HOLOMORPHIC NEIGHBOURHOOD RETRACTIONS OF AMPLE HYPERSURFACES

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Abstract. Let X be a hypersurface in a compact complex manifold M with $\dim M \geq 2$. We consider the neighbourhood structure of X and of coverings of X; specifically, the existence of holomorphic neighbourhood retractions. We show that if M has no non-trivial holomorphic vector fields and X is sufficiently ample, then no neighbourhood of X retracts holomorphically onto X. Let U be a tubular neighbourhood of X and $\pi : V \to U$ be a covering space. We show that if X is ample and linearly equivalent to some other hypersurface in M, and bounded holomorphic functions separate points locally on the universal covering of X, then any holomorphic retraction $V \to \pi^{-1}(X)$ is a lifting of a retraction $U \to X$. This implies that if X is a sufficiently ample curve in a surface M of general type with universal covering $\pi : \tilde{M} \to M$, then $\pi^{-1}(U)$ does not retract holomorphically onto $\pi^{-1}(X)$ for any neighbourhood U of X.

1. Introduction

Let X be a hypersurface in a compact complex manifold M of dimension at least 2. In this paper we will consider the neighbourhood structure of X and of covering spaces of X. Our first main result, theorem 2.3, states that if M has no non-trivial holomorphic vector fields and X is sufficiently ample in a certain precise sense, then no neighbourhood of X retracts holomorphically onto X. This is in contrast to the well-known existence of neighbourhood retractions in the smooth category.

To explain the idea of the proof, let us assume that X is very ample. We may also assume that X is smooth, since the image of a smooth retraction is always smooth.

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Then there is an embedding of M into some projective space $\mathbb{P}^N$ such that X is the transverse intersection of M with a hyperplane H in $\mathbb{P}^N$.

Suppose there is a neighbourhood U of X with a holomorphic retraction ρ onto X. This means that $\rho : U \to X$ is holomorphic and $\rho|X$ is the identity. Consider a hyperplane H' so close to H that it intersects M transversely and $X' = M \cap H' \subset U$. If H' is close enough to H, then $\rho : X' \to X$ is a diffeomorphism, being a small perturbation of the identity map of X, and hence a biholomorphism. This shows that all hyperplane sections of M sufficiently close to X are isomorphic to X. One would expect this to be the case only in exceptional circumstances. This is, in a sense, confirmed by theorem 2.3. The proof uses deformation theory and the Kodaira-Spencer map.

It was previously known that if X is a smooth ample hypersurface in a compact complex manifold M with $\dim M \geq 3$, then M does not retract holomorphically onto X [14].

We might try to unravel the complicated neighbourhood structure that precludes the existence of a holomorphic neighbourhood retraction by passing to coverings. Let U be a tubular neighbourhood of X, V be the universal covering of U and Y be the pullback of X to V. Then Y is the universal covering of X. Now we ask whether V retracts holomorphically onto Y. Let us assume that X is ample and not alone in its linear equivalency class. Suppose also that X is locally C-hyperbolic, which means that bounded holomorphic functions separate points locally on Y. This holds for instance if X is a curve of genus at least 2. Then our second main result, theorem 3.2, states that any holomorphic retraction $V \to Y$ is equivariant under the covering group, so it descends to a holomorphic retraction $U \to X$. Thus, in many cases, passing to coverings does not change the problem at all.

Our results may be combined as follows.

1.1. Main Theorem. *Let M be a compact complex manifold of dimension at least 2 with no non-trivial holomorphic vector fields. If X is a connected hypersurface in M, X is linearly equivalent to some other hypersurface in M, and*

$$H^1(M, [X]^\vee \otimes TM) = 0,$$

then no neighbourhood of X retracts holomorphically onto X.

If, in addition, X is ample and locally C-hyperbolic, then no covering space of a tubular neighbourhood of X retracts holomorphically onto the preimage of X.

The following corollary may serve to clarify the theorem.

1.2. Corollary. *Let M be a locally C-hyperbolic compact complex manifold of general type with $\dim M \geq 2$. If X is a sufficiently ample hypersurface in M, then no covering* *space of a tubular neighbourhood of X retracts holomorphically onto the preimage of X.*

The precise meaning of the term *sufficiently ample* is given below.

This work was motivated by the question whether retractions could be used to extend holomorphic or plurisubharmonic functions from preimages of ample curves in covering spaces of projective algebraic manifolds to uniformly thick neighbourhoods. The following corollary of the results in this paper shows that the answer is in general negative.

1.3. Corollary. *Let M be a surface of general type with universal covering space $\pi : \tilde{M} \to M$. Choose a hermitian metric on M and pull it up to $\tilde{M}$. Let X be a sufficiently ample smooth curve in M. Then the ϵ-neighbourhood of $\pi^{-1}(X)$ does not retract holomorphically onto $\pi^{-1}(X)$ for any $\epsilon > 0$.*

By the Lefschetz hyperplane theorem, the inclusion $X \hookrightarrow M$ induces an epimorphism of fundamental groups, so the preimage $Y = \pi^{-1}(X)$ is connected Riemann surface embedded in $\tilde{M}$.

Note that if the fundamental group of M is infinite, then Y is an open Riemann surface and therefore Stein, so by a theorem of Siu [13], there is a neighbourhood V of Y that retracts holomorphically onto Y. In fact, V is a holomorphic tubular neighbourhood. By our corollary, V cannot contain the preimage of any neighbourhood of X in M. Loosely speaking, the thickness of V must go to zero somewhere at infinity.

Let us clarify some of our terminology. By a surface we shall mean a connected compact complex manifold of dimension 2. A hypersurface in a complex manifold is a closed subvariety of pure codimension 1. A curve in a complex manifold is a closed subvariety of pure dimension 1. All our manifolds will be connected. Finally, when we say that a condition on a hypersurface X in a compact complex manifold M or its associated line bundle L holds if X or L is sufficiently ample, we mean that there is a constant $c > 0$, depending only on a given positive $(1,1)$-form ω on M, such that the condition holds if L has a hermitian metric with curvature form $\Theta \geq c\,\omega$. Then L is ample. Also, if L is ample, then a sufficiently high tensor power of L is sufficiently ample in this sense.

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2. Non-existence of neighbourhood retractions

Let X be a smooth connected hypersurface in a connected compact complex manifold M of dimension at least 2. Let $L = [X]$ be the line bundle defined by X. The zero locus of any holomorphic section of L is a hypersurface in M. Let

$$Z = \{(x,t) \in M \times \mathbb{P}H^0(M,L) : t(x) = 0\},$$

and let $p : Z \to \mathbb{P}H^0(M,L)$ be the projection, which is a proper holomorphic surjection. Its fibres are the zero loci of sections of L. Let B be the subset of points $t \in \mathbb{P}H^0(M,L)$ such that all points in the fibre $X_t = p^{-1}(t)$ are smooth points of Z and regular points of p. Then B is a Zariski-open subset of $\mathbb{P}H^0(M,L)$. There is $o \in \mathbb{P}H^0(M,L)$ such that $X = X_o$. It may be verified that $o \in B$, so B is non-empty. The restriction $p : E = p^{-1}(B) \to B$ is a submersion. Hence $p : E \to B$ is a smooth fibre bundle by Ehresmann's fibration theorem [4]. We call $p : E \to B$ the family associated to X or to L. It is a regular family of connected compact complex manifolds in the usual sense. Its fibres are precisely the smooth hypersurfaces in M which are linearly equivalent to X.

Let us assume that B is not a point. This is equivalent to any of the following three conditions.

(i) $\dim H^0(M,L) \geq 2$.
(ii) X is linearly equivalent to some other hypersurface in M.
(iii) X is the zero divisor of a meromorphic function on M.

The projection of $M \times \mathbb{P}H^0(M,L)$ onto M yields a holomorphic map $\xi : E \to M$ which restricts to the identity map of X_t onto itself for each $t \in B$.

We have a short exact sequence

$$0 \to TX \to TE \otimes \mathcal{O}_X \xrightarrow{dp} T_oB \otimes \mathcal{O}_X \to 0$$

of locally free sheaves over X [12], and an exact sequence

$$H^0(X, TE) \xrightarrow{dp} T_oB \xrightarrow{\kappa} H^1(X, TX).$$

The connecting homomorphism κ is the Kodaira-Spencer map. If the family p is locally holomorphically trivial, or equivalently, if the fibres X_t, $t \in B$, are mutually isomorphic, then $\kappa : T_tB \to H^1(X_t, TX_t)$ is trivial for every $t \in B$ [8, 10]. We shall need a condition to insure that this is not the case. A sufficient condition for κ to be injective will do.

Consider a pencil containing X, i.e., a line $\mathbb{P}^1$ in $\mathbb{P}H^0(M,L)$ containing o. Let $V = p^{-1}(\mathbb{P}^1 \cap B)$. Then $\xi : V \to M$ is injective outside the preimage of the base locus A of the pencil, which is a proper subvariety of X. Hence, $\xi|V \setminus \xi^{-1}(A)$ is an isomorphism onto a neighbourhood of $X \setminus A$ in M. The exact sequence

$$H^0(X, TV) \to T_o\mathbb{P}^1 \xrightarrow{\kappa} H^1(X, TX)$$

associated to the family $p : V \to \mathbb{P}^1 \cap B$ shows that κ restricted to the pencil is injective if $H^0(X, TV) = 0$. Now $\xi : X \to X$ is the identity and $d\xi : TV|X \to TM|X$ is an isomorphism outside A. Hence, if $TV|X$ has a non-trivial holomorphic section v, then $d\xi \circ v \circ (\xi|X)^{-1}$ is a non-trivial holomorphic section of $TM|X$.

This shows that $\kappa : T_oB \to H^1(X, TX)$ is injective if $H^0(X, TM) = 0$. Say X is the zero locus of a section s of L. We have the short exact sequence

$$0 \to L^\vee \xrightarrow{\cdot \otimes s} \mathcal{O}_M \xrightarrow{\text{restriction}} \mathcal{O}_M|X \to 0$$

of sheaves over M. Here, $L^\vee$ is the dual bundle of L. After tensoring by TM, we obtain the exact sequence

$$H^0(M, TM) \to H^0(X, TM) \to H^1(M, L^\vee \otimes TM).$$

The following theorem is now clear.

2.1. Theorem. *Let X be a smooth connected hypersurface in a compact complex manifold M with $\dim M \geq 2$. Let $p : E \to B$ be the associated family, and say $X = p^{-1}(o)$. If*

(1) $H^0(M, TM) = 0$ *and*
(2) $H^1(M, [X]^\vee \otimes TM) = 0$,

then the Kodaira-Spencer map $\kappa : T_oB \to H^1(X, TX)$ is injective.

Let us consider the conditions (1) and (2) in the theorem. If M is of general type, then the automorphism group of M is finite [1, 9]. Hence, M has no non-trivial holomorphic vector fields. This is also true if M is a K3 surface [2], and therefore if M is an Enriques surface. See also [6].

By the Kodaira-Nakano vanishing theorem [5, 11] and Serre duality, (2) is satisfied if the dual of the vector bundle $L^\vee \otimes TM$ is positive in the sense of Nakano. This is true if L is sufficiently ample. Hence, with the same notation as above, we have the following corollary.

2.2. Corollary. *Let X be a smooth hypersurface in a compact complex manifold with no non-trivial holomorphic vector fields, such as a manifold of general type. If X is sufficiently ample, then the Kodaira-Spencer map $\kappa : T_oB \to H^1(X, TX)$ is injective.*

We will now apply these results to show the non-existence of holomorphic neighbourhood retractions. We still consider a smooth connected hypersurface X in a compact complex manifold M. Let $L = [X]$ be the line bundle defined by X and assume that $\dim H^0(M, L) \geq 2$. Let $p : E \to B$ be the family associated to L and say $X = p^{-1}(o)$.

Let W be a neighbourhood of o in B, diffeomorphic to a ball. Then $p : p^{-1}(W) \to W$ is smoothly trivial, so we have a diffeomorphism

$$\psi : X \times W \to p^{-1}(W),$$

inducing smoothly varying diffeomorphisms $\psi_t : X \to X \times \{t\} \to X_t$ for each $t \in W$. We may assume that ψ_o is the identity map on X.

Suppose there is a neighbourhood U of X in M with a holomorphic retraction $\rho : U \to X$. We may assume that $X_t \subset U$ for each $t \in W$. Consider the smooth map $\beta_t = \rho \circ \psi_t : X \to X$. For $t \in W$ sufficiently close to o, β_t is so close to $\beta_o = \mathrm{id}_X$ in the smooth topology that the derivative of β_t is nonsingular, so β_t is a local diffeomorphism and hence a covering map. Also, β_t induces an automorphism of the fundamental group of X because it is homotopic to $\beta_o = \mathrm{id}_X$, so β_t is a one-sheeted covering map. Hence, β_t is a diffeomorphism, so $\rho : X_t \to X$ is a diffeomorphism. Being holomorphic, $\rho : X_t \to X$ is a biholomorphism.

This shows that the Kodaira-Spencer map $\kappa : T_oB \to H^1(X, TX)$ is trivial. If X is sufficiently ample and $H^0(M, TM) = 0$, this contradicts theorem 2.1. We have established the following result.

2.3. Theorem. *Let X be a connected hypersurface in a compact complex manifold M of dimension at least 2. If*

(1) $H^0(M, TM) = 0$,
(2) $\dim H^0(M, [X]) \geq 2$, *and*
(3) $H^1(M, [X]^\vee \otimes TM) = 0$,

then no neighbourhood of X retracts holomorphically onto X.

There is no need to assume that X is smooth in the statement of the theorem. If M is a smooth manifold and $f : M \to M$ is a smooth map with $f \circ f = f$, then $f(M)$ is a smooth closed submanifold of M [4].

2.4. Corollary. *If X is a sufficiently ample hypersurface in a compact complex manifold of general type, then no neighbourhood of X retracts holomorphically onto X.*

If X is a smooth, connected curve of genus at least 2 in a complex manifold, U is a neighbourhood of X and $\rho : U \to X$ is a holomorphic map such that $\rho|X$ is merely non-constant, then $\rho : X \to X$ is an isomorphism by the Riemann-Hurwitz formula. Hence, $\rho : U \to X$ is actually a retraction modulo an automorphism of X, so we have the following corollary.

2.5. Corollary. *Let X be a sufficiently ample smooth curve in a surface with no non-trivial holomorphic vector fields. If U is a neighbourhood of X and $\rho : U \to X$ is a holomorphic map, then $\rho|X$ is constant.*

3. Retractions of covering neighbourhoods

Let us state a few definitions. A complex manifold is said to be ultra-Liouville if it has no non-constant bounded continuous plurisubharmonic functions. A complex manifold is said to be Carathéodory hyperbolic or C-hyperbolic if it has a covering space on which bounded holomorphic functions separate points. We will say that a complex manifold M is locally C-hyperbolic if every point in the universal covering space $\tilde{M}$ of M has a neighbourhood U such that bounded holomorphic functions on $\tilde{M}$ separate points on U. Clearly, C-hyperbolicity implies local C-hyperbolicity.

Suppose M is compact. Choose a hermitian metric on M and pull it up to $\tilde{M}$. Define an equivalence relation on $\tilde{M}$ by declaring points x and y equivalent if $f(x) = f(y)$ for all bounded holomorphic functions f on $\tilde{M}$. The equivalence classes are subvarieties of $\tilde{M}$. We see that M is locally C-hyperbolic if and only if there is a constant $\delta > 0$ such that the distance between two distinct equivalent points in $\tilde{M}$ is at least δ.

The following lemma is a variant of the Borel-Narasimhan theorem [3].

3.1. Lemma. *Let M be an ultra-Liouville manifold and $\pi : X \to M$ be a covering space. Let Y be a complex manifold on which bounded holomorphic functions separate points locally. If $f, g : X \to Y$ are holomorphic maps and $f = g$ on $\pi^{-1}(p)$ for some $p \in M$, then $f = g$.*

Proof. Let h be a bounded holomorphic function on Y. Define a function u on M by the formula

$$u(m) = \sup_{\pi(x)=m} |h(f(x)) - h(g(x))|.$$

Then u is a bounded continuous plurisubharmonic function on M, and $u(p) = 0$, so $u = 0$. Hence, $h \circ f = h \circ g$.

If $x \in X$ and $f(x) = g(x) = y$, then there is a neighbourhood U of y on which bounded holomorphic functions on Y separate points, so $f = g$ on the neighbourhood $f^{-1}(U) \cap g^{-1}(U)$ of x. This shows that the closed subset $\{x \in X : f(x) = g(x)\}$ of X is open. By assumption, it is non-empty, so we conclude that $f = g$. $\square$

Now let X be a smooth ample hypersurface in a connected compact complex manifold M of dimension at least 2. Then M is projective algebraic, and by the Lefschetz hyperplane theorem, X is connected. Let $L = [X]$ be the line bundle defined by X and assume that $\dim H^0(M, L) \geq 2$. Let $p : E \to B$ be the associated family as described in the previous section and say $X = p^{-1}(o)$.

Recall that a neighbourhood U of X in M is called tubular if there is a diffeomorphism from the normal bundle of X in M onto U which restricts to the identity on X. By the tubular neighbourhood theorem [4, 7], X has a neighbourhood basis of tubular neighbourhoods.

Let U be a tubular neighbourhood of X and let $\pi : V \to U$ be the universal covering space with covering group Γ. The long exact homotopy sequence associated to the normal bundle of X shows that the inclusion $X \hookrightarrow U$ induces an isomorphism of fundamental groups. Hence, $Y = \pi^{-1}(X)$ is the universal covering space of X.

Let W be a neighbourhood of o in B, diffeomorphic to a ball, such that $X_t \subset U$ for each $t \in W$. Now $p : p^{-1}(W) \to W$ is smoothly trivial, so we have a diffeomorphism

$$\psi : X \times W \to p^{-1}(W)$$

inducing smoothly varying diffeomorphisms $\psi_t : X \to X \times \{t\} \to X_t = p^{-1}(t)$ for each $t \in W$. We may assume that ψ_o is the identity map on X.

For $t \in W$, let i_t be the inclusion $X_t \hookrightarrow U$. The map $i_t \circ \psi_t$ is homotopic to i_o, so i_t also induces an isomorphism $\pi_1(X_t) \to \pi_1(U)$. Hence, $Y_t = \pi^{-1}(X_t)$ is the universal covering space of X_t.

Suppose now that there is a holomorphic retraction $\rho : V \to Y$. Then $\rho : Y_t \to Y$ is the identity on $Y \cap Y_t$, which is non-empty, as we will now explain. Since L is ample, it has a hermitian metric of positive curvature with norm $\|\cdot\|$. Let s be a holomorphic section of L such that $X = \{s = 0\}$, and consider the function $u = -\log \|s\|$ on $M \setminus X$. The Levi form $i\partial\bar\partial u$ is the curvature form of the metric on L (up to multiplication by a positive constant), so u is strictly plurisubharmonic. Hence $M \setminus X$ has no compact subvarieties, so X_t must intersect X, and Y_t intersects Y. (In fact, u is also an exhaustion, so $M \setminus X$ is a Stein manifold.)

Now fix $\gamma \in \Gamma$ and $t \in W$. Consider the holomorphic maps $\gamma \circ \rho$ and $\rho \circ \gamma$ from Y_t to Y. They agree on $Y \cap Y_t$, which contains a Γ-orbit. If bounded holomorphic functions separate points locally on Y, then lemma 3.1 shows that $\gamma \circ \rho = \rho \circ \gamma$ on Y_t. Hence, ρ is Γ-equivariant on $\bigcup\{Y_t : t \in W\}$, which is a non-empty open subset of V, so ρ is Γ-equivariant on all of V and descends to a holomorphic retraction $U \to X$.

Note that a retraction of any covering space of U onto the preimage of X lifts to a retraction of V onto Y. Hence, we have proved the following theorem.

3.2. Theorem. *Let X be a locally C-hyperbolic ample hypersuface in a compact complex manifold M of dimension at least 2, such that X is linearly equivalent to some other hypersurface in M. Let $\pi : V \to U$ be a covering space of a tubular neighbourhood U of X. Then any holomorphic retraction $V \to \pi^{-1}(X)$ is a lifting of a retraction $U \to X$.*

For curves in surfaces, the theorem reads as follows.

3.3. Corollary. *Let X be an ample curve of genus at least 2 in a surface M such that X is linearly equivalent to some other curve in M. Let $\pi : V \to U$ be a covering space of a tubular neighbourhood U of X. Then any holomorphic retraction $V \to \pi^{-1}(X)$ is a lifting of a retraction $U \to X$.*

The genus formula states that a curve X in a surface M has genus

$$g = \tfrac{1}{2}(X^2 + X \cdot K) + 1,$$

where K is the canonical bundle of M. If M is not ruled and X is ample, then $g \geq 2$. Hence, the hypotheses of the corollary are satisfied if X is a very ample curve in a non-ruled surface M.

Theorems 2.3 and 3.2 yield the following result.

3.4. Theorem. *Let M be a compact complex manifold of dimension at least 2 with no non-trivial holomorphic vector fields. Let X be a locally C-hyperbolic ample hypersurface in M such that X is linearly equivalent to some other hypersurface in M, and*

$$H^1(M, [X]^\vee \otimes TM) = 0.$$

If $\pi : V \to U$ is a covering space of a tubular neighbourhood U of X, then V does not retract holomorphically onto $\pi^{-1}(X)$.

Finally, for curves, we obtain the following corollary.

3.5. Corollary. *Let X be a curve in a surface of general type and let $\pi : V \to U$ be a covering space of a tubular neighbourhood U of X. If X is sufficiently ample, then V does not retract holomorphically onto $\pi^{-1}(X)$.*

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