MODEL STRUCTURES AND THE OKA PRINCIPLE

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Abstract. We embed the category of complex manifolds into the simplicial category of prestacks on the simplicial site of Stein manifolds, a prestack being a contravariant simplicial functor from the site to the category of simplicial sets. The category of prestacks carries model structures, one of them defined for the first time here, which allow us to develop *holomorphic homotopy theory*. More specifically, we use homotopical algebra to study lifting and extension properties of holomorphic maps, such as those given by the Oka Principle. We prove that holomorphic maps satisfy certain versions of the Oka Principle if and only if they are fibrations in suitable model structures. We are naturally led to a simplicial, rather than a topological, approach, which is a novelty in analysis.

1. Introduction. This paper, like its predecessor [L], is about model structures in complex analysis. Model structures are good for many things, but here we view them primarily as a tool for studying lifting and extension properties of holomorphic maps, such as those given by the Oka Principle. More precisely, model structures provide a framework for investigating two classes of holomorphic maps such that the first has the right lifting property with respect to the second and the second has the left lifting property with respect to the first in the absence of topological obstructions. (It is more natural, actually, to consider homotopy lifting properties rather than plain lifting properties.) We seek to make the maps in the first class into fibrations and those in the second class into cofibrations, with weak equivalences being understood in the topological sense. The machinery of abstract homotopy theory can then be applied.

The version of the Oka Principle we focus on here involves the inclusion $T \to S$ into a Stein manifold of a closed complex submanifold and a holomorphic fibre bundle $X \to Y$ whose fibre is an elliptic manifold. Loosely speaking, ellipticity means receiving many

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holomorphic maps from Euclidean spaces; it is thus dual to being Stein. A deep theorem of Gromov [G, FP] implies that for any commuting square

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & Y \end{array}$$

in which $T \to X$ and $S \to Y$ are otherwise arbitrary holomorphic maps, the inclusion of the space of holomorphic liftings $S \to X$ into the space of continuous liftings is a weak equivalence in the compact-open topology. Since $T \to S$ is a topological cofibration and $X \to Y$ is a topological fibration, it follows by basic topology that there is a holomorphic lifting if one of the two maps is a homotopy equivalence. This looks very much like a holomorphic manifestation of Quillen's first axiom for a model category [Q, Ch. I, p. 0.1], so it is natural to ask whether there is a model category containing the category of complex manifolds in which Stein inclusions are cofibrations, weak equivalences are defined topologically, and being a fibration is equivalent to an Oka property, such as the one attributed to elliptic bundles by Gromov's theorem. The main result of this paper is that a stronger and perhaps more natural Oka property, in which we consider not a single square but a continuous family of them, is equivalent to fibrancy in a new model category containing the category of complex manifolds. Elliptic manifolds are fibrant in this new sense, but it is still an open question whether all elliptic bundles are fibrations.

We equip the category of Stein manifolds in a natural way with a simplicial structure and a compatible topology, turning it into a simplicial site, and embed the category of complex manifolds into the simplicial category of prestacks on this site. By a prestack we mean a contravariant simplicial functor from the site to the category of simplicial sets. We make use of recent work of Toën and Vezzosi [TV], generalizing the homotopy theory of simplicial presheaves on ordinary, discrete sites to prestacks on simplicial sites. The category of prestacks carries several interesting model structures. Strengthening the main result of [L], we show that the prestack represented by a complex manifold X is fibrant in the so-called projective structure (so X represents a stack, in the terminology of [TV]) if and only if X satisfies what we call the weak Oka property. This means that for every Stein manifold S, the inclusion of the space of holomorphic maps from S to X into the space of continuous maps is a weak equivalence in the compact-open topology. By Gromov's theorem, this holds if X is elliptic. We generalize the weak Oka property to holomorphic maps (viewing manifolds as constant maps) and show that it is equivalent to being a projective fibration.

We introduce a new simplicial model structure on the category of prestacks on the Stein site, in a sense the smallest one in which every Stein inclusion is a cofibration. We characterize the fibrations in this structure and show that a holomorphic map is a fibration if and only if it satisfies a new, stronger Oka property. This Oka property is defined explicitly in purely analytic terms, without reference to, but with guidance from, abstract homotopy theory. For a holomorphic map which is a homotopy equivalence, it turns out to be simply the homotopy right lifting property with respect to all Stein inclusions. By Gromov's theorem, elliptic manifolds are fibrant. I conjecture that this extends to nonconstant maps: that elliptic bundles are fibrations. So far, this is known for covering maps but remains open for nontrivial bundles in general.

The interface between complex analysis and homotopical algebra will be explored further in future work. For more motivation, see the final remarks at the end of the paper, and for more background, the introduction in [L] and the survey [F2].

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2. The embedding. Let $\mathcal{M}$ be the category of complex manifolds, second countable but not necessarily connected, and holomorphic maps. As the first step in the development of holomorphic homotopy theory, or more specifically a homotopy-theoretic study of the Oka Principle, we wish to embed $\mathcal{M}$ in a simplicial model category.

Now $\mathcal{M}$ has a natural simplicial structure (enrichment over the category $s\mathbf{Set}$ of simplicial sets), making it a simplicial object in the category of categories with a discrete simplicial class of objects. For complex manifolds X and Y, the mapping space $\mathrm{Hom}(X,Y)$ is the singular set $s\mathcal{O}(X,Y)$ of the space of holomorphic maps from X to Y with the compact-open topology.

Let $\mathcal{S}$ be the full subcategory of Stein manifolds with this simplicial structure. It is a small category, or at least equivalent to one, since a connected Stein manifold can be embedded into Euclidean space. A prestack on $\mathcal{S}$ (in the terminology of [TV]) is a contravariant simplicial functor (morphism of simplicial categories) $\mathcal{S} \to s\mathbf{Set}$. Let $\mathfrak{S}$ denote the category of prestacks on $\mathcal{S}$ with its own natural simplicial structure (in a sense that is stronger than the sense in which $\mathcal{M}$ is a simplicial category; see [GJ, IX.1]).

By the simplicial Yoneda lemma [GJ, IX.1.2], if S is an object of $\mathcal{S}$ and F is a prestack on $\mathcal{S}$, then there is a natural isomorphism of simplicial sets

$$F(S) \cong \mathrm{Hom}_{\mathfrak{S}}(\mathrm{Hom}_{\mathcal{M}}(\cdot, S), F).$$

(From now on we will usually omit the subscripts.) Hence there is a simplicially full embedding of $\mathcal{S}$ into $\mathfrak{S}$, taking an object S of $\mathcal{S}$ to the prestack $\mathrm{Hom}(\cdot, S)$ represented by S.

The embedding $\mathcal{S} \to \mathfrak{S}$ clearly extends to a functor $\mathcal{M} \to \mathfrak{S}$, taking a complex manifold X to the prestack $\mathrm{Hom}(\cdot, X)$ on $\mathcal{S}$ represented by X. This functor induces monomorphisms (injections at each level) of mapping spaces, as is easily seen by plugging in the terminal object of $\mathcal{S}$, the one-point manifold $\mathfrak{p}$. Hence, for complex manifolds X and Y, we have a monomorphism of mapping spaces

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}(\mathrm{Hom}(\cdot, X), \mathrm{Hom}(\cdot, Y)),$$

which is an isomorphism when X is Stein, and we have a simplicial embedding of $\mathcal{M}$ into $\mathfrak{S}$. Whether the embedding is full remains to be investigated.

3. Remarks. We would like to motivate the above construction and explain why it seems to produce an appropriate setting for applying homotopical algebra in complex analysis. Yoneda embeddings provide the canonical way of closing geometric categories under limits and colimits. This is the first step in the homotopy theory of schemes, for instance; I know of no alternative. In our paper [L], we embedded $\mathcal{M}$ into the category of all simplicial presheaves on $\mathcal{S}$, but there is every reason to take into account the topology on our hom-sets and restrict attention to those simplicial presheaves that respect it, now that the homotopy theory of simplicial presheaves on ordinary, discrete sites has been generalized to prestacks on simplicial sites by Toën and Vezzosi [TV]. Indeed, we want a full embedding of the category of complex manifolds into a simplicial model category, at least for Stein sources, and with plain simplicial presheaves we cannot expect this. Homotopy theory gives information about simplicial hom-sets and maps between them; to apply such results in complex analysis, we need to know that simplicial hom-sets essentially equal spaces of holomorphic maps. We get this at least when the source is Stein; this has proved sufficient so far.

It would seem simpler and more natural to use presheaves of topological spaces on $\mathcal{S}$ rather than simplicial presheaves. The homotopy theory of the former is not available in the literature — although it could presumably be developed in a straightforward manner for a suitable locally presentable category of topological spaces, now that one such has been discovered: J. Smith's category of I-spaces — but that is not why we use the latter. The reason is that we are aiming for a model structure in which the inclusion $T \hookrightarrow S$ of a closed complex submanifold T in a Stein manifold S is a cofibration (this is the intermediate structure, defined below). It is appropriate, then, to require such an inclusion to induce a pointwise cofibration, so in the topological setting we would need $\mathcal{O}(X,T) \to \mathcal{O}(X,S)$ to be a cofibration of topological spaces for every Stein manifold X. There are simple examples for which this fails. For instance, let S be the complex plane with a puncture, T be a one-point subset of S, and X be the complex plane with the integers removed. Then $\mathcal{O}(X,T) \to \mathcal{O}(X,S)$ is not a cofibration, not even in the weaker of the two senses considered by topologists, because the point $\mathcal{O}(X,T)$ in the space $A = \mathcal{O}(X,S)$ does not have a neighbourhood contractible in A. Indeed, there are uncountably many homotopy classes of holomorphic maps $X \to S$ (consider winding numbers around each integer), so A has uncountably many connected components, and every nonempty open subset of A contains uncountably many of these, so it is not contractible in A. However, the induced map $s\mathcal{O}(X,T) \to s\mathcal{O}(X,S)$ is a cofibration of simplicial sets, simply because it is injective at each level. Shifting our focus from the spaces of holomorphic maps themselves to the singular sets that catalogue continuous families of holomorphic maps with nice parameter spaces alleviates the difficulties associated with the compact-open topology for noncompact sources.

Thus we are, somewhat surprisingly, led to a simplicial approach, which is a novelty in analysis. Fortunately, there is often no loss involved in applying the singular functor to spaces of holomorphic maps, because the singular functor not only preserves but also reflects fibrations. For example, if A and B are spaces of holomorphic maps and $A \to B$ is a map such that the induced map $sA \to sB$ of mapping spaces is a Kan fibration, as might follow from some homotopy-theoretic arguments, then $A \to B$ itself is a Serre fibration (and conversely). Also, $sA \to sB$ is a weak equivalence if and only if $A \to B$ is.

4. The projective model structures. The category $\mathfrak{S}$ carries several interesting simplicial model structures. We begin by describing the most basic one, the coarse projective structure, originally defined by Dwyer and Kan [GJ, IX.1]. (We call it coarse because it is associated to the coarsest topology on $\mathcal{S}$, that is, the trivial topology; see below.) In this structure, which is cofibrantly generated and proper, weak equivalences and fibrations are defined pointwise, so a map $F \to G$ of prestacks on $\mathcal{S}$ is a weak equivalence or a fibration if the component maps $F(S) \to G(S)$ are weak equivalences or fibrations of simplicial sets, respectively, for all objects S in $\mathcal{S}$. In particular, a holomorphic map $X \to Y$, viewed as a map of the prestacks represented by X and Y, is a weak equivalence or a fibration in the coarse projective structure if the induced maps $\mathcal{O}(S, X) \to \mathcal{O}(S, Y)$ are weak equivalences or Serre fibrations of topological spaces, respectively, for all Stein manifolds S. Cofibrations are defined by a left lifting property. The prestacks represented by Stein manifolds are both cofibrant and fibrant.

Now we move to the projective structure on $\mathfrak{S}$, which is obtained by a left Bousfield localization of the coarse projective structure. There will be a larger class of weak equivalences, defined using a topology on the simplicial category $\mathcal{S}$, turning it into a simplicial site. The cofibrations are the same as in the coarse projective structure, so they can be referred to simply as projective cofibrations. The projective fibrations are determined by a right lifting property; they form a subclass of the class of pointwise fibrations.

The category of components $c\mathcal{S}$ (also called, at some risk of confusion, the homotopy category) of the simplicial category $\mathcal{S}$ has the same objects as $\mathcal{S}$, and its hom-sets are the sets of path components of the simplicial hom-sets of $\mathcal{S}$. We can also obtain $c\mathcal{S}$ from $\mathcal{S}$ by identifying maps in the underlying category of $\mathcal{S}$ that can be joined by a string of homotopies (provided by the simplicial structure). By precomposition by the morphism $\mathcal{S} \to c\mathcal{S}$, a presheaf on $c\mathcal{S}$ gives a presheaf on $\mathcal{S}$ such that equivalent maps in $\mathcal{S}$ induce the same restriction maps. Conversely, such a presheaf on $\mathcal{S}$ descends to $c\mathcal{S}$. Prestacks respect the simplicial structure, so they preserve homotopies, so the homotopy presheaves of a prestack on $\mathcal{S}$ naturally live on $c\mathcal{S}$ (or, more precisely, on overcategories thereof).

A topology on $\mathcal{S}$, turning it into a simplicial site (an S-site in the language of [TV]), is a Grothendieck topology in the usual sense on the category of components $c\mathcal{S}$. A map of prestacks is a weak equivalence, or acyclic, with respect to the topology, if it induces isomorphisms of homotopy sheaves in all degrees, that is, isomorphisms of the sheafifications (with respect to the given topology) of homotopy presheaves in all degrees. By a theorem of Toën and Vezzosi [TV, Thm. 3.4.1], the projective structure on $\mathfrak{S}$ is a cofibrantly generated, proper, simplicial model structure.

The projective structure specializes in two ways. It equals the coarse projective structure when the topology on $c\mathcal{S}$ is trivial. Also, when the simplicial structure on $\mathcal{S}$ is trivial (discrete), so $c\mathcal{S} = \mathcal{S}$, then $\mathcal{S}$ is an ordinary site and we obtain the well-known projective structure (sometimes called local) for simplicial presheaves on $\mathcal{S}$.

The topology we shall put on the Stein site $\mathcal{S}$ is the "usual" topology employed in [L], except we now view it as a topology on the category of components $c\mathcal{S}$, which is obtained from the plain category of Stein manifolds and holomorphic maps by identifying holomorphic maps $X \to Y$ that are homotopic in the usual sense that they can be joined by a continuous path in $\mathcal{O}(X, Y)$ with the compact-open topology. In other words,

$$\hom_{c\mathcal{S}}(X, Y) = \pi_0 \mathcal{O}(X, Y).$$

A cover of a Stein manifold S is a family of holomorphic maps into S such that by suitably deforming each map $X \to S$ inside $\mathcal{O}(X, S)$, we get a family of biholomorphisms onto Stein open subsets of S which cover S. This defines a Grothendieck topology on $c\mathcal{S}$.

The acyclic maps have a very simple description. First, for any map from the point $\mathfrak{p}$ to an open ball B, the map $\mathfrak{p} \to B \to \mathfrak{p}$ is the identity and the map $B \to \mathfrak{p} \to B$ is homotopic to the identity through holomorphic maps keeping the image point of the map $\mathfrak{p} \to B$ fixed. Hence, if F is a prestack on $\mathcal{S}$, the restriction map $F(\mathfrak{p}) \to F(B)$ is a homotopy equivalence, in fact the inclusion of a strong deformation retract. Since every cover has a refinement by balls, this implies that a map $F \to G$ of prestacks on $\mathcal{S}$ is acyclic if and only if $F(\mathfrak{p}) \to G(\mathfrak{p})$ is acyclic. Here it is crucial that prestacks respect the simplicial structure on $\mathcal{S}$; this does not work for arbitrary simplicial presheaves. It follows that a holomorphic map $f : X \to Y$ of complex manifolds, viewed as a map of the prestacks represented by X and Y, is acyclic if and only if it is a topological weak equivalence, that is, a homotopy equivalence.

5. The injective model structures. We will also need the so-called injective model structures on $\mathfrak{S}$ [TV, 3.6]. The coarse injective structure is a proper, simplicial model structure on $\mathfrak{S}$ in which weak equivalences and cofibrations are defined pointwise and fibrations are defined by a right lifting property. In the injective structure, which is also proper and simplicial, the cofibrations are the same, weak equivalences are acyclic with respect to the chosen topology on $\mathcal{S}$, and fibrations are defined by a right lifting property. Injective cofibrations are and will be referred to simply as monomorphisms.

6. A Quillen equivalence. Consider the functor $P : \mathfrak{S} \to \mathfrak{S}$ taking a prestack F to the prestack $PF = \mathrm{Hom}(s\cdot, F(\mathfrak{p}))$ and taking a map $f : F \to G$ to the map $Pf : PF \to PG$ induced by the map $F(\mathfrak{p}) \to G(\mathfrak{p})$. This functor is a projection: $P \circ P = P$. There is a natural transformation η from the identity functor on $\mathfrak{S}$ to P: if F is a prestack and S is an object of $\mathcal{S}$, the map (morphism of simplicial sets) $\eta_F(S) : F(S) \to PF(S) = \mathrm{Hom}(sS, F(\mathfrak{p}))$ comes from the map $sS = \mathrm{Hom}(\mathfrak{p}, S) \to \mathrm{Hom}(F(S), F(\mathfrak{p}))$ given directly by F. Here, again, it is crucial that prestacks respect the simplicial structure on $\mathcal{S}$; this does not work for arbitrary simplicial presheaves. The square

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\eta_F"'] & G \arrow[d, "\eta_G"] \\ PF \arrow[r, "Pf"] & PG \end{tikzcd}$$

commutes simply because maps of prestacks commute with restrictions. Note that the map $\eta_{PF} = P(\eta_F) : PF \to P^2F = PF$ is the identity. Also, $\eta_F : F \to PF$ is acyclic, since $\eta_F(\mathfrak{p})$ is the identity. The pair P, η is a key element of the structure of $\mathfrak{S}$ and plays an important role in our theory. It is an example of what is called a localization functor.

If A is a simplicial set, let $\tilde{A}$ denote the constant prestack with $\tilde{A}(S) = A$ for each S in $\mathcal{S}$ and with all restriction maps equal to the identity. Define a functor $R : s\mathbf{Set} \to \mathfrak{S}$ by $RA = P\tilde{A} = \mathrm{Hom}(s\text{•}, A)$. A map f from a prestack F to RA factors as

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\eta_F"'] & RA \arrow[d, equal, "\eta_{RA}"] \\ PF \arrow[r, "Pf"] & RA \end{tikzcd}$$

so f is determined by Pf, which is induced by the map $F(\mathfrak{p}) \to RA(\mathfrak{p}) = A$. Hence, we have a pair of adjoint functors

$$L : \mathfrak{S} \to s\mathbf{Set} : R, \qquad LF = F(\mathfrak{p}), \qquad RA = \mathrm{Hom}(s\text{•}, A),$$

with a natural bijection

$$\hom_{\mathfrak{S}}(F, RA) \cong \hom_{s\mathbf{Set}}(LF, A)$$

for every prestack F on $\mathcal{S}$ and every simplicial set A.

We see that a map $F \to RA$ is acyclic if and only if the corresponding map $LF \to A$ is. Also, it is clear that L takes monomorphisms to cofibrations and preserves weak equivalences. Hence, (L, R) is a pair of Quillen equivalences between the category of simplicial sets and the category of prestacks on $\mathcal{S}$ with the projective structure or the injective structure [H, 8.5]. Such a pair induces equivalences of homotopy categories, so the homotopy category of $\mathfrak{S}$ is the ordinary homotopy category of simplical sets or topological spaces. It also follows that R takes fibrations of simplicial sets to injective fibrations; in particular, if K is a fibrant simplicial set (a Kan complex), then the prestack $\mathrm{Hom}(s\text{•}, K)$ is injectively fibrant. Hence, if X is a complex manifold, so $\eta_X : X \to PX$ is a monomorphism, then η_X is an injectively cofibrant fibrant model for X.

7. Projective fibrations. The projective structure is the left Bousfield localization of the coarse projective structure on $\mathfrak{S}$ with respect to the class of acyclic maps of prestacks. The theory of the left Bousfield localization provides a useful characterization of projective fibrations.

Let $f : F \to G$ be a pointwise fibration of prestacks such that $F(\mathfrak{p})$ and $G(\mathfrak{p})$ are fibrant, so PF and PG are injectively and hence projectively fibrant. Then the square

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\eta_F"'] & G \arrow[d, "\eta_G"] \\ PF \arrow[r, "Pf"] & PG \end{tikzcd}$$

is a localization of f [H, 3.2.16]. The map f is a projective fibration if and only if this square is a homotopy pullback in the coarse projective structure [H, 3.4.8]. This means that the natural map from F to the homotopy pullback of $G \to PG \leftarrow PF$ is pointwise acyclic. Since $F(\mathfrak{p}) \to G(\mathfrak{p})$ is a fibration, Pf is a pointwise fibration, so the homotopy pullback is naturally pointwise weakly equivalent to the ordinary pullback (taken pointwise).

In summary, a map $F \to G$ of prestacks fibrant at $\mathfrak{p}$ is a projective fibration if and only if it is a pointwise fibration and the induced map $F \to G \times_{PG} PF$ is pointwise acyclic. In particular, a prestack F is projectively fibrant if and only if it is pointwise fibrant and η_F is pointwise acyclic.

8. Stacks on the Stein site and the weak Oka property. A pointwise fibrant prestack on the simplicial site $\mathcal{S}$ is called, in the language of [TV], a stack on $\mathcal{S}$ (with respect to the chosen topology) if it is projectively fibrant. Loosely speaking, this is a "homotopy sheaf condition", with the limits in the usual sheaf condition replaced by homotopy limits. The sheaf condition is not really relevant here; indeed, the prestacks and the topology live on different categories ($\mathcal{S}$ and $c\mathcal{S}$, respectively), so we will not be talking about a prestack being a sheaf in the usual sense.

We say that a complex manifold X satisfies the weak Oka property, or that X is weakly Oka, if the inclusion map $\mathcal{O}(S, X) \hookrightarrow \mathcal{C}(S, X)$ is a weak equivalence for all Stein manifolds S, where the spaces of holomorphic and continuous maps from S to X carry the compact-open topology. The main result of [L] characterizes the weak Oka property (there called the Oka-Grauert property) in terms of excision; the following theorem, using a better model structure, is more to the point.

9. Theorem. *A complex manifold is weakly Oka if and only if it represents a stack on the Stein site.*

Proof. A prestack F is projectively fibrant if and only if it is pointwise fibrant and the map $\eta_F : F \to PF$ is pointwise acyclic. If F is represented by a complex manifold X, so it is pointwise fibrant, this means that the map from $F(S) = s\mathcal{O}(S, X)$ to

$PF(S) = \mathrm{Hom}(sS, sX) = s\mathcal{C}(|sS|, X)$ is acyclic for every Stein manifold S. Since $PF(S)$ is homotopy equivalent to $s\mathcal{C}(S, X)$, this is nothing but the weak Oka property. □

It is an interesting open question whether the inclusions $\mathcal{O}(S, X) \hookrightarrow \mathcal{C}(S, X)$ have functorial homotopy inverses when X is weakly Oka. Since the spaces in question are not known to be cofibrant, even the existence of pointwise homotopy inverses is not clear [L, Thm. 2.2], but it is in the simplicial setting, so we ask whether the pointwise homotopy equivalence $\eta_X : X \to PX$ is in fact a simplicial homotopy equivalence of prestacks. This would follow if η_X was not only a monomorphism but actually a projective cofibration [H, 9.6.5], that is, if PX was a cofibrant fibrant model for X not only in the injective structure but also in the projective structure.

10. The weak Oka property for maps. Let us generalize the above discussion from objects to arrows. We say that a holomorphic map $f : X \to Y$ satisfies the weak Oka property, or that f is weakly Oka, if

(1) the induced map $\mathcal{O}(S, X) \to \mathcal{O}(S, Y)$ is a Serre fibration and
(2) the inclusion $\mathcal{O}(S, X) \hookrightarrow \{h \in \mathcal{C}(S, X) : f \circ h \in \mathcal{O}(S, Y)\}$ is acyclic

for every Stein manifold S.

In particular, if $f : X \to Y$ is weakly Oka, then every continuous map h from a Stein manifold to X such that $f \circ h$ is holomorphic can be continuously deformed through such maps to a holomorphic map. Clearly, a complex manifold X is weakly Oka if and only if the constant map $X \to \mathfrak{p}$ is weakly Oka.

11. Theorem. *A holomorphic map is weakly Oka if and only if it is a projective fibration.*

Proof. A holomorphic map $f : X \to Y$ is a projective fibration if and only if it is a pointwise fibration, meaning that the induced map $\mathcal{O}(S, X) \to \mathcal{O}(S, Y)$ is a Serre fibration for every Stein manifold S, and the induced map $X \to Y \times_{PY} PX$ is pointwise acyclic, which is equivalent to the map

$$\mathcal{O}(S, X) \to \mathcal{O}(S, Y) \times_{\mathcal{C}(S,Y)} \mathcal{C}(S, X)$$

being acyclic for every Stein manifold S. Finally, the space on the right is the space of continuous maps $h : S \to X$ such that $f \circ h$ is holomorphic. □

12. The intermediate model structure. We now introduce a new simplicial model structure on $\mathfrak{S}$, in between the projective and injective structures in the sense that it has fewer fibrations than the projective structure and more fibrations than the injective structure; for cofibrations it is the other way around. The weak equivalences are the same: the maps that are acyclic with respect to the chosen topology on $\mathcal{S}$.

By a Stein inclusion we mean the inclusion $T \hookrightarrow S$ of a closed complex submanifold T in a Stein manifold S (then T is also Stein). Let the set C consist of all the monomorphisms

$$S \times \partial\Delta^n \cup_{T \times \partial\Delta^n} T \times \Delta^n \to S \times \Delta^n,$$

in $\mathfrak{S}$, where $T \hookrightarrow S$ is a Stein inclusion and $n \geq 0$. Among these maps are the Stein inclusions $T \hookrightarrow S$ themselves (with $n = 0$), as well as the standard generating cofibrations $S \times \partial\Delta^n \to S \times \Delta^n$ for the projective structure (with $T = \varnothing$).

To avert confusion, we should make clear that by the prestack $\varnothing$ (as above when $T = \varnothing$, for instance) we mean the empty prestack $\varnothing_{\mathfrak{S}}$ (the initial object in $\mathfrak{S}$) but not the prestack represented by the empty manifold $\varnothing_{\mathcal{S}}$ (the initial object in $\mathcal{S}$): these prestacks differ over $\varnothing_{\mathcal{S}}$. If F is a prestack, we will sometimes write $F(\varnothing)$ for $\mathrm{Hom}(\varnothing_{\mathfrak{S}}, F)$, which is the terminal simplicial set, rather than for $\mathrm{Hom}(\varnothing_{\mathcal{S}}, F)$, which is the simplicial set of sections of F over $\varnothing_{\mathcal{S}}$ (these are of course the same if F is represented by a manifold).

Let $\mathcal{C}$ be the saturation of C, that is, the smallest class of maps in $\mathfrak{S}$ which contains C and is closed under pushouts, retracts, and transfinite compositions. The maps in $\mathcal{C}$ are called intermediate cofibrations; they are retracts of transfinite compositions of pushouts of maps in C. An intermediate fibration is defined to be a map with the right lifting property with respect to all acyclic intermediate cofibrations.

The idea of an intermediate structure in which Stein inclusions would be cofibrations came up in a discussion with Rick Jardine, who subsequently showed me how to obtain such a structure and later wrote up a proof in [J], which we follow below. The argument for a simplicial site is the same as for the special case of a discrete site, treated in [J]. Later, I learned that one can show that the intermediate structure exists and, moreover, is cofibrantly generated, using a very general argument due to T. Beke and J. Smith [B, Thm. 1.7], based solely on $\mathfrak{S}$ being locally presentable and the class of weak equivalences being accessible. (Cofibrant generation is also contained in a second version of [J].) Unfortunately, the generating set of acyclic cofibrations produced by this method is too large to be of much practical use.

13. Theorem. *There is a proper, simplicial model structure on $\mathfrak{S}$, called the intermediate structure, with cofibrations, fibrations, and weak equivalences defined as above.*

Proof. Consider factorization first. Since $\mathfrak{S}$ is locally presentable, a standard small object argument shows that a map $X \to Y$ of prestacks can be factored as $X \xrightarrow{j} Z \xrightarrow{p} Y$, where j is in $\mathcal{C}$ and p has the right lifting property with respect to every map in $\mathcal{C}$, so p is an acyclic intermediate fibration (note that we do not know the converse of this yet).

For the other factorization, we make use of the injective structure to factor $X \to Y$ as $X \xrightarrow{i} W \xrightarrow{q} Y$, where i is an acyclic injective cofibration and q is an injective fibration and hence an intermediate fibration. Then factor i as above as $X \xrightarrow{j} Z \xrightarrow{p} W$, where j is an intermediate cofibration and p is an acyclic intermediate fibration. Then j is acyclic too and qp is an intermediate fibration.

Consider now the lifting axiom. One half of it is immediate from the definition of a fibration. For the other half, say $X \xrightarrow{p} Y$ is an acyclic intermediate fibration. Factor p as $X \xrightarrow{j} Z \xrightarrow{q} Y$, where j is in $\mathcal{C}$ and q has the right lifting property with respect to every map in $\mathcal{C}$. Then, as before, q is an acyclic intermediate fibration, so j is acyclic, and by the definition of an intermediate fibration, we have a lifting in the square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "j"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q"] \arrow[ur, dashed] & Y \end{tikzcd}$$

Hence, p is a retract of q, so p also has the right lifting property with respect to every map in $\mathcal{C}$.

The remaining three axioms for a model structure are clear. Right properness follows from right properness of the injective structure, and left properness follows from left properness of the projective structure. Finally, Axiom SM7, relating the simplicial structure and the model structure, may be verified using [GJ, II.3.12]. $\square$

Without a useful generating set of acyclic intermediate cofibrations it is not easy to describe the intermediate fibrations, but for acyclic intermediate fibrations the following characterization is immediate.

14. Proposition. *An acyclic map $F \to G$ of prestacks is an intermediate fibration if and only if it has the homotopy right lifting property with respect to all Stein inclusions.*

Proof. By definition of the intermediate structure, an acyclic map $F \to G$ is an intermediate fibration if and only if there is a lifting in every square

$$\begin{tikzcd} S \times \partial\Delta^n \cup_{T \times \partial\Delta^n} T \times \Delta^n \arrow[r] \arrow[d] & F \arrow[d] \\ S \times \Delta^n \arrow[r] & G \end{tikzcd}$$

where $T \hookrightarrow S$ is a Stein inclusion and $n \geq 0$, that is, by adjunction, in every square

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & F(S) \arrow[d] \\ \Delta^n \arrow[r] & G(S) \times_{G(T)} F(T) \end{tikzcd}$$

This means precisely that the map $F(S) \to G(S) \times_{G(T)} F(T)$ is an acyclic fibration for every Stein inclusion $T \hookrightarrow S$. $\square$

15. The three structures are different. Two simple examples show that the projective, intermediate, and injective model structures on $\mathfrak{S}$ are all different. First consider the unit disc $\mathbb{D}$ (or rather the prestack on $\mathcal{S}$ it represents). Since $\mathbb{D}$ is holomorphically contractible, it is projectively fibrant by Theorem 9. On the other hand, by Liouville's

Theorem, the inclusion $\{0, \frac{1}{2}\} \hookrightarrow \mathbb{D}$ does not factor through the inclusion $\{0, \frac{1}{2}\} \hookrightarrow \mathbb{C}$, which is an intermediate cofibration, so $\mathbb{D}$ is not intermediately fibrant.

The complex plane $\mathbb{C}$ is projectively fibrant for the same reason that $\mathbb{D}$ is. Since $\mathbb{C}$ is elliptic, it is intermediately fibrant (see below). However, $\mathbb{C}$ is not injectively fibrant; in fact, no nondiscrete complex manifold X is. The inclusion of $\mathbb{D}$ into the disc of radius 2 is a pointwise acyclic monomorphism, but there are many holomorphic maps $\mathbb{D} \to X$ that do not factor through it, so X is not even coarsely injectively fibrant.

16. The Oka property for manifolds and maps. We say that a holomorphic map $f : X \to Y$ is Oka if it satisfies one of the following equivalent conditions for every Stein inclusion $j : T \hookrightarrow S$.

(i) The map f is a topological fibration and satisfies the *Parametric Oka Principle with Interpolation*, meaning that for every finite polyhedron P with subpolyhedron Q and every diagram

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & \mathcal{O}(S,X) \arrow[r] \arrow[d] & \mathcal{C}(S,X) \arrow[d] \\
P \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X) \arrow[r] & \mathcal{C}(S,Y) \times_{\mathcal{C}(T,Y)} \mathcal{C}(T,X)
\end{tikzcd}$$

of continuous maps, every lifting $P \to \mathcal{C}(S,X)$ in the big square can be deformed through liftings in the big square to a lifting that factors through $\mathcal{O}(S,X)$ and is thus a lifting in the left-hand square. (We recall that a Serre fibration between smooth manifolds is a Hurewicz fibration [C], so we will simply call such a map a topological fibration.)

(ii) A stronger version of condition (i), in which $Q \to P$ is any cofibration between cofibrant topological spaces and the conclusion is that the inclusion of the space of liftings $P \to \mathcal{O}(S,X)$ in the left-hand square into the space of liftings $P \to \mathcal{C}(S,X)$ in the big square is acyclic. (Here, and everywhere else in the paper, the notion of cofibrancy for topological spaces and continuous maps is the stronger one that goes with Serre fibrations rather than Hurewicz fibrations.)

(iii) The induced map

$$\mathcal{O}(S,X) \xrightarrow{(f_*, j^*)} \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X)$$

is a Serre fibration, and the inclusion

$$\mathcal{O}(S,X) \hookrightarrow \mathcal{C}_{f,T}(S,X) := \{h \in \mathcal{C}(S,X) : f \circ h \text{ and } h|T \text{ are holomorphic}\}$$

is acyclic. Note that $\mathcal{C}_{f,T}(S,X)$ is the pullback of the right-hand square in condition (i), so when f is a topological fibration, this inclusion being acyclic is equivalent to that square being a homotopy pullback.

(iv) The induced map

$$\mathcal{O}(S,X) \xrightarrow{(f_*,j^*)} \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X)$$

is a Serre fibration, and in any square of holomorphic maps

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the inclusion of the space of holomorphic liftings $S \to X$ into the space of continuous liftings is acyclic (where these spaces are, as usual, given the compact-open topology).

Before proving the equivalence of these conditions, we will make a few remarks.

Observe that the target of (f_*, j^*) is the space of commuting squares of holomorphic maps in which the map on the left is j and the map on the right is f. The fibre over such a square is its set of liftings. Taking $T = \varnothing$ in each of the conditions gives the weak Oka property, which we know is equivalent to f being a projective fibration.

Using the Stein inclusion $\varnothing \hookrightarrow \mathfrak{p}$, we see that an Oka map is a topological fibration, so its image is a union of connected components of the target. An Oka map has the right lifting property with respect to the inclusion of a point into a ball, so it is a submersion. In fact, if a holomorphic map $f : X \to Y$ is Oka, q is a point in a contractible Stein open subset V of Y, and $p \in f^{-1}(q)$, then condition (iv) implies that f has a holomorphic section (a right inverse) $V \to X$ taking q to p.

A complex manifold X is said to be Oka if the constant map $X \to \mathfrak{p}$ is Oka. This is equivalent to X being weakly Oka and the restriction map $\mathcal{O}(S,X) \to \mathcal{O}(T,X)$ being a Serre fibration for every Stein inclusion $T \hookrightarrow S$. Namely, if X is weakly Oka, the pullback $\{h \in \mathcal{C}(S,X) : h|T \in \mathcal{O}(T,X)\} \hookrightarrow \mathcal{C}(S,X)$ of the acyclic map $\mathcal{O}(T,X) \hookrightarrow \mathcal{C}(T,X)$ by the Serre fibration $\mathcal{C}(S,X) \hookrightarrow \mathcal{C}(T,X)$ is acyclic, and since $\mathcal{O}(S,X) \hookrightarrow \mathcal{C}(S,X)$ is also acyclic, condition (iii) follows.

Let us now prove the equivalence of the four conditions defining the Oka property.

(i) $\Rightarrow$ (iii): Since f is a topological fibration and $T \hookrightarrow S$ is a topological cofibration, the map $\mathcal{C}(S,X) \to \mathcal{C}(S,Y) \times_{\mathcal{C}(T,Y)} \mathcal{C}(T,X)$ is a Serre fibration, so every diagram of continuous maps as below has a lifting as indicated.

$$\begin{tikzcd} {[0,1]^n} \arrow[r] \arrow[d] & \mathcal{O}(S,X) \arrow[r] \arrow[d] & \mathcal{C}(S,X) \arrow[d] \\ {[0,1]^{n+1}} \arrow[r] \arrow[urr, dashed] & \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X) \arrow[r] & \mathcal{C}(S,Y) \times_{\mathcal{C}(T,Y)} \mathcal{C}(T,X) \end{tikzcd}$$

The Parametric Oka Principle with Interpolation now gives a lifting $[0,1]^{n+1} \to \mathcal{O}(S,X)$, showing that $\mathcal{O}(S,X) \to \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X)$ is a Serre fibration.

To prove that $\mathcal{O}(S,X) \hookrightarrow \mathcal{C}_{f,T}(S,X)$ is acyclic, apply the Parametric Oka Principle with Interpolation to diagrams of the form

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & \mathcal{O}(S,X) \arrow[r] \arrow[d] & \mathcal{C}_{f,T}(S,X) \arrow[dl] \\ P \arrow[r] \arrow[urr, dashed] & \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X) & \end{tikzcd}$$

taking $Q \to P$ to be either the inclusion of a point in the n-sphere, $n \geq 1$, or the inclusion of the n-sphere in the closed $(n+1)$-ball, $n \geq -1$.

(iii) $\Leftrightarrow$ (iv): Consider the diagram

$$\begin{tikzcd} \{\text{holomorphic liftings}\} \arrow[r] \arrow[d] & \{\text{continuous liftings}\} \arrow[d] \\ \mathcal{O}(S,X) \arrow[r] \arrow[d] & \mathcal{C}_{f,T}(S,X) \arrow[dl] \\ \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X) & \end{tikzcd}$$

The common first part of conditions (iii) and (iv) implies that the lower downward maps are Serre fibrations, so each horizontal map is acyclic if and only if the other one is.

(iii) $\Rightarrow$ (ii): Assume now that $Q \to P$ is any cofibration between cofibrant topological spaces and consider a diagram as in condition (i), or equivalently, a diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & \mathcal{O}(S,X) \arrow[r] \arrow[d] & \mathcal{C}_{f,T}(S,X) \arrow[d] \\ P \arrow[r] & \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X) \arrow[r, equal] & \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X) \end{tikzcd}$$

of continuous maps. Let us write $A = \mathcal{O}(S,X)$, $B = \mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X)$, $C = \mathcal{C}_{f,T}(S,X)$, $\mathcal{L}_A$ for the space of liftings $P \to A$ in the left-hand square, and $\mathcal{L}_C$ for the space of liftings $P \to C$ in the big square. Then we have a diagram

$$\begin{tikzcd} \mathcal{L}_A \arrow[r] \arrow[d] & \mathcal{C}(P,A) \arrow[r] \arrow[d] & \mathcal{C}(P,B) \times_{\mathcal{C}(Q,B)} \mathcal{C}(Q,A) \arrow[d] \\ \mathcal{L}_C \arrow[r] & \mathcal{C}(P,C) \arrow[r] & \mathcal{C}(P,B) \times_{\mathcal{C}(Q,B)} \mathcal{C}(Q,C) \end{tikzcd}$$

The right-hand horizontal maps are Serre fibrations by Axiom SM7, because $Q \to P$ is a cofibration and $A \to B$ and $C \to B$ are Serre fibrations. The middle vertical map is acyclic because P is cofibrant and $A \to C$ is acyclic. To see that the right-hand vertical map is acyclic, consider the cube

$$
\begin{tikzcd}
\text{pullback} \arrow[rr] \arrow[dr] \arrow[dd, dashed] & & \mathcal{C}(Q,A) \arrow[dr] \arrow[dd] & \\
& \mathcal{C}(P,B) \arrow[rr] \arrow[dd, equal] & & \mathcal{C}(Q,B) \arrow[dd, equal] \\
\text{pullback} \arrow[rr] \arrow[dr] & & \mathcal{C}(Q,C) \arrow[dr] & \\
& \mathcal{C}(P,B) \arrow[rr] & & \mathcal{C}(Q,B)
\end{tikzcd}
$$

The map $\mathcal{C}(Q,A) \to \mathcal{C}(Q,C)$ is acyclic because Q is cofibrant and $A \to C$ is acyclic. The maps $\mathcal{C}(Q,A) \to \mathcal{C}(Q,B)$ and $\mathcal{C}(Q,C) \to \mathcal{C}(Q,B)$ are Serre fibrations because Q is cofibrant and $A \to B$ and $C \to B$ are Serre fibrations. Hence, the top and bottom squares are homotopy pullbacks and we get an induced weak equivalence of the pullbacks. Therefore, finally, we get an induced weak equivalence $\mathcal{L}_A \to \mathcal{L}_C$.

17. Subellipticity and the Oka property. Subelliptic manifolds satisfy the Parametric Oka Principle with Interpolation. This theorem originated in Gromov's work [G] and was proved in detail by Forstnerič and Prezelj [FP, Thm. 1.4] for elliptic manifolds; for the extension from ellipticity to subellipticity, see [F1]. Hence, subelliptic manifolds are Oka.

I conjecture that this result extends to nonconstant maps: that a holomorphic map which is both a subelliptic submersion and a topological fibration is Oka. This is an open question even for nontrivial elliptic fibre bundles. Here is a small step in this direction, proving the conjecture in the case of discrete fibres, including the case of covering maps.

18. Proposition. *A holomorphic map which is a topological fibration and a local biholomorphism is Oka.*

Proof. Let $f : X \to Y$ be a topological fibration and a local biholomorphism and $T \hookrightarrow S$ be a Stein inclusion. First, the inclusion $\mathcal{O}(S,X) \hookrightarrow \mathcal{C}_{f,T}(S,X)$ is acyclic: it is in fact the identity map because f is a local biholomorphism. Second, the square

$$
\begin{tikzcd}
\mathcal{O}(S,X) \arrow[r] \arrow[d, "\alpha"'] & \mathcal{C}(S,X) \arrow[d, "\beta"] \\
\mathcal{O}(S,Y) \times_{\mathcal{O}(T,Y)} \mathcal{O}(T,X) \arrow[r] & \mathcal{C}(S,Y) \times_{\mathcal{C}(T,Y)} \mathcal{C}(T,X)
\end{tikzcd}
$$

is a pullback, because a continuous lifting in a square of holomorphic maps with right-hand map f is holomorphic, again because f is a local biholomorphism. Since f is a Serre fibration, so is β by Axiom SM7, and hence α. $\square$

We now come to the main result of this paper, describing the intermediate fibrations. Notice the similarity with the Oka property as expressed by condition (iv) above.

19. Theorem (characterization of intermediate fibrations). *A map $F \to G$ of prestacks is an intermediate fibration if and only if*

(1) *for every Stein inclusion $T \hookrightarrow S$, the induced map*

$$F(S) \to G(S) \times_{G(T)} F(T)$$

is a fibration, and

(2) *in any diagram*

$$\begin{tikzcd} T \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & PF \arrow[d] \\ S \arrow[r] & G \arrow[r] & PG \end{tikzcd}$$

the map of the simplicial set of liftings $S \to F$ into the simplicial set of liftings $S \to PF$, given by postcomposition with $F \to PF$, is acyclic.

We remark that taking $T = \varnothing$ in (1) and (2) yields precisely the description of projective fibrations between prestacks fibrant at $\mathfrak{p}$ given earlier: (1) says that $F \to G$ is a pointwise fibration and (2), using (1), says that the induced map $F \to G \times_{PG} PF$ is pointwise acyclic.

Proof. First suppose $F \to G$ is an intermediate fibration and let $T \hookrightarrow S$ be a Stein inclusion. Then (1) follows directly from Axiom SM7 for the intermediate structure. As for (2), consider the equivalent diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & E \arrow[dl] \\ S \arrow[r] & G & \end{tikzcd}$$

where $E = G \times_{PG} PF$. Since $F(\mathfrak{p}) \to G(\mathfrak{p})$ is a fibration, $PF \to PG$ and hence $E \to G$ is an injective fibration. Also, $F \to E$ is acyclic. Working in the over-under category $T \downarrow \mathfrak{S} \downarrow G$ with the model structure induced from the intermediate structure on $\mathfrak{S}$, we need to show that $\mathrm{Hom}_{T\downarrow\mathfrak{S}\downarrow G}(S, F) \to \mathrm{Hom}_{T\downarrow\mathfrak{S}\downarrow G}(S, E)$ is acyclic. Now $T \to F \to G$ and $T \to E \to G$ are fibrant in $T \downarrow \mathfrak{S} \downarrow G$ since $F \to G$ and $E \to G$ are fibrations in $\mathfrak{S}$.

By Brown's Lemma [H, 7.7], we may assume that $F \to E$ is an intermediate fibration in addition to being acyclic. Consider the fibration sequences

$$\begin{tikzcd} \mathrm{Hom}_{T\downarrow\mathfrak{S}\downarrow G}(S,F) \arrow[r] \arrow[d] & \mathrm{Hom}(S,F) \arrow[r] \arrow[d] & \mathrm{Hom}(S,G) \times_{\mathrm{Hom}(T,G)} \mathrm{Hom}(T,F) \arrow[d] \\ \mathrm{Hom}_{T\downarrow\mathfrak{S}\downarrow G}(S,E) \arrow[r] & \mathrm{Hom}(S,E) \arrow[r] & \mathrm{Hom}(S,G) \times_{\mathrm{Hom}(T,G)} \mathrm{Hom}(T,E) \end{tikzcd}$$

Since S and T are intermediately cofibrant, the middle and right-hand vertical maps are acyclic, so the left-hand vertical map is acyclic too.

Now suppose $F \to G$ satisfies (1) and (2). We need to show that $F \to G$ is an intermediate fibration. First, (1) implies that $\mathrm{Hom}(B,F) \to \mathrm{Hom}(B,G) \times_{\mathrm{Hom}(A,G)} \mathrm{Hom}(A,F)$ is a fibration for every intermediate cofibration $A \to B$ (this property is preserved under simplicial saturation).

Let us say that an intermediate cofibration $A \to B$ is *good* if in any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & PF \arrow[d] \\ B \arrow[r] & G \arrow[r] & PG \end{tikzcd}$$

in $\mathfrak{S}$, the map of the simplicial set of liftings $B \to F$ into the simplicial set of liftings $B \to PF$ is acyclic. By (2), Stein inclusions are good. With the help of (1), we will show that all intermediate cofibrations are good. Assuming this, the proof is complete. Namely, take a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F \arrow[d] \\ B \arrow[r] & G \end{tikzcd}$$

where $A \to B$ is an acyclic intermediate cofibration. By (1), $F(\mathfrak{p}) \to G(\mathfrak{p})$ is a fibration, so $PF \to PG$ is an injective fibration and there is a lifting $B \to PF$. Since $A \to B$ is good, there is also a lifting $B \to F$.

By piecing together arguments that have already been used in this paper, the reader can show that the generating cofibrations $S \times \partial\Delta^n \cup_{T\times\partial\Delta^n} T \times \Delta^n \to S \times \Delta^n$ are good. It is easy to see that being good is preserved under pushouts and retracts: pushouts give isomorphisms of lifting spaces and retracts give retractions of lifting spaces. It remains to show that a transfinite composition of good intermediate cofibrations is good.

Let $A \to B$ and $B \to C$ be good and consider a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F \arrow[r] \arrow[dd] & PF \arrow[dd] \\ B \arrow[d] & & \\ C \arrow[r] & G \arrow[r] & PG \end{tikzcd}$$

Let $\mathcal{L}_{AC}$ and $\mathcal{L}'_{AC}$ be the simplicial sets of liftings $C \to F$ and $C \to PF$ in the squares with left-hand map $A \to C$ and right-hand maps $F \to G$ and $PF \to PG$, respectively. We define $\mathcal{L}_{AB}$ and $\mathcal{L}'_{AB}$ similarly. The fibre over a lifting $B \to F$ of the map $\mathcal{L}_{AC} \to \mathcal{L}_{AB}$ given by precomposing with $B \to C$ is the simplicial set $\mathcal{L}_{BC}$ of liftings in the square with left-hand map $B \to C$, right-hand map $F \to G$, and this particular top map $B \to F$. We define $\mathcal{L}'_{BC}$ similarly. We have a pullback square

$$\begin{tikzcd} \mathcal{L}_{AC} \arrow[r] \arrow[d] & \mathcal{L}_{AB} \arrow[d] \\ \mathrm{Hom}(C,F) \arrow[r] & \mathrm{Hom}(C,G) \times_{\mathrm{Hom}(B,G)} \mathrm{Hom}(B,F) \end{tikzcd}$$

where the right-hand map takes a map in $\mathcal{L}_{AB}$ to the constant map $C \to G$ in $\mathrm{Hom}(C,G)$ and itself in $\mathrm{Hom}(B,F)$. Since the bottom map is a fibration, so is the top map $\mathcal{L}_{AC} \to \mathcal{L}_{AB}$. (It follows that the simplicial set of liftings in any square with right-hand map $F \to G$ whose left-hand map is an intermediate cofibration is fibrant: just take $A = B$ and $A \to B$ to be the identity map.)

By the same argument, $\mathcal{L}'_{AC} \to \mathcal{L}'_{AB}$ is also a fibration. Thus the rows in the diagram

$$\begin{tikzcd} \mathcal{L}_{BC} \arrow[r] \arrow[d] & \mathcal{L}_{AC} \arrow[r] \arrow[d] & \mathcal{L}_{AB} \arrow[d] \\ \mathcal{L}'_{BC} \arrow[r] & \mathcal{L}'_{AC} \arrow[r] & \mathcal{L}'_{AB} \end{tikzcd}$$

are fibration sequences. The left-hand and right-hand vertical maps are acyclic by assumption, so the middle one is too, which shows that $A \to C$ is good.

We now move to the transfinite case. Let λ be an ordinal and $A : \lambda \to \mathfrak{S}$ be a functor such that for every limit ordinal $\gamma < \lambda$, the induced map $\mathrm{colim}_{\alpha<\gamma} A_\alpha \to A_\gamma$ is an isomorphism, and such that for every ordinal α with $\alpha + 1 < \lambda$, the map $A_\alpha \to A_{\alpha+1}$ is a good intermediate cofibration. We will show by transfinite induction that the composition

$A_0 \to \operatorname{colim}_{\alpha<\lambda} A_\alpha$ is good. Suppose $\mu \leq \lambda$ and $A_0 \to \operatorname{colim}_{\alpha<\beta} A_\alpha$ is good for all $\beta < \mu$. We need to show that $A_0 \to \operatorname{colim}_{\alpha<\mu} A_\alpha$ is good.

Assume μ is a successor, say $\mu = \beta+1$. If β is a limit ordinal, then $A_0 \to \operatorname{colim}_{\alpha<\beta} A_\alpha = A_\beta = \operatorname{colim}_{\alpha<\mu} A_\alpha$ is good by the induction hypothesis. If β is a successor, say $\beta = \gamma+1$, then $A_0 \to \operatorname{colim}_{\alpha<\beta} A_\alpha = A_\gamma \to A_{\gamma+1} = \operatorname{colim}_{\alpha<\mu} A_\alpha$ is good, being the composition of two good maps.

Suppose now that μ is a limit ordinal and take a square

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & F \arrow[d] \\ \operatorname{colim}_{\alpha<\mu} A_\alpha \arrow[r] & G \end{tikzcd}$$

Define a μ-tower $\mathcal{L} : \mu^{\mathrm{op}} \to s\mathbf{Set}$ such that $\mathcal{L}_\alpha$ is the simplicial set of liftings in the square

$$\begin{tikzcd} A_0 \arrow[rr] \arrow[d] & & F \arrow[d] \\ A_\alpha \arrow[r] & \operatorname{colim}_{\alpha<\mu} A_\alpha \arrow[r] & G \end{tikzcd}$$

for $\alpha < \mu$. Define $\mathcal{L}'$ similarly for $PF \to PG$. Then $\mathcal{L}$ and $\mathcal{L}'$ are fibrant objects in the category of μ-towers with the pointwise cofibration simplicial model structure [GJ, VI.1], the main point being that for all $\alpha < \mu$, the map $\mathcal{L}_{\alpha+1} \to \mathcal{L}_\alpha$ is a fibration, as shown above. Thus, since the map $\mathcal{L} \to \mathcal{L}'$ is pointwise acyclic by the induction hypothesis, it induces an acyclic map from $\lim_{\alpha<\mu} \mathcal{L}_\alpha$ to $\lim_{\alpha<\mu} \mathcal{L}'_\alpha$, that is, from the simplicial set of liftings $\operatorname{colim}_{\alpha<\mu} A_\alpha \to F$ to the simplicial set of liftings $\operatorname{colim}_{\alpha<\mu} A_\alpha \to PF$. $\square$

Suppose that the prestacks F and G are represented by complex manifolds X and Y respectively. We have

$$\operatorname{Hom}(S, PX) = PX(S) = \operatorname{Hom}(sS, sX) = s\mathcal{C}(|sS|, X).$$

Using the homotopy equivalence $|sS| \to S$, we can verify that our characterization of the map $F \to G$ induced by a holomorphic map $X \to Y$ being an intermediate fibration means precisely that $X \to Y$ satisfies the Oka property as defined by condition (iv) above.

20. Corollary. *A holomorphic map is an intermediate fibration if and only if it is Oka.*

It follows that subelliptic manifolds are intermediately fibrant and that holomorphic covering maps are intermediate fibrations. Also, the class of Oka maps is preserved under composition, pullbacks, and retracts.

Our conjecture now looks like this.

21. Conjecture. *A subelliptic submersion is an intermediate fibration if and only if it is a topological fibration.*

This would be a new manifestation of the Oka Principle, in a sense dual to the usual formulations that refer to Stein manifolds, saying that for holomorphic maps satisfying the geometric condition of subellipticity there is only a topological obstruction to being a fibration in our new, holomorphic sense.

22. An alternative approach to the intermediate structure. We have gone from the coarse projective structure on $\mathfrak{S}$ to the intermediate structure via the projective structure by first enlarging the class of weak equivalences by a Bousfield localization, keeping the cofibrations fixed, and then enlarging the class of cofibrations, keeping the weak equivalences fixed. Alternatively, we could do this the other way around, passing through what we shall call the coarse intermediate structure on $\mathfrak{S}$. The cofibrations in this structure are the same as in the intermediate structure, but the weak equivalences are defined pointwise, and the proof of Theorem 13 goes through word for word.

A modification of the proof of Theorem 19 gives a characterization of the coarse intermediate fibrations. Take a map $F \to G$ of prestacks. Instead of $PF \to PG$, we now use a coarsely injectively fibrant model $\tilde{F} \to \tilde{G}$ of $F \to G$. In particular, $F \to \tilde{F}$ and $G \to \tilde{G}$ are pointwise acyclic and $\tilde{F} \to \tilde{G}$ is a coarse injective fibration. Suppose that $F \to G$ satisfies property (1) in Theorem 19. The key point is that Stein inclusions are now automatically good, that is, (1) implies (2), and we can go on to show that all intermediate cofibrations are good as before. Namely, consider a diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & \tilde{F} \arrow[d] \\ S \arrow[r] & G \arrow[r] & \tilde{G} \end{tikzcd}$$

and the induced diagram of fibration sequences

$$\begin{tikzcd} \{\text{liftings } S \to F\} \arrow[r] \arrow[d] & F(S) \arrow[r] \arrow[d] & G(S) \times_{G(T)} F(T) \arrow[d] \\ \{\text{liftings } S \to \tilde{F}\} \arrow[r] & \tilde{F}(S) \arrow[r] & \tilde{G}(S) \times_{\tilde{G}(T)} \tilde{F}(T) \end{tikzcd}$$

Since the middle and right-hand vertical arrows are acyclic, so is the left-hand vertical arrow. It follows that $F \to G$ is a coarse intermediate fibration if and only if it satisfies property (1), that is, for every Stein inclusion $T \hookrightarrow S$, the induced map $F(S) \to G(S) \times_{G(T)} F(T)$ is a fibration. Hence, the coarse intermediate fibrations are precisely those maps that have the right lifting property with respect to the maps

$$S \times \Lambda_k^n \cup_{T \times \Lambda_k^n} T \times \Delta^n \to S \times \Delta^n,$$

where Λ^n_k denotes the k-th horn of Δ^n, $0 \le k \le n$ (just look at the squares in the proof of Proposition 14). By a standard factorization and retraction argument, these maps form a generating set of acyclic coarse intermediate cofibrations.

Let us now pass to the intermediate structure by a Bousfield localization. As we saw for the projective structure earlier, a coarse intermediate fibration $F \to G$ between prestacks fibrant at $\mathfrak{p}$ is an intermediate fibration if and only if the square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d] \\ PF \arrow[r] & PG \end{tikzcd}$$

is a homotopy pullback in the coarse intermediate structure. Since $PF \to PG$ is an injective fibration, this is simply the old condition that the induced map $F \to G \times_{PG} PF$ be pointwise acyclic, or in other words, that $F \to G$ be a projective fibration. This gives one more characterization of the Oka property, namely condition (iii) above with $T = \varnothing$ in its second half, which we had previously observed to be equivalent to (iii) in the case of manifolds.

23. Final remarks. We conclude the paper with a few additional words of motivation. Model categories are highly nontrivial structures. Finding them in a new area of mathematics should be of interest in itself, especially when they can be shown to be relevant to a topic as deep and important as the Oka Principle. The gist of the results in this paper is that analytically defined Oka properties for complex manifolds and holomorphic maps fit into a homotopy-theoretic framework in a precise sense: they are equivalent to fibrancy in suitable model categories containing the category of complex manifolds. Our definitions of the Oka property and the weak Oka property for maps, extending familiar Oka properties of manifolds, are in fact dictated by abstract homotopy theory. In short, we take the point of view that the Oka Principle is about fibrancy.

It is hoped that this work will eventually have concrete applications in complex analysis. Here are three brief remarks in this direction. First, whether subelliptic submersions that are also topological fibrations are closed under composition is unknown. Subelliptic submersions are not closed under composition and neither is the class of maps with the property attributed to elliptic bundles by Gromov's theorem (the second half of condition (iv) above): just consider $\mathbb{D} \setminus \{0\} \hookrightarrow \mathbb{C} \to \mathfrak{p}$. Adding to this property a holomorphic version of Axiom SM7 (the first half of (iv)) yields our Oka property with all the functorial properties we could wish for. It readily implies that if the target is Oka, so is the source, and, if our conjecture is true, has being a subelliptic submersion and a topological fibration as a useful geometric sufficient condition.

Second, by the previous section, the Parametric Oka Principle with Interpolation, as expressed by condition (i), can be verified by only checking it for acyclic maps $Q \to P$ (giving coarse intermediate fibrancy) and for $T = \varnothing$ (giving projective fibrancy). I do not know a direct proof of this (except in the special case of manifolds, where it is easy).

Third, homotopy theory may shed light on the relationship between topological and holomorphic contractibility for Stein manifolds. I believe it is currently unknown whether the former implies the latter. If we had a suitable weak sufficient condition for coarse intermediate fibrancy (weaker than subellipticity) satisfied by a topologically contractible Stein manifold S which did not have the extension property with respect to some Stein inclusion, then S would not be intermediately and hence not projectively fibrant and therefore not holomorphically contractible. Candidates for such an example exist in the literature and are being investigated. The homotopy-theoretic side of this problem is to distinguish between coarse and fine intermediate fibrancy for complex manifolds.

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