# COMPACT QUOTIENTS OF LARGE DOMAINS IN COMPLEX PROJECTIVE SPACE

Finnur Lárusson

University of Western Ontario

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Abstract. We study compact complex manifolds covered by a domain in $n$-dimensional projective space whose complement $E$ is non-empty with $(2n-2)$-dimensional Hausdorff measure zero. Such manifolds only exist for $n \geq 3$. They do not belong to the class $\mathcal{C}$, so they are neither Kähler nor Moishezon, their Kodaira dimension is $-\infty$, their fundamental groups are generalized Kleinian groups, and they are rationally chain connected. We also consider the two main classes of known 3-dimensional examples: Blanchard manifolds, for which $E$ is a line, and the generalized Schottky coverings constructed by Nori. We determine their function fields and describe the surfaces they contain.

## Introduction

It has been known for a long time that if a bounded domain $\Omega$ in $\mathbb{C}^n$ is a Galois covering space of a compact manifold $M$, then $\Omega$ is a domain of holomorphy and $M$ is projective, meaning that $M$ is isomorphic to a subvariety of some complex projective space $\mathbb{P}^N$. In fact, the canonical bundle of $M$ is ample. Bounded domains in $\mathbb{C}^n$ can be viewed as domains in $\mathbb{P}^n$ with a large complement: the complement is so large that it contains a hyperplane in its interior. In this paper, we study the other end of the spectrum, following suggestions of Nori [Nor] and Yau [Yau] that this might lead to new and interesting compact complex manifolds, outside the well-known and much-studied classes of manifolds that are algebraic or in some sense close to being algebraic.

We will consider compact complex manifolds $M$ covered by a domain $\Omega$ in $\mathbb{P}^n$ whose complement $E = \mathbb{P}^n \setminus \Omega$ is non-empty and small in the sense that the $(2n-2)$-dimensional Hausdorff measure $\Lambda_{2n-2}(E)$ vanishes. This condition is just strong enough to exclude

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hypersurfaces in $E$. Little work seems to have been done on this subject. Among the few relevant papers in the literature are [Kat1], [Kat2], [Kat3], [Kat4], [Nor], and [Yam].

In section 1, we establish basic properties of manifolds $M$ of this kind. These include:

(1) The Kodaira dimension of $M$ is $-\infty$.
(2) The covering group $\mathrm{Aut}_M \Omega$ is a subgroup of the automorphism group of $\mathbb{P}^n$, so $\pi_1(M)$ is in fact a generalized Kleinian group.
(3) There is a lower bound on the size of $E$, which implies that no 2-dimensional examples exist.
(4) $M$ is rationally chain connected. The limit set $E$ can be described in terms of rational curves in $M$ that respect the unique projective structure on $M$.
(5) $M$ is not of class $\mathcal{C}$. In particular, $M$ is neither Kähler nor Moishezon.

In sections 2 and 3, we study in some detail two classes of 3-dimensional examples: the generalized Schottky coverings constructed by Nori, and Blanchard manifolds, for which $E$ is a line, which is the smallest it can be. We determine their fields of meromorphic functions, and describe the surfaces they contain. In section 4, we make some final remarks on the general 3-dimensional case.

Let us clarify a few terms. By a curve in a complex manifold, we shall mean a (closed analytic) subvariety of pure dimension 1. A surface is a subvariety of pure dimension 2, and a hypersurface is a subvariety of pure codimension 1. When we speak of a manifold, we assume that it is connected.

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## 1. Properties of the quotient manifolds

Let $M$ be an $n$-dimensional compact complex manifold, $n \geq 2$, covered by a domain $\Omega$ in complex projective space $\mathbb{P}^n$ such that the $(2n-2)$-dimensional Hausdorff measure $\Lambda_{2n-2}(E)$ of the complement $E = \mathbb{P}^n \setminus \Omega$ is zero. Then $\Omega$ is simply connected, so it is the universal covering space of $M$. Let $\pi : \Omega \to M$ be the covering map, and $\Gamma \cong \pi_1(M)$ be the covering group. We assume that $\Omega \neq \mathbb{P}^n$, so $\Gamma$ is infinite.

Let us note that if $U$ is a domain in $\mathbb{P}^n$, then $U \cap \Omega$ is connected. For this, it actually suffices to have $\Lambda_{2n-1}(E) = 0$. Hence, the compactification $\mathbb{P}^n$ of $\Omega$ is finer than the end compactification of $\Omega$. Indeed, the connected components of $E$ correspond bijectively to the ends of $\Omega$, which in turn correspond bijectively to the ends of $\Gamma$ since $M$ is compact. In particular, $E$ is connected if and only if $\Gamma$ has only one end.

We will make much use of the following extension theorem, due to Shiffman [Shi1, Shi2]. See also [HP].

**1.1. Theorem (Shiffman).** *Let $E$ be a closed subset of an $n$-dimensional complex manifold $X$. If $\Lambda_{2n-2}(E) = 0$, then holomorphic, meromorphic, and plurisubharmonic* *functions extend from $X \setminus E$ to $X$. If $\Lambda_{2n-3}(E) = 0$, then the closure of a hypersurface in $X \setminus E$ is a hypersurface in $X$.*

The theorem implies that $M$ inherits many properties from $\mathbb{P}^n$. We see that $\Omega$ has no non-constant holomorphic or plurisubharmonic functions, and no non-zero holomorphic $p$-forms for $p \geq 1$, so

$$H^{p,0}(M) = 0, \qquad p \geq 1,$$

and $M$ has a trivial Albanese. Also, no positive power of the canonical bundle of $M$ has any non-zero holomorphic sections, so $M$ has Kodaira dimension $-\infty$.

**1.2. Proposition.** *Let $\varphi : \Omega \to \mathbb{P}^n$ be a holomorphic map.*

(1) *$\varphi$ extends to a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$.*

(2) *If $\varphi$ is an immersion, then $\varphi$ extends to an automorphism of $\mathbb{P}^n$.*

*In particular, every automorphism of $\Omega$ is the restriction of a unique automorphism of $\mathbb{P}^n$, so*

$$\Gamma \subset \operatorname{Aut} \mathbb{P}^n = \mathrm{PGL}(n+1, \mathbb{C}).$$

By Selberg's theorem [Sel], the proposition implies that $\Gamma$ has a normal torsion-free subgroup of finite index. It also implies that $E \subset \mathbb{P}^n$ is a biholomorphic invariant of $M$, modulo automorphisms of $\mathbb{P}^n$.

*Proof.* The meromorphic function $(z_i/z_0) \circ \varphi$ on $\Omega$ extends to a meromorphic function $\psi_i$ on $\mathbb{P}^n$, and $\psi = [1, \psi_1, \ldots, \psi_n]$ is a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ extending $\varphi$.

Write $\psi = [q_0, \ldots, q_n]$, where $q_0, \ldots, q_n$ are homogeneous polynomials in $z_0, \ldots, z_n$ of the same degree $d$, and let $\Psi = (q_0, \ldots, q_n) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. Let $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the canonical projection. Suppose $\varphi$ is an immersion. Then $\Psi | p^{-1}(\Omega)$ is an immersion. The zero set of the Jacobian determinant $J = \det[\partial q_i / \partial z_j]$ of $\Psi$ is either empty or a hypersurface in $p^{-1}(E) \cup \{0\}$. Since $p^{-1}(E)$ cannot contain a hypersurface, $J$ is a non-zero constant. Also, $J$ is homogeneous of degree $(d-1)^{n+1}$. Hence $d = 1$, so $q_0, \ldots, q_n$ are linear, and $\psi$ is an automorphism of $\mathbb{P}^n$. $\square$

Part (2) of the proposition can also be deduced from [Iva1, Theorem 1].

We remark that $\Omega$ is maximal among domains in $\mathbb{P}^n$ on which $\Gamma$ acts with a Hausdorff quotient. Indeed, if $\Omega'$ is a domain containing $\Omega$ on which $\Gamma$ acts with a Hausdorff quotient $M' = \Omega'/\Gamma$, then $M \subset M'$ is both open and compact, and hence closed, so since $M'$ is connected, $M = M'$. Therefore, $\Omega' \subset \Gamma\Omega = \Omega$, so $\Omega' = \Omega$.

We now show that there is a lower bound on the size of $E$.

**1.3. Proposition.** *If $n$ is even, then $\Lambda_n(E) > 0$. If $n$ is odd, then $\Lambda_{n-1}(E) > 0$.*

*Proof.* Suppose $\Lambda_{2n-2k}(E) = 0$ for an integer $k$ in $[0, n]$. Then $\Omega$ contains a $k$-dimensional complex linear subspace $S$. Find a sequence $\gamma_i \to \infty$ in $\Gamma$. Then $\gamma_i(S)$ converge to a

$k$-dimensional linear subspace in $E$, so $\Lambda_{2k}(E) > 0$. Hence, $2k < 2n - 2k$, so $k < n/2$. This shows that if $k \geq n/2$, then $\Lambda_{2n-2k}(E) > 0$, and the proposition follows. $\square$

Since $\Lambda_{2n-2}(E) = 0$, the proof shows that $E$ contains a line.

**1.4. Corollary.** *If a domain $\Omega$ in $\mathbb{P}^2$ covers a compact complex manifold, then $\Omega = \mathbb{P}^2$ or $\Lambda_2(\mathbb{P}^2 \setminus \Omega) > 0$.*

The proposition is sharp in the sense that we may have $\Lambda_{n+\epsilon}(E) = 0$ when $n$ is even and $\Lambda_{n-1+\epsilon}(E) = 0$ when $n$ is odd for all $\epsilon > 0$. To see this, let $k$ be $n/2$ if $n$ is even and $(n-1)/2$ if $n$ is odd, and consider the automorphism $\varphi$ given by the formula

$$\varphi[z_0, \ldots, z_k, z_{k+1}, \ldots, z_n] = [2z_0, \ldots, 2z_k, z_{k+1}, \ldots, z_n].$$

The group $\Gamma$ of iterates of $\varphi$ acts freely and properly on $\Omega = \mathbb{P}^n \setminus E$, where $E$ is the union of the two linear subspaces $\{z_0, \ldots, z_k = 0\}$ and $\{z_{k+1}, \ldots, z_n = 0\}$, and the quotient manifold $M = \Omega/\Gamma$ is compact.

Next we show that $M$ contains many rational curves.

**1.5. Proposition.** *If $L$ is a line in $\Omega$, then $\pi(L)$ is a rational curve in $M$. Hence, $M$ is rationally chain connected.*

*For a very general point $p \in M$ and every $v$ in a dense set of tangent vectors at $p$, there is a smooth rational curve through $p$ which is tangent to $v$.*

*For very general points $p_1$, $p_2$ in $M$, there is a connected curve containing $p_1$ and $p_2$ which is the union of two smooth rational curves.*

Recall that a general point is a point outside a finite union of proper subvarieties, and a very general point is a point outside a countable union of proper subvarieties.

For the proof, we need the following lemma.

**1.6. Lemma.** *For an automorphism $\varphi$ of $\mathbb{P}^n$, $n \geq 3$, the following are equivalent.*

(1) *$L \cap \varphi(L) \neq \varnothing$ for all lines $L$ in $\mathbb{P}^n$.*
(2) *$\varphi$ has a hyperplane of fixed points.*

*Proof.* (2) $\Rightarrow$ (1) is clear. For the converse, represent $\varphi$ by a matrix $A$ in $\mathrm{GL}(n+1, \mathbb{C})$. First let $n = 3$. In suitable coordinates $z_0, z_1, z_2, z_3$ in $\mathbb{C}^4$, $A$ has the Jordan form

$$A = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ \epsilon_1 & a_1 & 0 & 0 \\ 0 & \epsilon_2 & a_2 & 0 \\ 0 & 0 & \epsilon_3 & a_3 \end{bmatrix},$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$. Suppose $S \cap AS \neq 0$ for all 2-dimensional subspaces $S$ in $\mathbb{C}^4$. We need to show that $A$ has a 3-dimensional eigenspace.

Suppose first that $A$ is diagonal. Let $S = \{z_{i_0} = z_{i_1}, z_{i_2} = z_{i_3}\}$ with $i_0, i_1, i_2, i_3$ mutually distinct. Since $S \cap AS \neq 0$, we get $a_{i_0} = a_{i_1}$ or $a_{i_2} = a_{i_3}$. This means that three of the diagonal entries must be the same, so $A$ has a 3-dimensional eigenspace.

Now suppose $A$ is not diagonal; say $\epsilon_3 = 1$, so $a_2 = a_3$. Let $S = \{z_0 = z_3, z_1 = 0\}$. Since $S \cap AS \neq 0$, there are $x, y \in \mathbb{C}$, not both zero, such that

$$A \begin{bmatrix} x \\ 0 \\ y \\ x \end{bmatrix} = \begin{bmatrix} a_0 x \\ \epsilon_1 x \\ a_2 y \\ y + a_3 x \end{bmatrix} \in S,$$

so $\epsilon_1 x = 0$ and $a_0 x = y + a_3 x$. This implies that $\epsilon_1 = 0$. Taking $S = \{z_0 = z_2, z_3 = 0\}$, we get $\epsilon_2 = 0$. Taking $S = \{z_0 = z_1, z_3 = 0\}$, we get $a_0 = a_1$. Finally, taking $S = \{z_0 = z_2, z_1 = z_3\}$, we get $a_0 = a_2$, so $A$ has only one eigenvalue and a 3-dimensional eigenspace corresponding to it.

To disprove (1) in general, it suffices to find a 4-dimensional $A$-invariant subspace in $\mathbb{C}^{n+1}$ which does not contain a 3-dimensional eigenspace. By examining Jordan forms, it is easy to see that such a subspace exists precisely when $A$ does not have an $n$-dimensional eigenspace, i.e., when (2) fails. $\square$

*Proof of proposition 1.5.* Let $L$ be a line in $\Omega$. Then $\pi(L)$ is an irreducible curve in $M$. Now $y \in L$ is in a fibre of $\pi|L$ with more than one element if and only if $y \in L \cap \gamma L$ for some $\gamma \in \Gamma$, $\gamma \neq \mathrm{id}$. Since $\Gamma$ acts properly on $\Omega$, there are at most finitely many $\gamma \in \Gamma$ with $L \cap \gamma L \neq \varnothing$. Also, if $\gamma L = L$, then $\gamma$ has a fixed point in $L$, so $\gamma = \mathrm{id}$. Hence, $\pi|L$ is injective outside a finite set, so $\pi(L)$ is rational.

For $\gamma \in \Gamma$, let $Y(\gamma)$ be the set of $y \in \mathbb{P}^n$ such that $L \cap \gamma L \neq \varnothing$ for all lines $L$ through $y$. Then $Y(\gamma)$ is a subvariety of $\mathbb{P}^n$. We have $\beta Y(\gamma) = Y(\beta\gamma\beta^{-1})$ for $\beta \in \Gamma$. Also, for a compact subset $K$ of $\Omega$, we have $Y(\gamma) \cap K \neq \varnothing$ for only finitely many $\gamma \in \Gamma$ since $\Gamma$ acts properly on $\Omega$. This implies that $X(\gamma) = \pi(Y(\gamma) \cap \Omega)$ is a subvariety of $M$. Now $n \geq 3$ by corollary 1.4, so if $\gamma \neq \mathrm{id}$, then $X(\gamma) \neq M$ by lemma 1.6. Let $X = \bigcup_{\gamma \neq \mathrm{id}} X(\gamma)$.

Let $p \in M \setminus X$ (so $p$ is a very general point) and $q \in \pi^{-1}(p)$. For $\gamma \in \Gamma$, $\gamma \neq \mathrm{id}$, let $\mathcal{L}_\gamma$ be the set of lines $L$ in $\Omega$ through $q$ such that $L \cap \gamma L = \varnothing$. Then $\mathcal{L}_\gamma$ is open and dense in the $(n-1)$-dimensional projective space of lines through $q$ in $\mathbb{P}^n$. By the Baire category theorem, the intersection $\bigcap \mathcal{L}_\gamma$ is dense. If $L$ is in the intersection, then $\pi|L$ is injective, so $\pi(L)$ is a smooth rational curve through $p$.

Now let $p_1, p_2 \in M \setminus X$ and $q_k \in \pi^{-1}(p_k)$, $k = 1, 2$. For $\gamma \in \Gamma$, $\gamma \neq \mathrm{id}$, and $k = 1, 2$, let $S(\gamma, k)$ be the union of lines $L$ in $\Omega$ through $q_k$ such that $L \cap \gamma L = \varnothing$. Then $S(\gamma, k)$ is open and dense in $\Omega \setminus \{q_k\}$, so the intersection $\bigcap_{\gamma,k} S(\gamma, k)$ is dense in $\Omega$. Hence there are intersecting lines $L_1$, $L_2$ in $\Omega$ through $q_1$, $q_2$ respectively, such that $\pi$ is injective on both $L_1$ and $L_2$. Then $\pi(L_1) \cup \pi(L_2)$ is a connected union of two smooth rational curves in $M$ containing both $p_1$ and $p_2$. $\square$

The proposition implies that if $n = 3$ and the algebraic dimension $a$ of $M$ is 0 or 1 (examples of which will be given in sections 2 and 3), then there is no holomorphic surjection $f$ from $M$ onto a 2-dimensional complex manifold. Namely, if $a = 0$, then there are only finitely many surfaces in $M$. If $a = 1$, then there is a complex manifold $X$, a compact Riemann surface $Y$ (the algebraic reduction of $M$), a proper modification $g : X \to M$, and a holomorphic surjection $h : X \to Y$, such that with only finitely many exceptions, an irreducible surface in $M$ is an irreducible component of $g(h^{-1}(y))$ for some $y \in Y$. See [FF]. In either case, by the proposition, there is a smooth rational curve $C$ in $M$ which is not contained in any surface in $M$. But $C$ is contained in the surface $f^{-1}(f(C))$, which is absurd.

Let us recall that an atlas of holomorphic charts on $M$ is called *projective* if the charts map to open sets of $\mathbb{P}^n$ and the coordinate changes are restrictions of automorphisms of $\mathbb{P}^n$. A projective atlas on $M$ defines an element of $H^1(M, \mathrm{PGL}(n+1, \mathbb{C}))$, called a *projective structure* on $M$. Equivalently, a projective structure on $M$ is given by a conjugacy class of group homomorphisms $\pi_1(M) \to \mathrm{PGL}(n+1, \mathbb{C})$. Clearly, $M$ has a projective structure. For more information and references, see [Sim]. A projective structure on $M$ yields a developing map, which is a holomorphic immersion from the universal covering space $\Omega$ of $M$ to $\mathbb{P}^n$. By proposition 1.2, any such map is an automorphism of $\mathbb{P}^n$, so the projective structure on $M$ is unique.

The projective structure on $M$ defines a germ $\mathcal{F}_p$ of a holomorphic foliation at each point $p$ in $M$, obtained by pulling back a pencil of lines by a projective chart. If $p \in M$ and $q \in \pi^{-1}(p)$, then the leaf space $D_p$ of $\mathcal{F}_p$ is naturally identified with the space $\mathbb{P}^{n-1}$ of lines through $q$, so we have a linear projection of $\mathbb{P}^n \setminus \{q\}$ onto $D_p$. Since $E$ is $\Gamma$-invariant, its image in $D_p$ is well defined, regardless of the choice of $q$.

We say that a curve in $M$ respects the projective structure on $M$ if it appears as a union of straight lines in each projective chart, i.e., if its germ at every point $p$ (or merely at some point in each of its irreducible components) is a union of germs in $D_p$. Note that the rational curves constructed in the proof of proposition 1.5 are of this kind.

Our next result relates $E$ to rational curves in $M$ that respect the projective structure.

**1.7. Proposition.** *A germ in $D_p$ extends to a rational curve in $M$ if and only if it does not lie in the image of $E$ in $D_p$.*

A germ in the image of $E$ in $D_p$ may or may not extend to a curve in $M$. This can be verified by explicit computations for the example of a Blanchard manifold of type A given in [Kat2] (see section 3). There, some germs extend to a torus in $M$, and others do not extend to a curve at all.

*Proof.* Let $L$ be the line through $q$ corresponding to a germ in $D_p$ outside the image of $E$, so $L \subset \Omega$. Then the curve $\pi(L)$ in $M$ is rational by proposition 1.5.

Conversely, suppose $L$ is a line through $q$ that intersects $E$, and that the corresponding germ in $D_p$ extends to a rational curve $C$ in $M$. Then there is a non-constant map

$\mathbb{P}^1 \to C \subset M$, which lifts by $\pi$ to a map $\mathbb{P}^1 \to \Omega$, whose image lies in $\pi^{-1}(C) \subset \Gamma(L \setminus E)$. Hence, the image lies in a connected component of $\gamma(L \setminus E)$ for some $\gamma \in \Gamma$, but such a component is isomorphic a domain in $\mathbb{C}$, which is absurd. $\square$

It remains to be seen if new information about $E$ can be obtained from this result. It does, however, say something about rational curves in $M$ through a given point that respect the projective structure.

**1.8. Corollary.** *The set of germs in $D_p$ that do not extend to a rational curve in $M$ is closed, nowhere dense, non-empty, and, when $n = 3$, connected.*

*Proof.* Only the last statement needs to be proved. Let $L$ be a line in $\Omega$. Let $e$ be an end of $\Gamma$ and $(\gamma_n)$ be a sequence in $\Gamma$ converging to $e$. Then the lines $\gamma_n(L)$ converge to the connected component $E_0$ of $E$ corresponding to $e$, so $E_0$ contains a line. This shows that if $E_0$ is a connected component of $E$, then the image of $E_0$ in $D_p \cong \mathbb{P}^{n-1}$ contains a line. If $n = 3$, then two such lines must intersect, so the image of $E$ in $D_p$ is connected. $\square$

We conclude this section by showing that $M$ is far from being projective.

**1.9. Proposition.** *$M$ does not carry a Kähler metric.*

It is easy to see that a domain in $\mathbb{P}^n$ that contains a complex line does not admit a Shafarevich map (also known as a $\tilde{\Gamma}$-reduction). Since universal covering spaces of compact Kähler manifolds have Shafarevich maps [Cam2], this implies proposition 1.9. Also, a domain in a complex manifold is locally Stein if it covers a compact Kähler manifold [Iva2]. Since $\Omega$ is not locally Stein, proposition 1.9 follows. We will give a detailed proof using much simpler means.

The Fubini-Study metric on $\mathbb{P}^n$ is Kähler and invariant under unitary transformations, so the following corollary is immediate.

**1.10. Corollary.** $\Gamma \not\subset \mathrm{PU}(n+1)$.

The following lemma is well known. We supply a proof for the convenience of the reader.

**1.11. Lemma.** *If $M$ is a compact Kähler manifold and $H^{2,0}(M) = 0$, then $M$ is projective.*

*Proof.* Let $\omega$ be a Kähler form on $M$. We can find $\alpha \in H^2(M, \mathbb{Q})$ so close to $[\omega] \in H^2(M, \mathbb{R})$ that $\alpha$ is positive (but *a priori* not necessarily of type $(1,1)$ any more). For some integer $k > 0$ we have $k\alpha \in H^2(M, \mathbb{Z})$. Since $H^{2,0}(M) = 0$, the long exact sequence obtained from the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$ shows that $k\alpha$ is the Chern class of a line bundle $L$ on $M$. Then $L$ is positive, so $M$ is projective. $\square$

*Proof of proposition 1.9.* Suppose $M$ is Kähler. By the lemma, $M$ is projective, and hence Moishezon, but this contradicts the following proposition. $\square$

**1.12. Proposition.** *$M$ is not Moishezon.*

*Proof.* By the extension theorem 1.1, the field $\mathcal{M}(M)$ of meromorphic functions on $M$ can be identified with the field of $\Gamma$-invariant meromorphic functions on $\mathbb{P}^n$. Suppose $M$ is Moishezon, so $\mathcal{M}(M)$ has transcendence degree $n$ over $\mathbb{C}$. Now $\mathcal{M}(\mathbb{P}^n) = \mathbb{C}(X_1, \ldots, X_n)$ also has transcendence degree $n$ over $\mathbb{C}$, so $\mathcal{M}(\mathbb{P}^n)$ is algebraic over $\mathcal{M}(M)$.

Let $f \in \mathcal{M}(\mathbb{P}^n)$. Then there are $g_1, \ldots, g_k \in \mathcal{M}(M)$ such that

$$f^k + g_1 f^{k-1} + \cdots + g_k = 0.$$

Say $p \in \Omega$ and $g_1, \ldots, g_k$ are all finite at $p$. Then $f(\gamma p)$, $\gamma \in \Gamma$, are roots of the same polynomial, so the set $f(\Gamma p)$ is finite. Taking $f = z_i/z_0$, $i = 1, \ldots, n$, we see that $\Gamma$ has a finite orbit in $\Omega$, which is absurd. $\square$

When $n = 3$, by a result of Kato [Kat1, page 53], if the algebraic dimension of $M$ is nonzero, then there is a plane $P$ in $\mathbb{P}^3$ which is invariant under a subgroup $\Gamma_0$ of finite index in $\Gamma$. Then we have an embedding of $\Gamma_0$ into $\operatorname{Aut} P \cong \mathrm{PGL}(3, \mathbb{C})$ by $\gamma \mapsto \gamma|P$. One might say, therefore, that the "truly" 3-dimensional examples have algebraic dimension zero. In the following sections, we will see 3-dimensional examples with algebraic dimensions 0, 1, and 2.

Let us recall that a compact reduced complex space $X$ belongs to the class $\mathcal{C}$ (as defined by Fujiki) if there is a compact Kähler manifold $Y$ and a holomorphic surjection $Y \to X$. Equivalently, $X$ is bimeromorphically equivalent to a compact Kähler manifold. All reduced Moishezon spaces are contained in $\mathcal{C}$. For more information, see [CP] and the references therein.

We need the following property of the class $\mathcal{C}$.

**1.13. Theorem (Campana** [Cam1]**).** *Let $X$ be an irreducible compact complex space of class $\mathcal{C}$. Then a very general point in $X$ is contained in a largest irreducible Moishezon subvariety.*

The following result combines and strengthens propositions 1.9 and 1.12.

**1.14. Theorem.** *$M$ is not of class $\mathcal{C}$.*

*Proof.* Let $p$ be a point in $M$. If there is a largest irreducible Moishezon subvariety $Y$ through $p$, then $Y$ must be $M$ itself by proposition 1.5. But $M$ is not Moishezon by proposition 1.12, so $M$ is not of class $\mathcal{C}$ by Campana's theorem. $\square$

## 2. Schottky coverings

Nori [Nor] has constructed higher-dimensional analogues of the classical Schottky coverings in the following way. Let $n = 2k + 1$, $k \geq 1$, and $g \geq 1$. Choose $2g$ mutually disjoint linear subspaces $L_1, \ldots, L_{2g}$ of dimension $k$ in $\mathbb{P}^n$. Fix an integer $i$ with $1 \leq i \leq g$ and choose a basis so that

$$L_i = \{z_0, \ldots, z_k = 0\}, \qquad L_{g+i} = \{z_{k+1}, \ldots, z_n = 0\}.$$

Define $\varphi_i : \mathbb{P}^n \to \mathbb{R}$ by the formula

$$\varphi_i[z_0, \ldots, z_n] = \frac{|z_0|^2 + \cdots + |z_k|^2}{|z_0|^2 + \cdots + |z_n|^2},$$

and define open neighbourhoods

$$V_i = \{x \in \mathbb{P}^n : \varphi_i(x) < \alpha\}, \qquad V_{g+i} = \{x \in \mathbb{P}^n : \varphi_i(x) > 1 - \alpha\},$$

of $L_i$, $L_{g+i}$ respectively, where $0 < \alpha < \frac{1}{2}$. Define an automorphism $\gamma_i$ of $\mathbb{P}^n$ by the formula

$$\gamma_i[z_0, \ldots, z_n] = [\lambda z_0, \ldots, \lambda z_k, z_{k+1}, \ldots, z_n],$$

where $\lambda \in \mathbb{C}$ and $|\lambda| = \frac{1}{\alpha} - 1$. Then $\gamma_i(V_i) = \mathbb{P}^n \setminus \overline{V}_{g+i}$. Let $\Gamma$ be the subgroup of $\mathrm{PGL}(n+1, \mathbb{C})$ generated by $\gamma_1, \ldots, \gamma_g$. Let $A$ be the complement of $V_1 \cup \cdots \cup V_{2g}$, and let $\Omega = \bigcup_{\gamma \in \Gamma} \gamma A$.

Suppose $\alpha$ is so small that the closures of the sets $V_1, \ldots, V_{2g}$ are mutually disjoint. Any positive power of $\gamma_i$ maps $\mathbb{P}^n \setminus \overline{V}_i$ into $V_{g+i}$, and any negative power of $\gamma_i$ maps $\mathbb{P}^n \setminus \overline{V}_{g+i}$ into $V_i$. Hence, any non-trivial word in $\gamma_1, \ldots, \gamma_g$ maps the interior of $A$ into its complement, so it is not the identity, and $\Gamma$ is free on the generators $\gamma_1, \ldots, \gamma_g$. Also, $\Omega$ is a domain on which $\Gamma$ acts freely and properly with compact quotient $M$. Let $\pi : \Omega \to M$ be the covering map. The compact manifold $M$ is precisely the quotient space of $A$ obtained by identifying the disjoint subsets $\partial V_i$ and $\partial V_{g+i}$ of $A$ by the transformation $\gamma_i$ for $i = 1, \ldots, g$.

The complement $E$ of $\Omega$ in $\mathbb{P}^n$ is the closure of the $\Gamma$-orbit of $L_1 \cup \cdots \cup L_{2g}$. Its connected components are $k$-dimensional linear subspaces. When $g \geq 2$, $E$ is a "Cantor set of $k$-dimensional linear subspaces". Given $\epsilon > 0$, $E$ has $(2k+\epsilon)$-dimensional Hausdorff measure zero if $\alpha$ is small enough. Suppose $\alpha$ is so small that $\Lambda_{2n-2}(E) = 0$. Then $\Omega$ is simply connected, so $\pi_1(M) = \Gamma$ is free on $g$ generators.

In this section, $M$ will denote a manifold constructed as above. We will call $M$ a *Schottky manifold*.

Note that if $g = 1$, then the functions

$$\frac{z_1}{z_0}, \ldots, \frac{z_k}{z_0}, \frac{z_{k+2}}{z_{k+1}}, \ldots, \frac{z_n}{z_{k+1}}$$

descend to algebraically independent meromorphic functions on $M$, so the algebraic dimension $a(M)$ of $M$ is at least $n-1$. Also, $E = L_1 \cup L_2$, so $\Lambda_{2n-2}(E) = 0$, and $a(M) = n-1$ by proposition 1.12.

Now let $M$ be a 3-dimensional Schottky manifold with $g \geq 2$. The remainder of this section will be concerned with determining the function field of $M$ and the surfaces contained in $M$ for small values of $\alpha$.

Let $Y$ be an irreducible surface in $\mathbb{P}^3$ which is invariant under a subgroup of finite index in $\Gamma$. Let $1 \leq i \leq g$. Choose a basis so that $L_i$ and $L_{g+i}$ are given as above. Now $Y$ is invariant under $\gamma_i^m$ for some natural number $m \geq 1$, so if $[z_0, z_1, z_2, z_3] \in Y \setminus (L_i \cup L_{g+i})$, then $[\lambda^{mj} z_0, \lambda^{mj} z_1, z_2, z_3] \in Y$ for all $j \in \mathbb{Z}$. Hence, $Y$ has infinitely many points in common with the line $L$ through the points $[0, 0, z_2, z_3] \in L_i$ and $[z_0, z_1, 0, 0] \in L_{g+i}$, so $L \subset Y$. This shows that $Y$ is a union of lines intersecting both $L_i$ and $L_{g+i}$, along with either $L_i$ or $L_{g+i}$. If $Y$ contains only one of the lines $L_i$, $L_{g+i}$, then $Y$ is a plane containing that line. The same will hold for other values of $i$, but that is absurd, since two lines in a plane intersect. Hence, $Y$ contains all the lines $L_1, \ldots, L_{2g}$, and for each $i$ with $1 \leq i \leq g$, $Y$ is a union of lines intersecting both $L_i$ and $L_{g+i}$.

If $Z$ is another irreducible surface in $\mathbb{P}^3$ which is invariant under a subgroup of finite index in $\Gamma$, then both $Y$ and $Z$ are invariant under the same subgroup $\Gamma'$ of finite index in $\Gamma$. Then $Y \cap Z$ contains the $\Gamma'$-orbit of $L_1 \cup \cdots \cup L_{2g}$, which is a union of an infinite number of mutually disjoint lines, so $Y = Z$. This shows that there is at most one surface in $\mathbb{P}^3$ invariant under a subgroup of finite index in $\Gamma$.

If $f$ is a meromorphic function on $M$ and $\Lambda_4(E) = 0$, then $f \circ \pi$ extends to a meromorphic function $h$ on $\mathbb{P}^3$. Applying the above to the level sets of $h$ gives the following result.

**2.1. Proposition.** *Let $M$ be a 3-dimensional Schottky manifold with $g \geq 2$. If $\Lambda_4(E) = 0$, which is the case if $\alpha$ is small enough, then $M$ has no non-constant meromorphic functions.*

In the remainder of the section, we assume that $\Lambda_3(E) = 0$.

Let $S$ be a smooth surface in $M$. The closure $Y$ of $\pi^{-1}(S)$ is a $\Gamma$-invariant surface in $\mathbb{P}^3$. If $Y$ is not smooth with singular locus $Z$, then $Z \subset E$ and $Z$ is $\Gamma$-invariant. Since a group with infinitely many ends acts on its space of ends with dense orbits [Kul], this contradicts $Z$ having only a finite number of connected components. Hence, $Y$ is smooth and irreducible. Since $Y$ is covered by rational curves, its degree is 1, 2, or 3. Since $Y$ contains disjoint lines, it cannot be a plane. Since $Y$ contains more than 27 lines, it cannot be a cubic. Hence, $Y$ is a quadric. In suitable projective coordinates, $Y$ is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding, and $Y$ has two rulings by lines, which are the only lines in $Y$. Two disjoint lines in $Y$ must belong to one of the rulings. In particular, the lines $L_1, \ldots, L_{2g}$ lie in a quadric.

Conversely, suppose $L_1, \ldots, L_{2g}$ lie in a smooth quadric $Q$ in $\mathbb{P}^3$. A line in $\mathbb{P}^3$ is $\Gamma$-invariant if and only if it intersects all the lines $L_1, \ldots, L_{2g}$. Since $Q$ is ruled by such lines, it is $\Gamma$-invariant. Also, $E$ is a union of lines in one of the rulings. This ruling consists of the fibres of a projection $p : Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. There is an induced action of $\Gamma$ on $\mathbb{P}^1 \setminus p(E)$, and the quotient map $\mathbb{P}^1 \setminus p(E) \to C$ is a 1-dimensional Schottky covering onto a compact Riemann surface of genus $g$. The image of $Q \setminus E$ in $M$ is a smooth surface, ruled over $C$.

We have proved the following result.

**2.2. Proposition.** *Let $M$ be a 3-dimensional Schottky manifold with $g \geq 2$. Suppose $\Lambda_3(E) = 0$, which is the case if $\alpha$ is small enough. Then $M$ contains at most one surface. Also, $M$ contains a smooth surface $S$ if and only if the lines $L_1, \ldots, L_{2g}$ lie in a smooth quadric. Then $S$ is a ruled surface of genus $g$.*

It is easy to see that four lines in $\mathbb{P}^3$ in general position do not lie in a quadric.

Finally, suppose $S$ is a non-smooth surface in $M$. Then the singular locus $Z$ of $Y$ is $\Gamma$-invariant and not contained in $E$. For each point $x \in Z \setminus E$ and each $i$ with $1 \leq i \leq g$, there is a line in $Z$ through $x$ intersecting both $L_i$ and $L_{g+i}$. The singular locus of $Z$ is finite, but also $\Gamma$-invariant, so it is empty and $Z$ is smooth. Hence, through every point in $Z \setminus E$ there is a line in $Z$ intersecting all the lines $L_1, \ldots, L_{2g}$, and $Z$ is a disjoint union of a finite number of such lines.

We see that if $M$ contains a surface, then there is a $\Gamma$-invariant line in $\mathbb{P}^3$. If $g \geq 3$ and $L_1, \ldots, L_{2g}$ are in general position, then no such line exists.

## 3. Blanchard manifolds

As before, we let $M$ be a compact complex manifold whose universal covering space is a domain $\Omega$ in $\mathbb{P}^n$, such that the complement $E = \mathbb{P}^n \setminus \Omega$ is non-empty with $\Lambda_{2n-2}(E) = 0$. The covering map is $\pi : \Omega \to M$, and the covering group $\Gamma \cong \pi_1(M)$ can be considered as a subgroup of $\mathrm{PGL}(n+1, \mathbb{C})$.

**3.1. Proposition.** *Suppose $E$ is a subvariety of $\mathbb{P}^n$. If there is a $\Gamma$-invariant curve $X$ in $\mathbb{P}^n$, not contained in $E$, then $\Gamma$ is a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$. Moreover, the normalization of each irreducible component of $\pi(X \setminus E)$ is a torus.*

*Proof.* Now $X \setminus E$ is $\Gamma$-invariant and has finitely many irreducible components. The elements of $\Gamma$ that leave each component invariant form a normal subgroup $\Gamma'$ of finite index. Let $C$ be an irreducible component of the curve $\pi(X \setminus E)$ in $M$, and let $Y$ be an irreducible component of $\pi^{-1}(C)$. Then $Y$ is an irreducible component of $X \setminus E$. Also, $Y$ is a non-compact covering space over $C$ whose covering group $\Gamma''$ contains $\Gamma'$.

Let $Z$ be the irreducible component of $X$ containing $Y$. Then $Z \setminus Y$ is finite and non-empty. The normalization $\hat{Y}$ of $Y$ is naturally identified with the complement of a non-empty finite set in the normalization of $Z$. Each element of $\Gamma''$ lifts to a unique automorphism of $\hat{Y}$. These are the deck transformations of the induced covering map $\hat{Y} \to \hat{C}$. Since $\hat{Y}$ is a compact Riemann surface with a finite non-zero number of points removed, and $\hat{Y}$ has an infinite automorphism group, $\hat{Y}$ is either the complex plane or the punctured plane. In the first case, $\Gamma'' = \mathbb{Z}^2$, and in the second case, $\Gamma'' = \mathbb{Z}$. In both cases, $\hat{C}$ is a torus. $\square$

**3.2. Corollary.** *Suppose $E$ is a curve (so $n = 3$). If $M$ contains two distinct irreducible surfaces $S_1$ and $S_2$ which intersect, then $\Gamma$ is a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$. The intersection $S_1 \cap S_2$ is a curve, and the normalization of each of its irreducible components is a torus.*

*Proof.* The closure $Y_k$ of $\pi^{-1}(S_k)$ is a surface in $\mathbb{P}^3$, and $Y_1 \cap Y_2 \not\subset E$ is a $\Gamma$-invariant curve. $\square$

Now we restrict ourselves to the lowest dimensional case, so we let $n = 3$. We have seen that the smallest $E$ can be is a line. If $E$ is a line, then $M$ is called a *Blanchard manifold*. The first example of such a manifold was given in [Bla]; see also [Kat1]. Kato [Kat2] has shown that if $M$ is a Blanchard manifold, then $\Gamma$ is torsion-free and contains a subgroup $\Gamma_0$ of finite index which is isomorphic to $\mathbb{Z}^4$. Furthermore, in a suitable system of homogeneous coordinates $z_0, z_1, z_2, z_3$, which we will henceforth adopt, we have $E = \{z_2, z_3 = 0\}$, and either $\Gamma_0 = \Gamma \cap G$ or $\Gamma_0 = \Gamma \cap H$, where

$$G = \{ \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{C}\}$$

and

$$H = \{ \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : a, b, c, d \in \mathbb{C}\},$$

considered as subgroups of $\mathrm{PGL}(4, \mathbb{C})$. Note that the groups $G$ and $H$ are abelian. A Blanchard manifold $M$ is said to be of type A if $\pi_1(M)$ contains an abelian subgroup of finite index which is conjugate to a subgroup of $G$, but $\pi_1(M)$ does not contain an abelian subgroup of finite index which is conjugate to a subgroup of $H$. If $M$ is not of type A, then $M$ is said to be of type B. Blanchard manifolds of different types are non-isomorphic. There are examples of both types [Kat2].

The Blanchard manifold $M$ constructed in [Bla] is of type B. It is diffeomorphic to the product of the 2-sphere and the real 4-torus [Kat1]. Hence, the underlying smooth manifold of $M$ carries a projective algebraic complex structure, in addition to the complex structure being considered here, which by theorem 1.14 is not of class $\mathcal{C}$. In particular, $M$ does not fail to be Kähler due to any topological obstruction. Note that there are no 2-dimensional examples of this kind, due to the fact that a compact complex surface carries a Kähler metric if and only if its first Betti number is even.

From now on, we let $M$ be a Blanchard manifold.

Consider the planes in $\mathbb{P}^3$ containing the line $E$. They foliate $\Omega$, and the leaf space is $\mathbb{P}^1$. The covering group $\Gamma$ permutes these planes, so there is an induced holomorphic foliation $\mathcal{F}$ of $M$. Its leaves are the images of the planes. The leaves are smooth, but not necessarily closed in $M$.

**3.3. Proposition.** *A plane $P$ in $\mathbb{P}^3$ containing $E$ is invariant under a subgroup of finite index in $\Gamma$ if and only if $T = \pi(P \setminus E)$ is closed in $M$. Then $T$ is a smooth connected surface in $M$, and $T$ is either a torus or a hyperelliptic surface. If $\Gamma_0 = \Gamma$, then $T$ is a torus.*

*Proof.* First of all, $T$ is closed in $M$ if and only if $\Gamma(P \setminus E) = \pi^{-1}(T)$ is closed in $\Omega$ if and only if the $\Gamma$-orbit of $P$ consists of finitely many planes if and only if $P$ is invariant under a subgroup of finite index in $\Gamma$.

Suppose the subgroup $\Gamma' = \{\gamma \in \Gamma : \gamma P = P\}$ is of finite index in $\Gamma$. If $p \in T$ and $x, y \in \pi^{-1}(p)$, then $y = \gamma x$ for some $\gamma \in \Gamma$, so $\gamma P \cap P \cap \Omega \neq \varnothing$, and $\gamma P = P$. This shows that $P \setminus E \cong \mathbb{C}^2$ is a Galois covering space of $T$ with covering group $\Gamma'$.

Now $P$ is given by an equation of the form $sz_2 + tz_3 = 0$ with $s, t \in \mathbb{C}$. Simple computations show that the zero-free holomorphic 2-form $dz_0 \wedge dz_1$ on $P \setminus E$ is $\Gamma_0$-invariant, so it descends to $T_0 = (P \setminus E)/(\Gamma' \cap \Gamma_0)$. Hence, the canonical bundle of $T_0$ is trivial, and the Kodaira dimension of $T_0$ is zero. Also, $\Gamma' \cap \Gamma_0$ is of finite index in $\Gamma_0$, which is isomorphic to $\mathbb{Z}^4$, so the first Betti number of $T_0$ is 4. Hence, by the Enriques-Kodaira classification [BPV], $T_0$ is a torus.

This shows that $T$ has a torus as a finite unbranched covering. Hence, $T$ is Kähler, the Kodaira dimension of $T$ is zero, and the fundamental group of $T$ is infinite. By the Enriques-Kodaira classification, $T$ is either a torus or a hyperelliptic surface. $\square$

We recall that every hyperelliptic surface $X$ is the quotient of a torus (in fact, a product of two smooth elliptic curves) by a finite group acting freely. In particular, $X$ is projective, so any torus covering $X$ must be an abelian variety. Furthermore, a torus is never homeomorphic to a hyperelliptic surface, e.g. since they have different first Betti numbers (4 and 2 respectively).

If $M$ is of type B, then all the planes containing $E$ are invariant under $\Gamma_0$, so by the proposition, $M$ is foliated by smooth surfaces, and each leaf is either a torus or a hyperelliptic surface. On the other hand, if $M$ is of type A, then only the plane $\{z_3 = 0\}$ is invariant under a subgroup of finite index in $\Gamma$, so by the proposition, $\mathcal{F}$ has only one closed leaf, and this leaf is a torus or a hyperelliptic surface.

Now let $T = \pi(P \setminus E)$, where $P = \{z_3 = 0\}$. We know that $T$ is a smooth connected surface in $M$. Suppose $S \neq T$ is a surface in $M$. The closure $X$ of $\pi^{-1}(S)$ is a $\Gamma$-invariant surface in $\mathbb{P}^3$ which intersects $P$ in a curve. By proposition 3.1 (applied with $\Gamma$ replaced by $\Gamma_0$), the intersection must be $E$. Also by proposition 3.1, $X \setminus E$ is smooth, so $S$ is smooth. Choose an irreducible component $Y$ of $X$. It is the zero locus of an irreducible homogeneous polynomial $g$. Since $Y \cap P = E$, we may take $g = z_2^m + z_3 h$, where $m \geq 1$ is the degree of $g$, and $h$ is a polynomial. Now $Y$ is invariant under a subgroup of finite index in $\Gamma$, so for every $\gamma$ in a subgroup of finite index in $\Gamma_0$ there is a constant $c \neq 0$ such that

$$cg = \gamma^* g = (z_2 + az_3)^m + z_3 \gamma^* h = z_2^m + z_3 k,$$

where $a \in \mathbb{C}$ and $k$ is a polynomial, so $c = 1$ and $g$ is $\gamma$-invariant. Hence, the non-constant meromorphic function $f = g/z_3^m$ on $\mathbb{P}^3$ is invariant under a normal subgroup $\Gamma_1$ of finite index in $\Gamma$. The indeterminacy locus of $f$ is $E$. The restriction $f|\Omega : \Omega \to \mathbb{P}^1$ is a $\Gamma_1$-invariant holomorphic map with smooth fibres, and it descends to a holomorphic map $f_1 : M_1 = \Omega/\Gamma_1 \to \mathbb{P}^1$. Let $f_2 : M_1 \to C_1$ be the Stein factorization of $f_1$. The 1-dimensional complex space $C_1$ is normal, and hence smooth, so it is a compact Riemann surface. The fibres of $f_2$ are the connected components of the fibres of $f_1$.

The finite group $\Gamma/\Gamma_1$ acts on $M_1$ with quotient $M$. By corollary 3.2, the fibres of $f_2$ are the only irreducible surfaces in $M_1$, so they are permuted by $\Gamma/\Gamma_1$. Hence, there is an induced action of $\Gamma/\Gamma_1$ on $C_1$ that makes $f_2$ equivariant. Passing to quotients, we obtain a non-constant holomorphic map $\psi : M \to C$, where $C$ is a normal complex space, and hence a compact Riemann surface. The fibres of $\psi$ are smooth, and they are the images of the fibres of $f_2$, so they are connected. The map $\psi$ lifts to a non-constant holomorphic map from $\Omega$ to the universal covering $\tilde{C}$ of $C$, but $\Omega$ carries no non-constant holomorphic functions, so $\tilde{C}$ must be $\mathbb{P}^1$, and $C = \mathbb{P}^1$.

So far we have proved the following.

**3.4. Theorem.** *If $M$ is a Blanchard manifold, then one of the following holds.*

(1) *$M$ contains only one surface, which is a torus or a hyperelliptic surface. Hence, $M$ has no non-constant meromorphic functions.*
(2) *There is a holomorphic map $\psi : M \to \mathbb{P}^1$ with smooth connected fibres, which are the only irreducible surfaces in $M$. Hence, $\mathcal{M}(M) = \mathcal{M}(\mathbb{P}^1) \circ \psi$, and the algebraic dimension of $M$ is 1.*

*If $M$ is of type B, then* (2) *holds, and the fibres of $\psi$ are the leaves of $\mathcal{F}$.*

We now wish to understand the dichotomy in the theorem when $M$ is of type A. First we consider the automorphism group of $M$. Let $N_\Gamma$ be the normalizer of $\Gamma$ in $\mathrm{PGL}(4, \mathbb{C})$. If $\nu \in N_\Gamma$, then $\Gamma\nu E = \nu\Gamma E = \nu E$, so the line $\nu E$ is $\Gamma$-invariant. Now $\nu E \not\subset \Omega$, so $\nu E$ must intersect $E$ in the unique $\Gamma$-fixed point $[1, 0, 0, 0]$ of $E$. Computations show that $E$ is the only $\Gamma$-invariant line through $[1, 0, 0, 0]$, so $\nu E = E$. This shows that $\Omega$ is $N_\Gamma$-invariant. Therefore, every $\nu \in N_\Gamma$ induces an automorphism of $M$, which is the identity if and only if $\nu \in \Gamma$. Conversely, every automorphism of $M$ lifts to an element of $N_\Gamma$, so

$$\mathrm{Aut}\, M = N_\Gamma/\Gamma.$$

Let $P = \{z_3 = 0\}$ be the unique plane in $\mathbb{P}^3$ containing $E$, which is invariant under a subgroup of finite index in $\Gamma$. Since $P$ is $\Gamma_0$-invariant, $\nu P$ is invariant under the group $\nu\Gamma_0\nu^{-1}$, which is of finite index in $\Gamma$, so $\nu P = P$. This shows that the surface $T = (P \setminus E)/\Gamma$ in $M$ is $\mathrm{Aut}\, M$-invariant.

Assume now that $\Gamma \subset G$. This amounts to replacing $M$ by the finite unbranched covering space $M_0 = \Omega/\Gamma_0$. Clearly, (1) in theorem 3.4 holds for $M$ if and only if it holds for $M_0$.

Since $G$ is abelian, $G \subset N_\Gamma$. This shows that the dimension of the complex Lie group $\operatorname{Aut} M$ is at least 3. It is easy to see that $G$ acts transitively on $\Omega \setminus P$ (with trivial stabilizers) and on $P \setminus E$, so $G$ acts transitively on $M \setminus T$ and on $T$. Hence, $\operatorname{Aut} M$ has precisely two orbits, namely $M \setminus T$ and $T$. In particular, $M$ is almost homogeneous.

Let

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The commuting matrices $N$, $N^2$, and $N^3$ form a basis for the Lie algebra $\mathfrak{g}$ of $G$, and the exponential map $\exp : \mathfrak{g} \to G$,

$$rN + sN^2 + tN^3 \mapsto I + rN + (\tfrac{1}{2}r^2 + s)N^2 + (t + \tfrac{1}{6}r^3 + rs)N^3,$$

is an algebraic group isomorphism with inverse $\log : G \to \mathfrak{g}$,

$$I + aN + bN^2 + cN^3 \mapsto aN + (b - \tfrac{1}{2}a^2)N^2 + (c - ab + \tfrac{1}{3}a^3)N^3.$$

Suppose now that $T$ is not the only surface in $M$. By theorem 3.4, there is a holomorphic map $\psi : M \to \mathbb{P}^1$ with smooth connected fibres, which are the only irreducible surfaces in $M$. Say $T = \psi^{-1}(\infty)$. Then $G$ permutes the fibres of $\psi$, and we get an induced action of $G$ on $\mathbb{P}^1$ which is transitive on $\mathbb{C}$ and fixes $\infty$. The stabilizer of $0 \in \mathbb{C}$ is a proper algebraic subgroup $H$ of $G$, and $\Gamma \subset H$. In particular, $\Gamma$ is not Zariski-dense in $G$. We have $H = \exp \mathfrak{h}$, where $\mathfrak{h}$ is a 2-dimensional subspace of $\mathfrak{g}$. If $\mathfrak{h}$ is defined by an equation of the form $ar + bs = 0$ with $a, b \in \mathbb{C}$, then the polynomial function

$$f[z, w, 1, 0] = bz - \tfrac{1}{2}bw^2 + aw$$

on $P \setminus E$ is $H$-invariant, and hence $\Gamma$-invariant, so $f$ descends to a non-constant holomorphic function on $T$, but this is absurd since $T$ is compact. Hence, $\mathfrak{h}$ is defined by an equation of the form $t = ar + bs$ with $a, b \in \mathbb{C}$.

For $r \in \mathbb{C}$, let $w_r$ be a solution of the quadratic equation

$$w^2 + (r + b)w + \tfrac{1}{3}r^2 + \tfrac{1}{2}br - a = 0,$$

and let

$$s_r = -r(\tfrac{1}{2}r + w_r).$$

Then

$$h_r = \exp(rN + s_rN^2 + (ar + bs_r)N^3) \in H$$

maps $p_r = [0, 0, w_r, 1]$ to $[0, 0, w_r + r, 1]$. Since $w_r \to \infty$ and $w_r + r \to \infty$ as $r \to \infty$, both $p_r$ and $h_r p_r$ converge to the point $[0, 0, 1, 0]$ in $\Omega$. This shows that $H$ does not act properly on $\Omega$.

Now $\Gamma = \exp \Lambda$, where $\Lambda$ is a lattice in $\mathfrak{h}$. There is a constant $c_0 > 0$ such that for every $v \in \mathfrak{h}$ there is $u \in \Lambda$ with $|u - v| < c_0$. Hence, there is a constant $c > 0$ such that for every $h \in H$ there is $\gamma \in \Gamma$ such that all the off-diagonal entries of $\gamma h^{-1}$ have absolute value less than $c$. Let $\gamma_r \in \Gamma$ be associated to $h_r$ in this way. Then $\gamma_r \to \infty$ as $r \to \infty$. Let $U$ be the neighbourhood of the point $[0, 0, 1, 0]$ defined by the inequality

$$|z_0| + |z_1| + |z_3| < |z_2|.$$

Then $U$ is relatively compact in $\Omega$. There is $\rho > 0$, depending only on $c$, such that if $g \in G$ and all the off-diagonal entries of $g$ have absolute value less than $c$, then $gU$ lies in the compact subset $K$ of $\Omega$ defined by the inequality

$$|z_0| + |z_1| \leq \rho(|z_2| + |z_3|).$$

For $r \in \mathbb{C}$ sufficiently large, we have $h_r p_r \in U$, so $\gamma_r p_r = (\gamma_r h_r^{-1}) h_r p_r \in K$. This shows that $\Gamma$ does not act properly on $\Omega$, contrary to hypothesis.

We have proved the following.

**3.5. Theorem.** *If $M$ is a Blanchard manifold of type A, then $M$ contains only one surface, and $\Gamma$ is Zariski-dense in $G$.*

This shows that the classification of Blanchard manifolds by type is the same as their classification by algebraic dimension.

Note that $\Gamma$ is Zariski-dense in $G$ if and only if $\log(\Gamma \cap G)$ spans $\mathfrak{g}$ as a $\mathbb{C}$-vector space.

It would be interesting to have a characterization of those discrete Zariski-dense subgroups $\Gamma$ of $G$ of rank 4 that act freely and properly on $\Omega$ with compact quotient, so that $\Omega/\Gamma$ is a Blanchard manifold of type A.

Proposition 3.3 and theorems 3.4 and 3.5 imply the following.

**3.6. Corollary.** *An irreducible surface in a Blanchard manifold is a torus or a hyperelliptic surface. If $\Gamma_0 = \Gamma$, then it is a torus.*

If $M$ is of type B, then the map $\psi : M \to \mathbb{P}^1$ in theorem 3.4 is a proper submersion, and hence a smooth fibre bundle by Ehresmann's fibration theorem [BJ], over the complement of its finite set of critical values. Hence, all but finitely many of the surfaces in $M$ are mutually diffeomorphic, so either all but finitely many surfaces in $M$ are tori, or all but finitely many surfaces in $M$ are hyperelliptic.

**3.7. Example.** We will now show that hyperelliptic surfaces can occur in Blanchard manifolds of type A.

In [Kat2], Kato constructs a Blanchard manifold $M_0 = \Omega/\Gamma_0$ of type A, such that $\Gamma_0$ is the subgroup of $G$ generated by $I + N$, $I + iN$, $I + N^2$, and $I + iN^2$.

Let

$$\varphi = \begin{bmatrix} 1 & 1 & i/2 & 0 \\ 0 & -1 & -1 & -i/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \in \mathrm{PGL}(4, \mathbb{C}).$$

Then

$$\varphi N = -N\varphi,$$

so $\varphi$ commutes with $I + N^2$ and $I + iN^2$. Also,

$$\varphi(I+N)\varphi^{-1} = I - N \qquad \text{and} \qquad \varphi(I+iN)\varphi^{-1} = I - iN$$

are in $\Gamma_0$, since their logarithms lie in the $\mathbb{Z}$-span of the logarithms of the given generators of $\Gamma_0$. Hence, $\Gamma_0$ is a normal subgroup of the subgroup $\Gamma$ of $\mathrm{PGL}(4,\mathbb{C})$ spanned by $\Gamma_0$ and $\varphi$. Moreover, $\varphi^2 = I + (i-1)N^2 \in \Gamma_0$, so $\Gamma_0$ is of index 2 in $\Gamma$.

This shows that $\varphi$ induces a holomorphic involution of $M_0$. It has a fixed point in $M_0$ if and only if there are $\gamma \in \Gamma_0$ and $x \in \Omega$ such that $\varphi x = \gamma x$. Then

$$\varphi^2 x = \varphi\gamma x = (\varphi\gamma\varphi^{-1})(\varphi x) = \varphi\gamma\varphi^{-1}\gamma x,$$

so $\varphi^2 = \varphi\gamma\varphi^{-1}\gamma$ since $\Gamma_0$ acts freely on $\Omega$. Hence, $\varphi\gamma^{-1} \in \Gamma$ is of order 2. We will show that this cannot happen.

For $\gamma = I + aN + bN^2 + cN^3 \in \Gamma_0$, we have

$$(\varphi\gamma)^2 = \begin{bmatrix} 1 & 0 & 2b - a^2 - 1 + i & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so if $(\varphi\gamma)^2 = \mathrm{id}$, then $a^2+1 = 2b+i$. Now $\log\gamma$ is a $\mathbb{Z}$-linear combination of the logarithms of the given generators of $\Gamma_0$, so

$$a = n_1 + in_2, \qquad b = \tfrac{1}{2}(n_1 + in_2)^2 + \tfrac{1}{2}(n_2 - n_1) + n_3 + in_4,$$

where $n_1, \ldots, n_4 \in \mathbb{Z}$. Hence, the imaginary part of $a^2+1$ is an even integer, whereas the imaginary part of $2b+i$ is an odd integer. This shows that $\Gamma$ has no elements of order 2, so $\varphi$ has no fixed points in $M_0$, and $M = M_0/\varphi = \Omega/\Gamma$ is a Blanchard manifold of type A with fundamental group $\Gamma$.

The unique surface $T = (P \setminus E)/\Gamma$ in $M$ has the torus $(P\setminus E)/\Gamma_0$ in $M_0$ as a 2-sheeted unbranched covering space, and its fundamental group is $\Gamma$. Let $\alpha : \Gamma \to \Gamma' = \Gamma/[\Gamma,\Gamma]$ be the abelianization map. Then $\alpha(\Gamma_0)$ has finite index (1 or 2) in $\Gamma'$. Also, $\Gamma_0$ contains non-trivial commutators, such as $\varphi(I-N)\varphi^{-1}(I-N)^{-1}$, so the rank of $\Gamma'$, i.e., the first Betti number of $T$, is less than 4. Hence, $T$ is hyperelliptic.

It has been pointed out that certain Blanchard manifolds of type B are twistor spaces, as are certain 3-dimensional Schottky manifolds with $g=1$. See [Bes, Chapter 13.D,E] and [Hit]. On the other hand, we have the following result.

**3.8. Proposition.** *A Blanchard manifold of type A is not a twistor space.*

*Proof.* Suppose a Blanchard manifold of type A is a twistor space. The twistor structure pulls up to the universal covering $\Omega$, so $\Omega$ is smoothly fibred by smooth rational curves, and $\Omega$ has a $\Gamma$-equivariant fixed-point-free anti-holomorphic involution $\tau$ preserving the fibres, which extends to a $\Gamma$-equivariant anti-holomorphic involution of $\mathbb{P}^3$. The normal bundle $N$ of a fibre $C$ is $H_1 \oplus H_1$, where $H_k$ denotes the hyperplane bundle on $\mathbb{P}^k$.

By the short exact sequence

$$0 \to TC \to T\mathbb{P}^3|C \to N \to 0$$

on $C$, we have

$$4c_1(H_3|C) = c_1(T\mathbb{P}^3|C) = c_1(TC) + c_1(N) = 2c_1(H_1) + 2c_1(H_1)$$

in $H^2(C, \mathbb{Z}) = \mathbb{Z}$, so $\deg C = \deg(H_3|C) = 1$, and $C$ is a line.

Let $P = \{z_3 = 0\}$ be the unique $\Gamma_0$-invariant plane in $\mathbb{P}^3$ containing $E$. Now $\tau(z) = \varphi(\bar{z})$ for some $\varphi \in \operatorname{Aut} \mathbb{P}^3$, so $\tau P$ is a plane in $\mathbb{P}^3$ containing $\tau E = E$, invariant under $\tau \Gamma_0 \tau^{-1} = \Gamma_0$, so $\tau P = P$.

Let $C$ be a fibre in $\Omega$. Clearly, $C \not\subset P$, so $C$ intersects $P$ in a single point $x$. Since $C$ and $P$ are $\tau$-invariant, $\tau x = x$, which is absurd. $\square$

## 4. The 3-dimensional case

In conclusion, we will make some remarks on the general 3-dimensional case. Let us make the following definition, motivated by [Kat2]. A 3-dimensional compact complex manifold $M$ is called a *Kato manifold* if

(1) the universal covering space of $M$ is a domain $\Omega$ in $\mathbb{P}^3$ such that every connected component of the complement $\mathbb{P}^3 \setminus \Omega$ is a line,
(2) the fundamental group of $M$ has a torsion-free subgroup of finite index, and
(3) $M$ contains a domain which is biholomorphic to a neighbourhood of a line in $\mathbb{P}^3$.

The following main result of [Kat2] classifies Kato manifolds and describes their fundamental groups. For terms left undefined here, we refer the reader to [Kat2].

**4.1. Theorem (Kato).** *Let $M$ be a Kato manifold and let $\Gamma_0$ be a torsion-free subgroup of finite index in $\pi_1(M)$. Then $\Gamma_0$ is isomorphic to the free product*

$$\Gamma_1 * \cdots * \Gamma_r * \Gamma_{r+1} * \cdots * \Gamma_s,$$

*where $0 \le r \le s$, $\Gamma_1, \ldots, \Gamma_r$ are infinite cyclic, and $\Gamma_{r+1}, \ldots, \Gamma_s$ contain $\mathbb{Z}^4$ as a subgroup of finite index. Also, the finite covering of $M$ with fundamental group $\Gamma_0$ is a Klein combination of $r$ primary L-Hopf manifolds and $s - r$ Blanchard manifolds.*

Now we let $M$ be a compact complex manifold whose universal covering space is a domain $\Omega$ in $\mathbb{P}^3$, such that the complement $E = \mathbb{P}^3 \setminus \Omega$ is non-empty with $\Lambda_4(E) = 0$. The covering map is $\pi : \Omega \to M$.

By proposition 1.2 and Selberg's theorem, $M$ satisfies (2). As for (3), by the proof of proposition 1.5, there is a line $L$ in $\Omega$ such that $\pi|L$ is injective. If $\pi$ is not injective on any neighbourhood of $L$, then there are $p_n \to p \in L$ and $q_n \to q \in L$ with $p_n \neq q_n$ such that $\pi(p_n) = \pi(q_n)$, so $\pi(p) = \pi(q)$ and $p = q$, but this is absurd because $\pi$ is locally injective. Hence, $M$ satisfies (3).

This shows that $M$ is a Kato manifold if and only if all the connected components of $E$ are lines. Recall that in the proof of corollary 1.8, we observed that every connected component of $E$ contains a line, so roughly speaking, $M$ is a Kato manifold if the connected components of $E$ are as small as they can possibly be.

If $M$ is Blanchard or Schottky, then $M$ is Kato. However, $M$ need not be Kato. Kato [Kat3] has constructed examples where $E$ is a smooth submanifold of $\mathbb{P}^3$ of real dimension 3 (in fact, $E$ is a circle of projective lines), and $\Gamma$ is the fundamental group of a hyperbolic compact Riemann surface. These examples are twistor spaces of algebraic dimension zero.

All the examples of compact manifolds covered by "large" domains in projective space that I know of have been mentioned in this paper. There are clearly very few of them, especially in dimensions greater than 3, and they do not seem to give any clues to a possible classification. In particular, we have seen that although the class of these manifolds intersects the classes of twistor spaces and Kato manifolds, it is not contained in either of them. The work that has been done to date gives a glimpse of a rich and varied theory, but it is only a beginning.

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Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada

*E-mail address*: `larusson@uwo.ca`