal{F}$-chart in $X' \setminus E$ we have

$$\det(\tau_{i\bar{j}}) = \tau^n |\rho_1|^2 \det(\rho_{\alpha\bar{\beta}}),$$

where $i, j = 1, \ldots, n$ and $\alpha, \beta = 2, \ldots, n$. The function ρ_1 is holomorphic, so outside D we have

$$0 = \partial\bar{\partial} \log \det(\tau_{i\bar{j}}) = n\, \partial\bar{\partial}\rho + \partial\bar{\partial} \log \det(\rho_{\alpha\bar{\beta}}).$$

Now let $z_1, \ldots, z_n$ be local coordinates on an open set U in X'. Let $H_z = (\partial^2\rho/\partial z_i \partial\bar{z}_j)$ be the complex Hessian of ρ with respect to these coordinates. For a matrix A, let $A^{j,k}$ denote the minor obtained by removing the j-th row and k-th column of A. It is a fact of linear algebra that if H is a singular hermitian matrix with $\det H^{k,k} = 0$ for all k, then $\det H^{i,j} = 0$ for all i, j, so corank $H \geq 2$. Hence, $E \cap U$ is the subset of U where the determinants of $H_z^{k,k}$, $k = 1, \ldots, n$, all vanish.

Let $w_1, \ldots, w_n$ be $\mathcal{F}$-coordinates near a point $x \in U \setminus D$. Then

$$H_z = J H_w J^*,$$

where $J = (\partial w_i/\partial z_j)$ is the Jacobian of the change of coordinates near x. Since the first line and first column of H_w are identically zero, we have

$$H_z^{j,k} = J^{j,1} H_w^{1,1} (J^*)^{1,k},$$

so

$$\partial\bar{\partial} \log \det H_z^{k,k} = \partial\bar{\partial} \log \det H_w^{1,1} + \partial\bar{\partial} \log |\det J^{k,1}|^2 = -n\, \partial\bar{\partial}\rho$$

on the set where all the determinants are non-zero. Let

$$u_k = \log \det H_z^{k,k} + n\rho, \qquad k = 1, \ldots, n.$$

Then u_k is pluriharmonic on $U \setminus u_k^{-1}(-\infty)$, so by Proposition 7.2, $u_k^{-1}(-\infty)$ is analytic in U. Hence

$$E \cap U = \bigcap_{k=1}^{n} u_k^{-1}(-\infty)$$

is analytic.

Finally, if $\operatorname{codim} E = 1$, then by the definition of E, ρ is Monge-Ampère homogeneous on the smooth locus of E. By Lemma 7.5, $\rho|E$ must be unbounded below, which is impossible since τ is strict at its zeros. $\square$

7.7. Lemma. *The degeneracy locus D is transverse to the foliation $\mathcal{F}$ outside a thin analytic subset of D.*

Proof. Otherwise there is an irreducible component A of D which fails to be transverse to $\mathcal{F}$ at every point. We will show that A is saturated; then ρ is Monge-Ampère homogeneous on the smooth locus of A, which by Lemma 7.5 contradicts strictness of τ at its zeros.

Let (U, z) be an $\mathcal{F}$-chart at $p \in A \setminus E$. Say $A = \{g = 0\}$, where $g \in \mathcal{O}(U)$ is square-free in $\mathcal{O}_p$, so g divides any germ $h \in \mathcal{O}_p$ with $h|A = 0$. For $x \in A \cap U$ we have $\partial g/\partial z_1(x) \neq 0$ if and only if the leaf through x is transverse to A at x.

By assumption, $\partial g/\partial z_1 = 0$ on $A \cap U$. Hence g divides $\partial g/\partial z_1$, so g divides $\partial^i g/\partial z_1^i$ for all i. Therefore all z_1-derivatives of g vanish at p, so g is locally constant on the leaf M through p at p and $M \subset A$. □

7.8. Theorem. *The τ-connection ∇ in $T(X \setminus D)$ extends to a regular meromorphic connection in TX. Moreover, ∇ has no monodromy locally at points of D.*

Proof. By Proposition 7.6 and Lemma 7.7, it suffices to prove this at a point $p \in D \setminus E$ where D is transverse to $\mathcal{F}$.

Let (U, z) be an $\mathcal{F}$-chart at p such that $D = \{z_1 = 0\}$ and let M be a plate in U. Since M is totally geodesic, the maps in the canonical short exact sequence

$$0 \to T\mathcal{F} \to TX \to N \to 0 \qquad \text{over } M$$

preserve the canonical connections of the τ-metrics in these bundles. This means that the restriction of ∇ to TM is the canonical connection of the τ-metric on M, so the analysis of section 5 can be applied to $\nabla|TM$. Likewise, the restriction of ∇ to N is the canonical connection of the quotient τ-metric in N, which, by Proposition 3.1, is the ρ-metric multiplied by τ, so $\nabla|N$ is smooth on all of U.

Since ∇ is regular on TM by section 5 and smooth on N, it is regular on $TX|M$. Therefore, ∇ is regular on TX.

The fundamental group of $U \setminus D$ is generated by a loop γ about p based at a point x in the plate M through p. We need to show that the monodromy operator μ on T_xX given by γ is the identity. Since M is totally geodesic, the monodromy operators μ_1 on $T_xM \subset T_xX$ and μ_2 on $N_x = T_xX/T_xM$ are induced by μ. By section 5 there is no monodromy in TM, and since $\nabla|N$ is smooth on all of U there is no monodromy in N either, so both μ_1 and μ_2 are the identity. Since μ is unitary, this implies that μ is the identity. □

Roughly speaking, a holomorphic connection with singularities is said to be *regular* if it is meromorphic with simple poles. This means that horizontal sections grow polynomially near the singular set. For the theory of regular connections, we refer to [Del].

In fact, we stated the regularity result not because we need it, but simply because it required no additional effort. What is important for us is vanishing of the monodromy of ∇ at D. This implies that the monodromy factors through $\pi_1(X)$, so the unitary local system $\ker\nabla$ on $X \setminus D$ extends to a unitary local system $\mathcal{L}$ on X. This local system is induced by a flat hermitian metric in the holomorphic vector bundle $L = \mathcal{L} \otimes \mathcal{O}$ on X, which is unique up to a choice of a unitary frame at any single point.

In explicit terms, there is an open cover (U_i) of all of X and holomorphic unitary frames e_i for $T(U_i \setminus D)$. The cocycle (a_{ij}) of constant unitary transition matrices satisfying $e_i = a_{ij}e_j$ determines $\ker\nabla$ as well as its extension $\mathcal{L}$. A section s of L is represented by a family of holomorphic maps $f_i : U_i \to \mathbb{C}^n$ such that $f_j = f_i a_{ij}$. A flat metric in L compatible with $\mathcal{L}$ is defined by setting $\|s\|^2 = f_i f_i^* = |f_i|^2$ on U_i.

7.9. Theorem. *The flat bundle L has a holomorphic section s such that $\tau = \|s\|^2$.*

Proof. On U_i, write $\operatorname{grad}\tau = f_i e_i$ with $f_i : U_i \setminus D \to \mathbb{C}^n$ holomorphic. Then $\tau = \|\operatorname{grad}\tau\|^2 = |f_i|^2$. In particular, f_i extends holomorphically to U_i. Since $f_j e_j = f_i e_i = f_i a_{ij} e_j$, we have $f_j = f_i a_{ij}$, so the maps f_i represent a holomorphic section s of L such that $\|s\|^2 = |f_i|^2 = \tau$. $\square$

Note that τ is in particular real-analytic.

7.10. Corollary. *There is an analytic covering $f : X \to B_R^n$ such that $\tau = |f|^2$ if and only if the monodromy of ∇ is trivial.*

Proof. If the monodromy is trivial, then L has a global unitary frame $\{e_1, \ldots, e_n\}$. Write $s = \sum f_i e_i$, where the f_i are holomorphic functions on X. Then the map $f = (f_1, \ldots, f_n)$ satisfies $|f|^2 = \|s\|^2 = \tau$, so $f : X \to B_R^n$ is proper and hence an analytic covering.

Conversely, if $\tau = |f|^2$, then the pullbacks $f^*(\partial/\partial z_i)$, $i = 1, \ldots, n$, form a global holomorphic unitary frame for $T(X \setminus D)$. $\square$

7.11. Corollary. *Locally, $\tau = |f|^2$, where f is an analytic covering of an open set in $\mathbb{C}^n$.*

Proof. By the proof of Theorem 7.9, locally, $\tau = |f|^2$ where f is a holomorphic map into $\mathbb{C}^n$. Fibres of f lie in the level sets of τ, which are compact real-analytic subvarieties of X, so f is discrete by the following lemma and thus locally an analytic covering. $\square$

7.12. Lemma (Diederich-Fornæss [DiFo]**).** *A compact real-analytic subvariety of a Stein manifold does not contain any non-trivial germs of complex-analytic varieties.*

Proof. Diederich and Fornæss prove this for $\mathbb{C}^m$. By the embedding theorem for Stein manifolds the generalization is obvious. $\square$

8. The structure theorem for parabolic exhaustions.

We are now ready to prove our main result.

8.1. Main Theorem. *Let X be an n-dimensional Stein manifold and $\tau : X \to [0, R)$, $0 < R \leq +\infty$, be a parabolic exhaustion which is strictly plurisubharmonic at its zeros. Then the following conditions are equivalent.*

(i) *The τ-metric on $X \setminus D$ is flat.*

(ii) *τ is real-analytic.*
(iii) *ρ is unbounded below on every leaf of the Monge-Ampère foliation $\mathcal{F}$ of $X' \setminus E$.*
(iv) *Locally, $\tau = |f|^2$, where f is a holomorphic map into $\mathbb{C}^n$.*
(v) *$\mathcal{F}$ extends to a holomorphic foliation of X'.*
(vi) *There exists a holomorphic section s of a flat hermitian vector bundle of rank n over X such that $\tau = \|s\|^2$.*

Moreover, there is an analytic covering $f : X \to B_R^n$ such that $\tau = |f|^2$ if and only if these conditions are satisfied and the canonical connection of the τ-metric has trivial monodromy.

Proof. (ii) $\Rightarrow$ (i): Let U be a connected component of $X \setminus D$ containing a zero of τ. By Proposition 4.3 and real-analyticity, the τ-metric is flat on U. Let $p \in \partial U \subset D$ and B be a coordinate ball about p. The fact that the function $\lambda = \log\det(\tau_{i\bar{j}})$ on B is pluriharmonic on $U \cap B$ translates into a differential equation for e^λ, which by real-analyticity holds on all of B. Hence, λ is pluriharmonic on $B \setminus D$ and plurisubharmonic on B. Therefore, $D \cap B$ is a pluripolar set and thus does not disconnect B, so $B \setminus D \subset U$. This shows that $U = X \setminus D$.

(iii) $\Rightarrow$ (i): This is Corollary 6.2.

(i) $\Rightarrow$ (vi): This is Theorem 7.9.

(vi) $\Rightarrow$ (v): As shown in section 7, we have an open cover (U_i) of X' and analytic coverings $f_i : U_i \to V_i$ with $V_i \subset \mathbb{C}^n \setminus \{0\}$ open, such that $|f_i|^2 = \tau$ and $f_i = a_{ij} \circ f_j$ on $U_i \cap U_j$ with $a_{ij} \in U(n)$. Let $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ be the projection and $\phi_i = \pi \circ f_i : U_i \to \mathbb{P}^{n-1}$.

Note that the critical locus of ϕ_i is $E \cap U_i$ and that ϕ_i is open since π and f_i are. Also, ρ is harmonic along the fibres of $\phi_i | U_i \setminus E$. So to prove that the maps ϕ_i form an atlas for the desired foliation, we need only show that they are simple.

Suppose ϕ_i is not simple for some i. Then by Theorem 2.6, E contains an irreducible component of some fibre of ϕ_i. Let $\mathcal{C}$ be the set of all such components for any i. By assumption, $\mathcal{C} \neq \varnothing$. If $C \in \mathcal{C}$, then $\rho|C$ is non-constant and harmonic on the smooth locus of C, so $\rho(C)$ is open in $(-\infty, \sup\rho)$. Hence, $\rho(\bigcup\mathcal{C})$ is open.

Let us show that $\rho(\bigcup\mathcal{C})$ is also closed in $(-\infty, \sup\rho)$. Let (x_k) be a sequence in $\bigcup\mathcal{C}$ such that $\rho(x_k) \to a < \sup\rho$. Since ρ is an exhaustion, we may assume that (x_k) converges to a limit p. Say $p \in U_i$. We may also assume that $x_k \in U_i$ and $x_k \neq p$ for all k. For each k, $x_k \in C \in \mathcal{C}$, where C is a component of $f_j^{-1}(\ell)$ for a radial line ℓ in $\mathbb{C}^n$ (with j depending on k). Then $f_i(C) = a_{ij}(f_j(C)) \subset a_{ij}(\ell)$, so $\ell_k = a_{ij}(\ell)$ is a radial line such that a component C_k of $f_i^{-1}(\ell_k)$ containing x_k intersects $C \subset E$ in a non-discrete set. Hence, $C_k \subset E$. After passing to a subsequence, we obtain a locally analytic subset $A = \bigcup C_k$ of U_i, which is closed in $U_i \setminus F$, where $F = \phi_i^{-1}(\phi_i(p))$, such that $p \in \overline{A}$. Now $\overline{A}$ is clearly not analytic in any neighbourhood of p, so by the Remmert-Stein theorem [ReSt], $\overline{A}$ contains an irreducible component C of F with $p \in C$. Since $A \subset E$ and E is closed, $C \subset \overline{A} \subset E$, so $C \in \mathcal{C}$. Hence, $a = \rho(p) \in \rho(\bigcup\mathcal{C})$.

This shows that ρ is unbounded below on $\bigcup\mathcal{C} \subset E$, which contradicts strictness of τ

at its zeros.

(v) $\Rightarrow$ (iii): Suppose that ρ is bounded below on a smooth leaf M contained in an irreducible component C of a leaf of the extended foliation. Then $\overline{C} = \overline{M}$ and $\rho|\overline{C}$ has a minimum at some point $p \in \overline{C}$. By Lemma 2.7, $\overline{C}$ contains an irreducible component C' of the leaf through p and $\rho|C'$ has a minimum at p. Since $\rho|C'$ is harmonic on the smooth locus of C', it is constant. In particular, C' is relatively compact in X.

Let u be a strictly plurisubharmonic function on X. Then u has a maximum at some point $q \in \overline{C'}$ and $\overline{C'}$ contains an irreducible component C'' of the leaf through q. Then $u|C''$ has a maximum at q, so $u|C''$ is constant, which is absurd.

(vi) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (ii) are obvious.

Now Corollary 7.10 concludes the proof. $\square$

The assumption that τ be strict at its zeros is essential to our method of proof. It is used to prove the implications (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i), (i) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (v). In fact, without it, the theorem is no longer true.

Let τ on X be any parabolic exhaustion as in the theorem, satisfying conditions (i)–(vi), for instance $\tau(z,w) = |z|^2 + |w|^2$ on $\mathbb{C}^2$. Then τ^2 is also a parabolic exhaustion on X and

$$\tfrac{1}{2}\partial\bar\partial\tau^2 = \partial\tau \wedge \bar\partial\tau + \tau\partial\bar\partial\tau,$$

so τ^2 is not strict at its zeros. By formula (3.2), in any $\mathcal{F}$-chart in $X' \setminus E$ we have

$$\det((\tau^2)_{i\bar\jmath}) = \tau^{2n}|2\rho_1|^2 \det(2\rho_{\alpha\bar\beta}) = 2^{n+1}\tau^{2n}|\rho_1|^2 \det(\rho_{\alpha\bar\beta}) = 2^{n+1}\tau^n \det(\tau_{i\bar\jmath}),$$

where $i,j = 1,\dots,n$ and $\alpha,\beta = 2,\dots,n$. In particular, the degeneracy locus of τ^2 is $D \cup \tau^{-1}(0)$. Since the τ-metric is flat, the τ^2-metric has Ricci curvature

$$-i\partial\bar\partial \log\det((\tau^2)_{i\bar\jmath}) = -ni\partial\bar\partial\rho,$$

which is non-zero, so condition (i) fails for τ^2. Hence conditions (iv) and (vi) fail as well. On the other hand, τ^2 satisfies conditions (ii), (iii) and (v) because τ does.

Burns [Bu1] has proved a structure theorem for parabolic exhaustions which are strictly plurisubharmonic on the complements of their zero sets. It could be used to investigate exhaustions such as τ^2.

9. The Monge-Ampère foliation.

In this section we make some additional remarks about the Monge-Ampère foliation $\mathcal{F}$ associated to a parabolic exhaustion τ as in the Main Theorem 8.1.

First observe that $\mathcal{F}$ cannot be extended smoothly to any neighbourhood of any point p in E. Otherwise, we could define a function $u = n\rho + \log\det(\rho_{\alpha\bar\beta})$ in terms of an $\mathcal{F}$-chart at p and show that $E = u^{-1}(-\infty)$ is a hypersurface as in the proof of Proposition 7.6. Indeed, E is the set of critical points of the $\mathcal{F}$-charts obtained in the proof of the Main Theorem 8.1, so E is the singularity set of $\mathcal{F}$ in the sense of section 2.

It is clearly not possible to extend $\mathcal{F}$ across the zeros of τ to a holomorphic foliation on all of X. However, it does extend according to the alternative definitions of Baum-Bott and Gómez-Mont (see section 2). The extension of $T\mathcal{F}$ to X is simply the coherent analytic subsheaf of TX generated over the structure sheaf $\mathcal{O}$ by the holomorphic vector field $\operatorname{grad}\tau$ at the zeros of τ.

By Lemma 2.3, a leaf M of $\mathcal{F}$ is closed, and hence an analytic subvariety of X', if and only if M has only finitely many plates at every zero (in the picture of $\mathcal{F}$ as a pencil of pointed discs).

We can consider *the projective monodromy group* G at a zero p, i.e., the image of the representation $\pi_1(X,p) \to PU(n)$ obtained by parallel-translating radial lines in T_pX along loops at p. These radial lines can in fact be identified with plates of leaves at p. By total geodesy, if a radial line is tangent to a leaf M at one point, then it is tangent to M everywhere. Thus two plates of the same leaf at p lie in the same G-orbit. Hence $\mathcal{F}$ has closed leaves if G is finite.

Suppose now that the projective monodromy vanishes (so the monodromy itself is scalar-valued). Then every leaf is parallel to a unique radial line in T_pX, so we obtain a holomorphic map $\psi : X' \to \mathbb{P}^{n-1}$, which is a submersion on $X' \setminus E$. In fact, letting $\omega_o = i\partial\bar\partial \log|Z|^2$ be the Kähler form of the Fubini-Study metric on $\mathbb{P}^{n-1}$, we have $\psi^*(\omega_o) = i\partial\bar\partial\rho$ near p by the local picture of $\mathcal{F}$ at p. These forms are invariant under parallel translation along leaves, so they are equal on the union of leaves abutting p. On the union of leaves abutting another zero q, we have $\psi = \alpha \circ \psi$, where $\alpha : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is given by parallel translation of the pencil at q to the pencil at p. This map is an automorphism of $\mathbb{P}^{n-1}$ and preserves the τ-metric, so $\alpha^*(\omega_o) = \omega_o$. Therefore, $\psi^*(\omega_o) = i\partial\bar\partial\rho$.

Claim: The map ψ lifts by the natural projection $\pi : \mathbb{C}^n\setminus\{0\} \to \mathbb{P}^{n-1}$ to a holomorphic map $f : X \to \mathbb{C}^n$.

The map ψ lifts if and only if the pullback by ψ of the fibre bundle π has a section over X'. Now π is actually obtained from the universal bundle $J \to \mathbb{P}^{n-1}$ by removing the zero section, so ψ lifts if and only if ψ^*J has a zero-free section, i.e., is trivial.

The Chern class of J is represented by the form $-\omega_o$, so the Chern class of ψ^*J is represented by $\psi^*(-\omega_o) = -i\partial\bar\partial\rho$, and thus is trivial. It is easy to see that ψ^*J extends across the zeros of τ to a holomorphic line bundle on X. Now $H^2(X') \cong H^2(X)$, so the extension also has vanishing Chern class. Since X is Stein, this shows that ψ^*J is trivial.

Finally note that ρ and $\log|f|^2$ differ by a pluriharmonic function u on X, and the image of u in $H^1(X,\mathbb{U})$ is, not surprisingly, the cocycle representing the monodromy.

10. Affine algebraic manifolds.

In their paper [GrKi] on Nevanlinna theory for algebraic varieties, Griffiths and King showed that any affine algebraic manifold possesses a parabolic exhaustion with finitely many critical values (and some further properties). In [CoGr], Cornalba and Griffiths asked whether this characterizes affine algebraic manifolds, observing that it does in dimension one. This problem apparently has not been solved. Similar questions are posed in [Bu1], [Bu2] and [Dem]. See also [Fo]. We would like to present a result in this vein.

10.1. Theorem. *A Stein manifold X is affine algebraic if and only if X has a parabolic exhaustion τ such that*

(a) *τ is strict at its zeros,*
(b) *the leaves of the Monge-Ampère foliation $\mathcal{F}$ of $X' \setminus E$ are affine curves, and*
(c) *the canonical connection of the τ-metric has finite monodromy.*

An affine curve is obtained from a compact curve by removing a finite set. Thus, if M is a smooth leaf, and $M \not\subset D$ so $\rho|M$ is not constant, then condition (b) implies that $\rho|M$ is unbounded above and below. Hence, by Corollary 6.2, conditions (a) and (b) imply that the τ-metric is flat, so condition (c) makes sense.

Proof. $\Rightarrow$: Embed X as an algebraic submanifold of $\mathbb{C}^m$ for some m. The generic linear projection $f : X \to \mathbb{C}^n$, where $n = \dim X$, yields a parabolic exhaustion $\tau = |f|^2$ satisfying (a), (b) and (c); the monodromy is in fact trivial.

$\Leftarrow$: Let us first assume that the monodromy is trivial. By Corollary 7.10, there is an analytic covering $f : X \to \mathbb{C}^n$ such that $\tau = |f|^2$. By the *Fortsetzungssatz* of Grauert-Remmert [GrRe], if the critical locus $A = f(D)$ of f is algebraic in $\mathbb{C}^n$, then f extends to an analytic covering of $\mathbb{P}^n$, so X is in particular affine algebraic.

Let M be a smooth leaf and consider a puncture of M, which we identify with the punctured unit disc $\Delta^\times$. There is a holomorphic function h on $\Delta^\times$ and $c \in \mathbb{R}$ such that

$$\rho = c \log |z|^2 + 2 \operatorname{Re} h$$

on $\Delta^\times$, so $\rho_z = c/z + h'$. At the puncture, ρ has a limit in $[-\infty, +\infty]$, so h must have a removable singularity at 0. Hence the critical set of $\rho|\Delta^\times$, which is just $\Delta^\times \cap D$, is finite. Therefore, $M \cap D$ is finite.

If ℓ is a radial line in $\mathbb{C}^n$, then $f^{-1}(\ell)$ has only a finite number of irreducible components (in fact no more than the degree of f), so $f^{-1}(\ell)$ is a finite union of leaves, and each leaf is the union of a finite set and a finite number of smooth leaves. Hence, $A \cap \ell = f(D \cap f^{-1}(\ell))$ is finite. By the following lemma, this implies that A is algebraic.

In general, there is a finite unbranched covering $\pi : Y \to X$ such that the parabolic exhaustion $\tau \circ \pi$ of Y satisfies conditions (a) and (b) and has trivial monodromy. Then Y is affine algebraic and a holomorphic function on Y is rational if and only if it has polynomial growth with respect to $\tau \circ \pi$. The deck transformations of Y preserve $\tau \circ \pi$, so they are algebraic. Hence, by a result of Chevalley [Bor; No], X is affine algebraic. $\square$

10.2. Lemma. *Let A be a closed analytic hypersurface in $\mathbb{C}^n$ such that $A \cap \ell$ is finite for every line ℓ through the origin in $\mathbb{C}^n$. Then A is algebraic.*

Proof. For $k \in \mathbb{N}$ let E_k be the set of lines $\ell \in \mathbb{P}^{n-1}$ such that $|z| \leq k$ for every $z \in A \cap \ell$. Then E_k is closed in $\mathbb{P}^{n-1}$ because the projection $A \setminus \{0\} \to \mathbb{P}^{n-1}$ is discrete and hence open. Also, $\mathbb{P}^{n-1} = \bigcup E_k$, so $E_m^\circ \neq \varnothing$ for some m by the Baire Category Theorem. Therefore, the closure of A in $\mathbb{P}^n$ does not contain the hyperplane at infinity, which implies that A is algebraic [Ch]. $\square$

11. Parabolic functions which are not exhaustions.

Let X be an n-dimensional complex manifold and $f : X \to \mathbb{C}^n$ be a holomorphic map. We have observed that $\tau = |f|^2$ is a parabolic function on X, that f is proper if and only if τ is an exhaustion, and that f is an immersion if and only if τ is strictly parabolic.

It is of interest to ask which parabolic functions arise in this manner. It turns out to be more natural to ask for which parabolic functions τ there exists a flat hermitian vector bundle of rank n over X with a holomorphic section s such that $\tau = \|s\|^2$. To pass from s to an actual map f then becomes a separate problem about vanishing of monodromy. Of course, these questions are the same when X is simply connected.

We have already discussed the case when τ is an exhaustion. Let us now see how far we can go without this assumption. So let τ be a parabolic function on X. To get started, I must assume that τ is strictly plurisubharmonic at some point $p \in \tau^{-1}(0)$. (In particular, the zero set must not be empty.) By [St], in suitable coordinates z centred at p we have

$$\tau(z) = |z|^2 + O(|z|^3).$$

Hence, $\tau : U \to \tau(U)$ is proper for some neighbourhood U of p. By Stoll's Theorem 4.2, $\tau(w) = |w|^2$ near p in some (possibly different) coordinates w; in particular, the τ-metric is flat near p.

Suppose now that the τ-metric is flat on all of $X \setminus D$, e.g. either by condition (ii) or (iii) in the Main Theorem 8.1 (the proofs that these imply flatness did not depend on τ being an exhaustion). For condition (iii), we actually need τ to be strict at every point of $\tau^{-1}(0)$. Then the proofs of Propositions 7.3 and 7.6 show that D and E are analytic, but we cannot show that $\operatorname{codim} E \geq 2$ or that $\ker \nabla$ extends to a unitary local system on X.

We can make two general statements though. By a theorem of Deligne [Del], $T(X \setminus D)$ with ∇ has a unique meromorphic extension to an $\mathcal{O}_X[D]$-coherent sheaf on X with a regular meromorphic connection. Also, if $\ker \nabla$ happens to be trivial, then we get a holomorphic map $f : X \setminus D \to \mathbb{C}^n$ with $\tau = |f|^2$, which extends to a holomorphic map on X.

So we see that if we omit the condition that τ be proper, the singularities become difficult to handle. If we assume on the other hand that τ is strictly parabolic, then all singularities disappear and we can prove results such as the following.

11.1. Proposition. *Let τ be a strictly parabolic real-analytic function on a simply connected complex manifold X of dimension n such that $\tau^{-1}(0) \neq \varnothing$. Then there exists an immersion $f : X \to \mathbb{C}^n$ such that $\tau = |f|^2$.*

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