THE MARTIN BOUNDARY ACTION OF GROMOV HYPERBOLIC COVERING GROUPS AND APPLICATIONS TO HARDY CLASSES

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1. Introduction.

Hardy theory deals with holomorphic and harmonic functions whose growth is bounded in a certain precise sense that has turned out to be fruitful, important and appropriate for many purposes. The Hardy class $H^p(X)$, $0 < p < \infty$, of a Riemann surface X is the space of holomorphic functions f on X such that $|f|^p$ has a harmonic majorant, and $H^\infty(X)$ is the space of bounded holomorphic functions on X. In this paper, we will study Hardy classes on infinite-sheeted Galois covering spaces of compact Riemann surfaces. Such covering spaces may be thought of as surfaces with a large group of automorphisms. They are infinitely connected, with the obvious exceptions of the disc, the plane and the punctured plane [Gri]. We will mostly assume that the covering group is hyperbolic in the sense of Gromov [Gro1], because only in this case do we have enough information about the action of automorphisms on the Martin boundary, where Hardy functions are represented by measures.

The Hardy classes of the unit disc have been studied intensively ever since Hardy introduced them in 1915, leading to many developments in function theory, functional analysis and harmonic analysis. The theory has been extended with considerable success to Riemann surfaces with smooth boundary, non-planar as well as planar, see e.g. [Hei]. On the other hand, very little is known about Hardy classes of infinitely connected surfaces. The exception is the special class of Parreau-Widom surfaces, see [Has], which

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have been shown to have a Hardy theory similar to that of the disc. Apart from relatively compact domains in open surfaces, this is the only class of surfaces known to have non-constant bounded holomorphic functions. We will show that most covering surfaces, including those of greatest interest to us, are not Parreau-Widom.

Before describing the contents of the paper, we will discuss the possible applications that motivated our work and give some evidence that the restriction to Gromov hyperbolic coverings is natural and mild enough to be of interest.

Besides being of interest in itself, the Hardy theory of covering surfaces has potential applications to the Shafarevich conjecture, a central problem in higher dimensional analytic geometry. This conjecture states that the universal covering space of any projective algebraic manifold is holomorphically convex. There are no known counterexamples to the conjecture, and it has been verified only in a number of fairly special cases. For a survey of results related to the Shafarevich conjecture prior to 1985, see [Gur]. For more recent results in this area, see [ABR, Cam1, Cam2, Cam3, Kat, Kol1, Kol2, Nap1, Nap2, NR, Ram]. In our paper [Lár], we approached the problem of constructing holomorphic functions on covering spaces by proving an extension theorem that we will now describe.

Let M be a projective algebraic manifold of dimension $n \geq 2$ and $\pi : Y \to M$ be an infinite covering space. Suppose M is embedded in some projective space by sections of a very ample line bundle L. The generic linear subspace of codimension $k < n$ intersects M transversely in a submanifold C of codimension k. By the Lefschetz hyperplane theorem, C is connected and the inclusion of C in M induces an epimorphism of fundamental groups, so the preimage $X = \pi^{-1}(C)$ is connected. The extension theorem states that if L is sufficiently positive, then a holomorphic function f on X extends to all of Y if it does not grow too fast. More precisely, we must have $|f| \leq ce^{\epsilon r}$ for $\epsilon > 0$ small enough, where r is the distance from a fixed point in X with respect to any metric pulled back from C. In particular, if f is bounded, then f extends to Y.

Now take $k = n-1$, so that X is a Riemann surface. Then Harnack's inequality easily implies that functions f in the Hardy class $H^p(X)$ satisfy the above bound on growth and therefore extend to Y if p is large enough. The relevance of Hardy theory for coverings of compact Riemann surfaces is now clear. In a sense, many H^p functions on X for p large give many holomorphic functions on Y. It is an interesting open problem to decide whether H^p-convexity of some or all X in Y as above, for p sufficiently large, implies holomorphic convexity of Y. By H^p-convexity of X we mean that for every infinite subset S of X without limit points there is an H^p function on X which is unbounded on S.

Let us briefly discuss the restriction to Gromov hyperbolic coverings. First we recall that a simply connected complete Kähler manifold of non-positive sectional curvature is a Stein manifold [Wu]. Hence the Shafarevich conjecture is true for a compact Kähler manifold of negative sectional curvature. Such manifolds have hyperbolic fundamental groups. Hyperbolicity of $\pi_1(M)$ is a condition on the large-scale geometry of the universal covering space of M. The class of hyperbolic coverings is therefore a natural one to consider, but how substantial is the generalization? In other words, which projective manifolds have a hyperbolic fundamental group? As far as I know, this question has not been given the attention it deserves, but we can make a few elementary observations.

As mentioned above, if M is negatively curved, e.g. a ball quotient, then $\pi_1(M)$ is hyperbolic. From such manifolds we can construct many more examples. Namely, $\pi_1(M_2)$ is hyperbolic if $\pi_1(M_1)$ is hyperbolic and one of the following holds:

(1) M_2 is an ample divisor in M_1 and $\dim M_2 \geq 2$.
(2) $M_2 = M_1 \times M$ with $\pi_1(M)$ finite, e.g. M simply connected.
(3) There is a finite unbranched covering $M_1 \to M_2$ or $M_2 \to M_1$.

It is easy to see that if M is obtained from a negatively curved manifold in this way, then the universal covering of M is holomorphically convex, but not necessarily Stein, so M may not be non-positively curved.

We might add that hyperbolicity of the fundamental group of a compact Kähler manifold M does not seem closely related to other important notions of hyperbolicity. It is claimed, though, that if $\pi_1(M)$ is hyperbolic and $\pi_2(M) = 0$, then M is Kähler hyperbolic and hence Kobayashi hyperbolic [Gro2].

Additional motivation for studying Gromov hyperbolic coverings of compact Riemann surfaces comes from the work of Ancona [Anc], to be described in section 2, which exhibits a strong relationship between geometry and potential theory for such spaces. In this paper we attempt to bring function theory into the picture as well.

Let us now describe the contents of the paper. In section 2 we summarize the necessary background material on Gromov hyperbolicity and the Martin boundary and give new examples of Gromov spaces. Section 3 contains our results on the boundary action of a cocompact hyperbolic covering group in the general setting of Riemannian manifolds. In section 4 these results are applied to prove the following theorem on Hardy classes, which gives a sufficient condition for the existence of as many H^p functions as there can possibly be.

Main Theorem. *Let X be a Galois covering space of a compact Riemann surface with a non-elementary hyperbolic covering group. Then either:*

(1) *every positive harmonic function on X is the real part of a holomorphic function, or*
(2) *if $u \geq 0$ is the real part of an H^1 function on X, then the boundary decay of u at a zero on the Martin boundary of X is no faster than its radial decay.*

In case (1), *X is H^p-convex for each $p < \infty$.*

Positive harmonic functions are easily constructed as Poisson integrals of measures on the boundary. Hence the first alternative in this dichotomy provides a large supply of holomorphic functions of slow growth, which are in general very hard to construct. The second alternative is characteristic of the higher dimensional case. It gives a necessary condition for a boundary function to extend to the real part of a holomorphic function. Section 4 contains a further discussion of this, as well as examples.

The dichotomy needs to be clarified, and many questions remain unanswered. For instance, I do not know if group theoretic properties of the covering group determine which alternative holds, or if (1) always holds for covering spaces with one end. We are primarily interested in covering spaces with one end, because the universal covering space of a compact Kähler manifold with infinite fundamental group has one end [ABR]. In section 5 we show that there are hyperbolic covering spaces of compact Riemann surfaces of any genus greater than 1 with infinitely many ends that have no non-constant bounded holomorphic functions, although they do have an infinite dimensional space of bounded harmonic functions. These covering spaces are domains of discontinuity of certain Schottky groups.

Section 5 also contains our proof that a Parreau-Widom covering space of a compact Riemann surface is either the disc or homeomorphic to a sphere with a Cantor set removed. I expect, but cannot prove, that the latter possibility does not occur.

Finally, in section 6, we give a very short proof of a theorem of Kifer and Toledo, illustrating the advantages of working on the boundary. This theorem states that if a Galois covering space of a compact Riemannian manifold M has a non-constant bounded harmonic function, then it has an infinite dimensional space of such functions. It actually suffices to assume that M is parabolic. We also prove an analogous theorem for positive harmonic functions.

Most of the results we state for a Galois covering space X with a covering group Γ apparently still hold if Γ is only assumed to act properly discontinuously on X. In other words, the assumption that Γ act freely on X seems largely superfluous. We have not pursued this generalization in detail.

All covering spaces considered in this paper will be Galois, meaning that the covering group acts transitively on the fibres. All our manifolds will be connected.

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2. Gromov hyperbolicity and the Martin boundary.

In this section we will describe some necessary background material and establish notation. For the theory of Gromov hyperbolic spaces and groups we refer the reader to Gromov's original paper [Gro1], the expositions [CDP] and [GH], and the summary in [CP]. The Martin boundary of Riemann surfaces is discussed in [CC] and [Has]. Finally, one should consult [Anc] on the Martin boundary of Riemannian manifolds and its relationship to the Gromov boundary.

Let X be a metric space with distance function d. We will write $|x-y|$ for $d(x,y)$ and $|x|$ for $d(x,o)$, where the base point $o \in X$ is fixed once and for all. A geodesic in X is an isometry from an interval in $\mathbb{R}$ into X, or the image of such an isometry.

The Gromov product of $x, y \in X$ with respect to the base point o is

$$(x|y) = (x|y)_o = \tfrac{1}{2}(|x| + |y| - |x-y|).$$

We call X Gromov hyperbolic or simply Gromov if there is $\delta \geq 0$ with

$$(x|y) \geq \min\{(x|z), (y|z)\} - \delta, \qquad x, y, z \in X.$$

If any two points in X can be joined by a geodesic segment, then this means that geodesic triangles in X are uniformly thin. Among the basic examples of Gromov spaces are simply connected complete Riemannian manifolds with sectional curvature bounded above by a negative constant. Trees form another important class of Gromov spaces.

The following result, prompted by a question of Fausto Di Biase, gives new examples of Gromov spaces. They will not play a role elsewhere in the paper.

2.1. Proposition. *Let X be a simply connected, smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^n$. Consider X as a metric space with the Bergman metric, the Carathéodory metric or the Kobayashi metric. Then X is Gromov.*

Proof. The Bergman metric has negative sectional curvature bounded away from zero outside a compact set [Kle], so X is Gromov in the Bergman metric. It is well known that for a C^2 bounded strictly pseudoconvex domain, the three distances are mutually comparable outside a compact; see [JP] and the references therein. Hence X is also Gromov in the Carathéodory and Kobayashi metrics. □

This result may be of interest because it gives examples of Gromov metrics that are very different from Riemannian metrics. There are domains X as in the proposition (necessarily not strongly convex) with infinitely many Kobayashi geodesics joining the same two points and infinitely many Kobayashi geodesics having the same tangent vector at a point [Myung-Yull Pang, private communication].

It is not clear to what extent the conditions on X can be relaxed. Bounded symmetric domains of rank at least 2 are simply connected and weakly pseudoconvex, but not Gromov, since they contain flats. An annulus in $\mathbb{C}$ is not simply connected, but still Gromov in the Poincaré metric.

From now on, we will assume that X is a complete Riemannian manifold with the distance function given by the Riemannian metric.

Suppose X is Gromov. Scaling the metric on X by the functions $e^{-\epsilon|\cdot|}$ with $\epsilon > 0$ sufficiently small, we obtain a class of metrics $|\cdot - \cdot|_\epsilon$ on X, called visual metrics. Completing X with respect to any one of these metrics gives the Gromov compactification $\bar{X}$ of X. The Gromov boundary $\partial X = \bar{X} \setminus X$ can also be constructed as the set of ends of geodesic rays issuing from o. The Gromov product extends to $\bar{X}$ in the following way. For $x, y \in \bar{X}$,

$$(x|y) = \inf \liminf_{n\to\infty} (x_n|y_n),$$

where the infimum is taken over all sequences (x_n) in X converging to x and (y_n) converging to y. (This is the definition of [CDP]. The definition in [GH] is slightly different, but essentially equivalent.) Clearly, there are $x_n \to x$ and $y_n \to y$ in X such that $(x_n|y_n) \to (x|y)$, and $(x|y)$ is the smallest such limit. For any $x_n \to x$ and $y_n \to y$ in X,

$$\liminf_{n\to\infty} (x_n|y_n) - 2\delta \le (x|y) \le \liminf_{n\to\infty} (x_n|y_n).$$

We have

$$|x-y|_\epsilon \le ce^{-\epsilon(x|y)}, \qquad x, y \in \bar{X},$$

where c depends only on δ and ϵ. On ∂X, $|x-y|_\epsilon$ and $e^{-\epsilon(x|y)}$ are actually comparable.

Next we describe the Martin boundary of X (for the Laplace-Beltrami operator). Assume that X is non-parabolic, i.e., that X has a non-constant negative subharmonic function. This means that X has a Green kernel G, which yields the Martin kernel

$$k_y(x) = k(x,y) = G(x,y)/G(o,y), \qquad x, y \in X.$$

The Martin compactification X^* of X is the unique compactification of X to which all the functions $y \mapsto k(x,y)$, $x \in X$, extend continuously such that the extensions separate the points of the Martin boundary $\Delta = X^* \setminus X$. The Martin compactification is metrizable. The Martin functions k_y, $y \in \Delta$, are positive harmonic functions on X with $k_y(o) = 1$. Among them are all the minimal such functions; they correspond to points in the minimal Martin boundary Δ_1, which is a non-empty G_δ subset of Δ. Recall that a positive harmonic function u is called minimal if every positive harmonic minorant of u is a constant multiple of u.

Let $h^p(X)$, $1 \le p < \infty$, be the space of real harmonic functions u on X such that $|u|^p$ has a harmonic majorant, and let $h^\infty(X)$ be the space of real bounded harmonic functions on X. The space $h^1(X)$ is the space of functions that can be written as the difference of two positive harmonic functions. Such functions are represented by finite real Borel measures on the Martin boundary by means of a generalized Poisson integral. More precisely, for every $u \in h^1(X)$ there is a unique measure μ on Δ_1 such that

$$u(x) = \int_{\Delta_1} k_y(x)\mu(y), \qquad x \in X.$$

Then we write $u = H[\mu]$. Conversely, every measure μ on Δ_1 defines a function $u \in h^1(X)$ in this way, so $\mu \mapsto H[\mu]$ is an order-preserving isomorphism from the vector space $M(\Delta_1)$ of finite real Borel measures on Δ_1 onto $h^1(X)$. The constant function 1 is represented by the harmonic measure σ on Δ_1. The Lebesgue spaces $L^p(\Delta)$ are defined with respect to σ. Every $u = H[\mu]$ in $h^p(X)$ has a fine boundary function $\hat{u} \in L^p(\Delta)$ such that $\hat{u}\sigma$ is the absolutely continuous part of μ. If $\mu = \hat{u}\sigma$, then u is called quasi-bounded. This is always the case if $p > 1$. For an absolutely continuous measure $v\sigma$, $v \in L^1(\Delta)$, we simply write $H[v]$ instead of $H[v\sigma]$.

From now on, we will assume that X is a Galois covering space of a compact manifold with a covering group Γ of isometries. Then X is Gromov if and only if Γ is hyperbolic. This means that the word metric on Γ, defined using any finite set of generators, makes Γ into a Gromov space. Also, X is non-parabolic if and only if Γ has more than quadratic growth, which means that Γ is neither finite nor a finite extension of $\mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}$ [Anc, Var, VSC]. Then the Green function G_o with pole o vanishes at infinity [CF, Var].

Assume finally that Γ is hyperbolic and non-elementary, i.e., neither finite nor a finite extension of $\mathbb{Z}$. Then Γ is non-amenable, so X has a non-constant bounded harmonic function [LS]. In particular, X is non-parabolic. Also, Γ acts on the Gromov boundary ∂X, which is uncountable and perfect as a metric space, by homeomorphisms which are Lipschitz and quasi-conformal with respect to the visual metrics. We highlight the following important fact, proved in [GH].

2.2. Proposition. *Every point of the Gromov boundary has a dense Γ-orbit.*

The key to the main result of this paper is the following theorem of Ancona [Anc], which relates the potential theory of X to its geometry.

2.3. Theorem (Ancona). *Let X be a Galois covering space of a compact Riemannian manifold with a non-elementary hyperbolic covering group. Then the Gromov compactification of X is naturally homeomorphic to the Martin compactification of X.*

More precisely, there is a Γ-equivariant homeomorphism $\phi:\bar X\to X^*$ such that $\phi|X$ is the identity and for $x\in\partial X$, the Martin function $k_{\phi(x)}$ is the unique positive harmonic function u on X with $u(o)=1$ that is bounded above by a multiple of G_o on the complement of any neighbourhood of x in $\bar X$. This implies that k_y, $y\in\Delta$, extends continuously to $X^*\setminus\{y\}$ and vanishes on $\Delta\setminus\{y\}$. All the Martin functions are minimal, so $\Delta_1=\Delta$.

Hence the geometrically defined Gromov boundary and the analytically defined Martin boundary are in fact the same. We will identify them by means of the homeomorphism ϕ and denote both of them by ∂X.

Ancona also proves that the Dirichlet problem is solvable on X for any continuous function on ∂X. In other words, if u is a continuous function on ∂X, then

$$\lim_{x\to a} H[u](x)=u(a),\qquad a\in\partial X.$$

3. The boundary action of a hyperbolic group.

In this section, X will denote a Galois covering space of a compact Riemannian manifold M with a non-elementary hyperbolic covering group Γ. Then X is a Gromov space and the Gromov boundary is naturally homeomorphic to the Martin boundary. We will investigate the action of Γ on the compactification $\bar X$ and the boundary ∂X. The main question to be addressed is: When does a positive harmonic function on X have a dense Γ-orbit in $h^1(X)$? Our results will be used in section 4 to study the Hardy classes of X when M is a compact Riemann surface.

3.1. Lemma. *The harmonic measure σ on ∂X has no atoms.*

Proof. If $a \in \partial X$ has positive mass, then k_a is bounded because

$$\sigma(a)k_a(x) \le \int_{\partial X} k_y(x)\sigma(y) = 1, \qquad x \in X.$$

But on a geodesic ray from o to a,

$$c^{-1} \le G_o k_a \le c$$

[Anc]. Since G_o vanishes at infinity, k_a is unbounded near a. □

3.2. Lemma. *Let $a \in \partial X$. Then there are $\gamma_n \in \Gamma$, $n \in \mathbb{N}$, such that $\gamma_n(o) \to a$ and the map*

$$\mathbb{N} \to X, \quad n \mapsto \gamma_n(o),$$

is a quasi-isometry with

$$|n-m| - 2D \le |\gamma_n(o) - \gamma_m(o)| \le |n-m| + 2D, \tag{3.1}$$

where D is the diameter of M. Hence, every boundary point is a conical limit point.

Suppose $\gamma_{n_k}^{-1}(o) \to b$ for a subsequence (n_k). Then $b \in \partial X$ and $\gamma_{n_k} \to a$ locally uniformly on $\bar{X} \setminus \{b\}$. In fact, for every compact $K \subset \bar{X}$ with $b \notin K$ there is $c > 0$ such that

$$\min_K (\gamma_{n_k}|a) \ge n_k - c.$$

Finally, if $\gamma_{n_k}^{-1}(a) \to a'$, then $a' \ne b$.

A boundary point a is a conical limit point if it can be approached non-tangentially, i.e., within a bounded distance of any geodesic ray ending at a, by points in any Γ-orbit.

If a is the attracting fixed point of a hyperbolic isometry $\gamma \in \Gamma$, then it is well known that the sequence of iterates γ^n, $n \in \mathbb{N}$, has all the properties that we have asserted for the sequence (γ_{n_k}), *mutatis mutandis*, with b being the repelling fixed point of γ. Our lemma states that *every* boundary point can be approached by a sequence of transformations having the same essential properties as the iterates of a hyperbolic isometry. The last statement of the lemma will not be needed in the following, but is included to strengthen the generalization.

Proof. Let $\alpha : [0,\infty[\to X$ be a geodesic ray from o to a. For every $n \in \mathbb{N}$ there is $\gamma_n \in \Gamma$ with $|\alpha(n) - \gamma_n(o)| \le D$. Then (3.1) is clear.

Suppose $\gamma_{n_k}^{-1}(o) \to b$ and let $K \subset \bar{X}$ be compact with $b \notin K$. Let $x \in K$. Then

$$(\gamma_n(x)|a) \ge \min\{(\gamma_n(x)|\gamma_n(o)), (\gamma_n(o)|a)\} - \delta.$$

We will consider each of the two products on the right hand side in turn. First we have

$$\begin{aligned}(\gamma_n(o)|a) &\geq \liminf_{t\to\infty} (\gamma_n(o)|\alpha(t)) - 2\delta \\ &= \tfrac{1}{2} \liminf_{t\to\infty} (|\gamma_n(o)| + |\alpha(t)| - |\gamma_n(o) - \alpha(t)|) - 2\delta \\ &\geq n - D - 2\delta.\end{aligned}$$

Next, let U be a neighbourhood of K and V be a neighbourhood of b such that $\bar{U} \cap \bar{V} = \varnothing$. This implies that there is $c > 0$ with

$$(y|z) \leq c, \qquad y \in U, z \in V.$$

Suppose $x_j \to x$ in X. Then

$$(\gamma_{n_k}(x)|\gamma_{n_k}(o)) = (x|o)_{\gamma_{n_k}^{-1}(o)} \geq \liminf_{j\to\infty} (x_j|o)_{\gamma_{n_k}^{-1}(o)} - 2\delta.$$

For j and k sufficiently large,

$$\begin{aligned}2(x_j|o)_{\gamma_{n_k}^{-1}(o)} &= |x_j - \gamma_{n_k}^{-1}(o)| + |\gamma_{n_k}^{-1}(o)| - |x_j| \\ &= |x_j| + |\gamma_{n_k}^{-1}(o)| - 2(x_j|\gamma_{n_k}^{-1}(o)) + |\gamma_{n_k}^{-1}(o)| - |x_j| \\ &\geq 2|\gamma_{n_k}^{-1}(o)| - 2c = 2|\gamma_{n_k}(o)| - 2c,\end{aligned}$$

so

$$(\gamma_{n_k}(x)|\gamma_{n_k}(o)) \geq |\gamma_{n_k}(o)| - c - 2\delta \geq n_k - D - c - 2\delta.$$

Therefore,

$$(\gamma_{n_k}(x)|a) \geq n_k - c,$$

with c independent of $x \in K$, so in any of the visual metrics on $\bar{X}$,

$$\|\gamma_{n_k} - a\|_{\epsilon,K} \leq c_\epsilon \max_{x\in K} e^{-\epsilon(\gamma_{n_k}(x)|a)} \leq c_\epsilon e^{-\epsilon(n_k - c)} \to 0 \qquad \text{as } k \to \infty.$$

Finally, $\gamma_{n_k}^{-1}(o)$ and $\gamma_{n_k}^{-1}(a)$ both converge to the same point if and only if

$$(o|a)_{\gamma_{n_k}(o)} = (\gamma_{n_k}^{-1}(o)|\gamma_{n_k}^{-1}(a)) \to \infty,$$

but

$$(o|a)_{\gamma_{n_k}(o)} \leq \liminf_{t\to\infty} (o|\alpha(t))_{\gamma_{n_k}(o)}$$

and

$$\begin{aligned}2(o|\alpha(t))_{\gamma_{n_k}(o)} &= |\gamma_{n_k}(o)| + |\alpha(t) - \gamma_{n_k}(o)| - |\alpha(t)| \\ &\leq |\alpha(n_k)| + |\alpha(t) - \alpha(n_k)| - |\alpha(t)| + 2D \\ &= 2D\end{aligned}$$

for $t \geq n_k$. $\square$

We make $h^1(X)$ into a Banach space by defining the norm $\|u\|$ of $u \in h^1(X)$ to be $v(o)$, where v is the least harmonic majorant of $|u|$. If $u = H[\mu]$, where μ is a measure on ∂X, then $v = H[|\mu|]$, where $|\mu|$ is the total variation of μ, so

$$\|u\| = v(o) = \int_{\partial X} |\mu| = \|\mu\|.$$

Hence, $\mu \mapsto H[\mu]$ is an isometry of the Banach space $M(\partial X)$ of finite real Borel measures on ∂X onto $h^1(X)$.

The action of Γ on $h^1(X)$ by precomposition corresponds to an action by Banach automorphisms $\mu \mapsto \gamma^*\mu$ on $M(\partial X)$ with

$$H[\gamma^*\mu] = H[\mu] \circ \gamma.$$

The action on absolutely continuous measures $u\sigma$, $u \in L^1(\partial X)$, is given by

$$\gamma^*(u\sigma) = (u \circ \gamma)\sigma.$$

3.3. Lemma. *The action of Γ on $M(\partial X)$ is continuous in the weak-star topology.*

Note that the proof does not use hyperbolicity.

Proof. Let λ be a weak-star continuous functional on $M(\partial X)$, say $\lambda(\mu) = \int_{\partial X} h\mu$ with h continuous on ∂X. Let $\gamma \in \Gamma$. We need to show that the linear functional $\lambda \circ \gamma^*$ is also weak-star continuous.

Consider the continuous functions $y \mapsto k_y(x)$, $x \in X$, on ∂X. They span a subspace E in $\mathcal{C}(\partial X)$. If $\mu \in M(\partial X)$ and

$$H[\mu](x) = \int_{\partial X} k_y(x)\mu(y) = 0$$

for all $x \in X$, then $\mu = 0$. Hence, E is weakly dense, and thus norm dense, in $\mathcal{C}(\partial X)$.

For $x \in X$ and $\mu \in M(\partial X)$,

$$\begin{aligned}\int_{\partial X} k_y(x)(\gamma^*\mu)(y) = H[\gamma^*\mu](x) = H[\mu](\gamma x) &= \int_{\partial X} k_y(\gamma x)\mu(y) \\ &= \int_{\partial X} k_y(\gamma o) k_{\gamma^{-1}(y)}(x)\mu(y),\end{aligned}$$

because, as is easily verified,

$$k_y \circ \gamma = k_y(\gamma o) k_{\gamma^{-1}(y)}, \qquad y \in \partial X. \tag{3.2}$$

Hence,

$$\int_{\partial X} h(\gamma^* \mu) = \int_{\partial X} k.(\gamma o) h \circ \gamma^{-1} \mu$$

for every $h \in E$, and therefore for every $h \in \mathcal{C}(\partial X)$. This shows that the linear functional $\lambda \circ \gamma^*$ is represented by the continuous function $k.(\gamma o) h \circ \gamma^{-1}$. $\square$

Suppose now that $u \geq 0$ is an integrable function on ∂X. If u has a zero at a point $a \in \partial X$, then we can use the visual metrics on $\bar{X}$ to compare the boundary decay of u at a to the radial decay of $H[u]$ at a.

Let $\alpha : [0, \infty[\to X$ be a geodesic ray from o to a. Let ξ be the infimum of the numbers $\zeta > 0$ such that for some $c > 0$,

$$H[u](\alpha(t)) \geq c\zeta^{-|\alpha(t)|} = c\zeta^{-t}, \qquad t \geq 0.$$

If $\int_{\partial X} u\sigma > 0$, so $H[u] > 0$, then such numbers ζ exist by Harnack's inequality, and there is an upper bound for ξ depending only on X. Since

$$(\alpha(t)|a) \geq t = \lim_{s\to\infty} (\alpha(t)|\alpha(s)) \geq (\alpha(t)|a) + 2\delta,$$

ξ is also the infimum of the numbers $\zeta > 0$ such that for some $c > 0$,

$$H[u](\alpha(t)) \geq c\zeta^{-(\alpha(t)|a)} \geq c_\epsilon |\alpha(t) - a|_\epsilon^{(\log\zeta)/\epsilon}, \qquad t \geq 0.$$

We say that the boundary decay of u at a is faster than its radial decay if there is a neighbourhood $V \subset \partial X$ of a such that

$$u(x) \leq c_\epsilon |x - a|_\epsilon^s, \qquad x \in V,$$

with $s > (\log\xi)/\epsilon$, in any or all of the visual metrics. Equivalently,

$$u(x) \leq c\eta^{-(x|a)}, \qquad x \in V,$$

with $\eta > \xi$.

3.4. Theorem. *Let X be a Galois covering space of a compact Riemannian manifold with a non-elementary hyperbolic covering group Γ. Let u be a non-negative integrable function on ∂X. If $u(a) = 0$ and the boundary decay of u at a is faster than its radial decay, then the Γ-invariant subspace spanned by $u\sigma$ in $M(\partial X)$ is weak-star dense.*

Proof. We may assume that $H[u] > 0$. Find $\gamma_n \in \Gamma$ such that $\gamma_n(o) \to a$ as in lemma 3.2, and suppose $\gamma_{n_k}^{-1}(o) \to b$. Let

$$v_n = u \circ \gamma_n / \int_{\partial X} u \circ \gamma_n \sigma.$$

Then $v_n \geq 0$ and $\int_{\partial X} v_n \sigma = 1$. We claim that $v_{n_k}\sigma$ converges to the Dirac measure δ_b as $k \to \infty$ in the weak-star topology.

We need to show that if $U \subset \partial X$ is open and $b \notin \bar{U}$, then $\int_U v_{n_k}\sigma \to 0$, i.e.,

$$\frac{\int_U u \circ \gamma_{n_k} \sigma}{\int_{\partial X} u \circ \gamma_{n_k} \sigma} \to 0 \qquad \text{as } k \to \infty.$$

Since $|\alpha(n) - \gamma_n(o)|$ is bounded (see the proof of lemma 3.2), we have

$$\int_{\partial X} u \circ \gamma_n \sigma = H[u \circ \gamma_n](o) = H[u](\gamma_n(o)) \geq cH[u](\alpha(n)) \geq c\zeta^{-n}.$$

for some $\zeta < \eta$. There is $c > 0$ such that

$$(\gamma_{n_k}(x)|a) \geq n_k - c \qquad \text{for all } x \in \bar{U}.$$

Hence, $\gamma_{n_k}(U) \subset V$ for k large enough, and

$$\int_U u \circ \gamma_{n_k} \sigma \leq c \int_U \eta^{-(\gamma_{n_k}(x)|a)} \sigma(x) \leq c\eta^{-n_k},$$

so

$$\frac{\int_U u \circ \gamma_{n_k} \sigma}{\int_{\partial X} u \circ \gamma_{n_k} \sigma} \leq c \left(\frac{\eta}{\zeta}\right)^{-n_k} \to 0 \qquad \text{as } k \to \infty.$$

We have shown that the weak-star closure $\bar{E}$ of the Γ-invariant subspace E spanned by $u\sigma$ in $M(\partial X)$ contains the Dirac measure δ_b. By lemma 3.3, $\bar{E}$ is Γ-invariant. For $\gamma \in \Gamma$,

$$\gamma^*(\delta_b) = k_b(\gamma(o))\delta_{\gamma^{-1}(b)}$$

by (3.2). Hence $\bar{E}$ contains all the Dirac measures $\delta_{\gamma(b)}$, $\gamma \in \Gamma$. Since each Γ-orbit in ∂X is dense by proposition 2.2, this implies that E is weak-star dense in $M(\partial X)$. $\square$

What does the condition that the boundary decay be faster than the radial decay mean when X is the unit disc $\mathbb{D}$ with boundary $\mathbb{T}$? We have $H[u](z) \geq c(1 - |z|)$ by the Hopf lemma or by Harnack's inequality at the boundary, so as $z \to a$ radially, $H[u](z) \geq c|z - a|$. Here, $|\cdot|$ denotes the ordinary absolute value. The condition means that there is $\epsilon > 0$ with $u(z) \leq c|z - a|^{1+\epsilon}$ for $z \in \mathbb{T}$ close to a. In this particular situation it may in fact be shown that the conclusion of the theorem holds if u is only assumed to be differentiable at a.

Let us view this from a representation-theoretic perspective. We are considering a representation of $\mathrm{Aut}(\mathbb{D})$ or $\mathrm{SL}(2, \mathbb{R})$ on $M(\mathbb{T})$ with the weak-star topology, restricted to a cocompact lattice Γ. This representation is different from the well-known unitary representations of $\mathrm{SL}(2,\mathbb{R})$. As usual, one is interested in invariant subspaces. Let us start by dividing out by the obvious Γ-invariant subspace of constant multiples of the harmonic measure (which is just the normalized Lebesgue measure on $\mathbb{T}$), and call the quotient W. It may well happen that the representation of Γ on W is reducible. If there is a normal subgroup H in Γ such that the intermediate covering $Y = \mathbb{D}/H$ has a non-trivial Martin boundary Δ, which is for instance the case if Γ/H is non-amenable, then the pullback of $M(\Delta)$ gives a non-trivial weak-star closed Γ-invariant subspace of W. What we have claimed, however, is that a proper weak-star closed Γ-invariant subspace of W cannot contain $u\sigma$ for any non-constant differentiable function u on $\mathbb{T}$.

4. Hardy classes on covering surfaces.

This section contains our results on the Hardy classes of Gromov covering spaces of compact Riemann surfaces. We want to construct H^p functions on such spaces. We take the point of view that h^p functions are easy to obtain as Poisson integrals of L^p functions on the boundary. The problem we wish to address is to decide when such functions are real parts of holomorphic functions. Our main result, theorem 4.2, states that *all* h^1 functions are real parts of holomorphic functions unless the boundary decay of the real part of an H^1 function at a boundary minimum is never faster than the radial decay. This latter condition is characteristic of the higher dimensional case.

We will prove the theorem for all dimensions, although it is an interesting dichotomy only for Riemann surfaces. In higher dimensions, harmonic functions are of course not pluriharmonic in general. Then the theorem gives a necessary condition for a boundary function to extend to the real part of a holomorphic function. Such a restriction is to be expected. For example, consider the open unit ball B_n in $\mathbb{C}^n$, $n \geq 2$, and let u be a pluriharmonic function on B_n. If L is a complex line, then u is harmonic on $B_n \cap L$, which is a disc in L, and the maximum value in the disc is taken on the boundary. Hence, the interior decay of u at a boundary minimum controls its boundary decay.

Let X be a non-parabolic Kähler manifold with Martin boundary Δ. The Kähler condition implies that real parts of holomorphic functions are harmonic. Let $u \in h^1(X)$ be represented by a measure μ on Δ_1. We know that u is the real part of a holomorphic function if and only if the (1,0)-derivative ∂u is d-exact. This happens if and only if all the periods of u are zero. We will investigate the periods by representing them on the boundary.

Let α be a loop in X. The period of u over α is the real number

$$i \int_\alpha \partial u.$$

Using the integral representation of u, we get

$$\begin{aligned} i\int_\alpha \partial u &= i\int_\alpha \partial_x\big(\int_\Delta k_y(x)\mu(y)\big) \\ &= \int_\Delta \big(i\int_\alpha \partial_x k_y(x)\big)\mu(y) \\ &= \int_\Delta h_\alpha \mu, \end{aligned}$$

where

$$h_\alpha(y) = i\int_\alpha \partial k_y, \qquad y \in \Delta.$$

Using Harnack's inequality and the continuity of the function $(x, y) \mapsto k_y(x)$ on $X \times \Delta$, it is easy to show that we can differentiate under the boundary integral and then interchange the two integrals. Also, the function h_α is continuous, so the period functional of α is weak-star continuous on $M(\Delta_1)$.

We have proved the following lemma.

4.1. Lemma. *Let X be a non-parabolic Kähler manifold. The subspace of real parts of holomorphic functions is weak-star closed in $M(\Delta_1)$.*

Now we state our main theorem. Recall from section 2 that $u \in h^1(X)$ is called quasi-bounded if its boundary measure is absolutely continuous with respect to the harmonic measure. Then $u = H[\hat{u}]$, where $\hat{u} \in L^1(\Delta)$ is the fine boundary function of u.

4.2. Theorem. *Let X be a Galois covering space of a compact Kähler manifold with a non-elementary hyperbolic covering group. Then either:*

(1) *every h^1 function on X is the real part of a holomorphic function, or*
(2) *if a quasi-bounded harmonic function $u \geq 0$ on X is the real part of a holomorphic function, then the boundary decay of u at a zero in ∂X is no faster than its radial decay.*

In case (1), if X is a Riemann surface, then X is H^p-convex for each $p < \infty$.

We may rephrase the dichotomy as follows. If a non-constant integrable function on ∂X with a sufficiently flat minimum extends to the real part of a holomorphic function on X, then every measure on ∂X does.

Proof. Let $u = H[\hat{u}]$ be a quasi-bounded harmonic function on X which is the real part of a holomorphic function. Let E be the Γ-invariant subspace spanned by $\hat{u}\sigma$ in $M(\partial X)$. Then $H[\mu]$ is the real part of a holomorphic function for every $\mu \in E$. By lemma 4.1, $H[\mu]$ is the real part of a holomorphic function for every μ in the weak-star closure of E. If $u \geq 0$ and the boundary decay of u at a zero in ∂X is faster than its radial decay, then the weak-star closure of E is all of $M(\partial X)$ by theorem 3.4.

Now suppose that (1) holds and $\dim X = 1$. Let $1 < p < \infty$ and $z \in \partial X$. By lemma 3.1, $\sigma(z) = 0$, so it is easy to construct a continuous L^p function $v : \partial X \to [0, \infty]$ with $v(z) = \infty$. Then $H[v](x) \to \infty$ as $x \to z$ in X. Namely, given $M > 0$, find a finite-valued continuous function $w < v$ on ∂X such that $w(z) > M$. Since the boundary is regular, $H[w](x) \to w(z)$ as $x \to z$ in X. Hence, $H[v](x) > H[w](x) > M$ if $x \in X$ is sufficiently close to z.

By (1), there is a holomorphic function f on X with $H[v] = \operatorname{Re} f$. We need to show that $f \in H^p(X)$ or that $w = \operatorname{Im} f \in h^p(X)$. Let $\psi : \mathbb{D} \to X$ be the universal covering. Since $H[v] \in h^p(X)$, we have $H[v] \circ \psi \in h^p(\mathbb{D})$. By M. Riesz' theorem, the harmonic conjugate $w \circ \psi$ of $H[v] \circ \psi$ is also in $h^p(\mathbb{D})$. Since $|w \circ \psi|^p$ is invariant under the covering group of ψ, so is its least harmonic majorant. Hence, $|w|^p$ has a harmonic majorant on X. $\square$

The remainder of this section will be devoted to a discussion of the one-dimensional case.

The disc is of course an example of a Riemann surface X with $h^1(X) \subset \operatorname{Re} \mathcal{O}(X)$. It is in fact the only surface X of finite topological type with non-constant h^1 functions such that $h^\infty(X) \subset \operatorname{Re} \mathcal{O}(X)$. Namely, suppose X has finite topological type. Then X is isomorphic to a compact surface with a finite number of points and discs removed. Assume that at least one disc has been removed, for otherwise X has no non-constant positive harmonic functions. Consider the surface Y obtained by removing a slightly smaller disc. It is homeomorphic to X. If X is not the disc, then X and Y are not simply connected, so there is a harmonic function u on Y which is not the real part of a holomorphic function. Then neither is $u|X$, and $u|X$ is bounded.

Let us describe a method for constructing infinitely connected examples of Riemann surfaces X with $h^1(X) \subset \operatorname{Re} \mathcal{O}(X)$. Suppose we have a non-constant holomorphic map $C \to S$ between compact Riemann surfaces of genus at least 2. Pull back the universal covering $\mathbb{D} \to S$ to a covering $X \to C$. It is easy to show that X is connected if and only if the induced morphism $\pi_1(C) \to \pi_1(S)$ is surjective. We assume this is the case. Then X is Gromov in the Poincaré metric (or any other metric pulled back from C), because the covering group $\pi_1(S)$ is hyperbolic. The induced map $\phi : X \to \mathbb{D}$ is a surjective quasi-isometry, so it extends to a homeomorphism $\partial X \to \mathbb{T}$ of the boundaries, which preserves the Hölder structure defined by the visual metrics. This means that

$$\tfrac{1}{c}(x|y) - c' \le (\phi(x)|\phi(y)) \le c(x|y) + c', \qquad x, y \in \partial X.$$

Fix $a \in \partial X$. Given any $\theta > 1$, there is $g \in H^1(\mathbb{D})$ such that the boundary function $\hat{v}$ of $v = \operatorname{Re} g$ satisfies $\hat{v} \ge 0$, $\hat{v}(\phi(a)) = 0$ and

$$\hat{v}(z) \le \theta^{-(z|\phi(a))}, \qquad z \in \mathbb{T}.$$

We can simply define $\hat{v}(z)$ to equal $\theta^{-(z|\phi(a))}$. Let $f = g \circ \phi \in H^1(X)$ and $u = \operatorname{Re} f = v \circ \phi$. Then $\hat{u} \ge 0$, $\hat{u}(a) = 0$ and

$$\hat{u}(x) = \hat{v}(\phi(x)) \le \theta^{-(\phi(x)|\phi(a))} \le \theta^{c'} \theta^{-(x|a)/c}, \qquad x \in \partial X.$$

By choosing θ large enough, we can get

$$\hat{u}(x) \leq c\eta^{-(x|a)}, \qquad x \in \partial X,$$

with η arbitrarily large. This shows that X violates (2) and thus satisfies (1) in theorem 4.2.

The map $\phi : X \to \mathbb{D}$ is proper and its fibres are uniformly bounded in the Poincaré metric on X. These properties alone imply that $h^1(X) \subset \operatorname{Re}\mathcal{O}(X)$ and X is H^p-convex, as shown by the following result.

4.3. Proposition. *Let X and Y be Riemann surfaces and $\phi : X \to Y$ be a proper holomorphic map with uniformly bounded fibres in the Poincaré metric on X. Then ϕ induces a homeomorphism $\Delta_1^X \to \Delta_1^Y$ between the minimal Martin boundaries of X and Y, and*

$$h^1(X) = h^1(Y) \circ \phi.$$

Hence,

$$h^1(X) \subset \operatorname{Re}\mathcal{O}(X) \qquad \textit{iff} \qquad h^1(Y) \subset \operatorname{Re}\mathcal{O}(Y),$$

and for $1 \leq p < \infty$,

$$X \textit{ is } H^p\textit{-convex} \qquad \textit{iff} \qquad Y \textit{ is } H^p\textit{-convex.}$$

Here, the Martin boundaries are defined with respect to base points $o_X \in X$ and $o_Y \in Y$ such that $\phi(o_X) = o_Y$.

Proof. The map ϕ is a finite branched covering. Let k be the number of sheets of ϕ. For a harmonic function u on X, let

$$(\phi_* u)(y) = \frac{1}{k} \sum_{\phi(x)=y} u(x)$$

for $y \in Y$ outside the branch locus of ϕ. Extended across the branch points, $\phi_* u$ is a harmonic function on Y. For a function v on Y, let $\phi^* v = v \circ \phi$. Then $\phi_*\phi^* v = v$ and $\phi^*\phi_* u$ is the average of u over fibres.

Now let $u > 0$ be harmonic on X. Then $\tilde{u} = \phi^*\phi_* u > 0$ is also harmonic. Since the fibres of ϕ are uniformly bounded, by Harnack's inequality there is a constant $c > 0$ such that

$$\tilde{u} \leq cu.$$

If u is minimal, this implies that $\tilde{u}$ and u are proportional, so $u = \tilde{u} = (\phi_* u) \circ \phi$. It is easy to verify that $\phi_* u$ is minimal on Y.

If $u \in h^1(X)$, then there is a measure μ on Δ_1^X such that

$$u(x) = \int_{\Delta_1^X} k_y(x)\mu(y), \qquad x \in X.$$

We have seen that the functions k_y, $y \in \Delta_1^X$, are constant on fibres, so u is too. Therefore, $h^1(X) = h^1(Y) \circ \phi$.

Now we can easily show that if v is a minimal positive harmonic function on Y, then so is $\phi^* v$ on X. Hence, the map $\phi_* : \Delta_1^X \to \Delta_1^Y$ has inverse ϕ^*, and these maps are clearly continuous in the topology of locally uniform convergence. □

The proof shows that the proposition holds for any metric on X with a uniform Harnack inequality, for instance a metric that pulls up to a metric with bounded geometry on the disc. The Poincaré metric is such a metric. More generally, a uniform Harnack inequality holds for any complete Riemannian metric with Ricci curvature bounded below [Yau]. (It is actually an infinitesimal version of the Harnack inequality that is proved in [Yau].)

The squaring map $z \mapsto z^2$ on the unit disc shows that a proper holomorphic map may not have uniformly bounded fibres.

Let $\phi : X \to \mathbb{D}$ be a proper holomorphic map with uniformly bounded fibres. Proposition 4.3 implies that points in X in the same ϕ-fibre cannot be separated by h^1 functions, and hence not by bounded holomorphic functions either, whereas points in different ϕ-fibres obviously can. Similar examples can be constructed using the extension theorem in [Lár] described in the introduction. Let S be a compact Riemann surface of genus at least 2. Consider a sufficiently ample curve C in the surface $S \times S$ and let X be its pullback in the covering space $\mathbb{D} \times S$. Then points in X in the same slice $\{z\} \times S$, $z \in \mathbb{D}$, cannot be separated by a bounded holomorphic function. Such a function would extend to a holomorphic function on $\mathbb{D} \times S$ and be constant on the slice. Somehow, the Hardy theory of X detects compact subvarieties in the ambient space.

All the surfaces X with $h^1(X) \subset \operatorname{Re} \mathcal{O}(X)$ obtained from the disc using proposition 4.3 have one-dimensional minimal Martin boundary; the boundary is just the circle. I do not know if examples with higher dimensional boundary exist.

An attractive case to consider would be a surface X in the ball B_n in $\mathbb{C}^n$ produced by pulling up an ample curve from a compact quotient of the ball. The inclusion $X \hookrightarrow B_n$ is a cobounded quasi-isometry between Gromov spaces, so the boundary of X is the sphere $\partial B_n = S^{2n-1}$. I do not know if $h^1(X) \subset \operatorname{Re} \mathcal{O}(X)$. A first step towards a solution would be to determine if X has a bounded holomorphic function which is not the restriction of a bounded holomorphic function on the ball.

It is not hard, though, to show that X is H^p-convex for any $p < \infty$, so we do have examples of H^p-convex surfaces with boundaries of arbitrarily high dimension.

5. Parreau-Widom covering surfaces.

In this section, we recall the six homeomorphism classes of non-compact covering spaces of compact Riemann surfaces. (As before, all coverings are assumed to be Galois.)

We show that five of these classes do not contain any Parreau-Widom surfaces other than the disc. One of these five classes contains the covering surfaces that are of primary interest from the point of view of the Shafarevich conjecture. This shows that previously existing Hardy theory has little bearing on our investigation. We are unable to determine if there are any Parreau-Widom surfaces in the sixth class, but we do give examples from this class of surfaces with no non-constant bounded holomorphic functions.

Let us first recall that a non-compact orientable surface is determined up to homeomorphism by the following topological invariants.

The genus (the number of handles).

The space of ends, which is a totally disconnected compact Hausdorff space.

The open subset of planar ends.

This is a result of Kerékjártó. For a proof, see [Ric]. Every orientable surface is homeomorphic to the complement of a closed totally disconnected subset E of the sphere S^2 with a countable number of handles attached, such that only a finite number of handles are attached to any compact subset of $S^2 \setminus E$ [Ric].

Now let X be a non-compact covering space of a compact Riemann surface. In [Gri], Kerékjártó's classification is used to determine the possible topological types of X. We will sketch the argument. First, X has one or two ends or its space of ends is homeomorphic to the Cantor set [Hop]. If X has non-zero genus, then by moving a handle all over X by covering transformations we see that X has infinite genus and all its ends are non-planar. Therefore X is homeomorphic to exactly one of the following six surfaces.

(1) The plane, so X is isomorphic to $\mathbb{C}$ or $\mathbb{D}$.
(2) The punctured plane, so X is isomorphic to $\mathbb{C} \setminus \{0\}$.
(3) The sphere with a Cantor set removed.
(4) An orientable surface with a single non-planar end.
(5) An orientable surface with two non-planar ends.
(6) An orientable surface with a Cantor set of non-planar ends.

We mention in passing that all six topological types are represented by coverings with a hyperbolic covering group.

As we explained in the introduction, we are primarily interested in non-compact covering surfaces X obtained by pulling up a sufficiently ample curve C in a projective manifold M to the universal covering space $\tilde{M}$. Since $\tilde{M}$ has one end [ABR], so does X. In general, the epimorphism $\pi_1(C) \to \pi_1(M)$ has a large kernel, so $X \to C$ is not the universal covering and X is of type (4). Theorem 5.1 below shows that a covering surface of type (4) is never Parreau-Widom. The class of covering surfaces of type (4) is quite vast. It contains a continuum of different quasi-conformal types [Gri]. The function theory of this class is uncharted territory where many interesting problems, such as the one posed at the end of the previous section, await solution.

Let us recall that a non-parabolic Riemann surface X with a Green function G_o is called Parreau-Widom if

$$\int_0^\infty b(t)\,dt < \infty,$$

where $b(t)$ is the first Betti number of the set $\{x \in X : G_o(x) > t\}$. We interpret this to mean that the topology of X grows slowly as measured by the Green function. If X has finite topological type, then X is obviously Parreau-Widom, but one is of course mainly interested in the infinitely connected case. If G_o vanishes at infinity, then X is Parreau-Widom if and only if $\sum G_o(z) < \infty$, where the sum is taken over all the critical points z of G_o, counted with multiplicities. Parreau-Widom surfaces have been studied extensively by a number of authors. They are the only infinitely connected surfaces for which Hardy theory has been developed to any extent. They have a wealth of bounded holomorphic functions. In particular, such functions separate points and directions. We refer the reader to [Has] for an exposition.

5.1. Theorem. *Let X be a Galois covering space of a compact Riemann surface which is not of topological type* (3). *If X is Parreau-Widom, then X is the disc.*

I expect, but cannot prove, that a covering surface of type (3) is never Parreau-Widom.

Proof. Assume that X is non-parabolic of infinite genus. We know that the Green function G_o with pole o vanishes at infinity. We will show that X is not Parreau-Widom.

Find a regular value s of G_o such that the smoothly bounded subsurface $S = \{G \geq s\}$ has non-zero genus. By Harnack's inequality, there is $a < s$ with $0 < a < 1$ such that

$$a \max_K G_o < \min_K G_o$$

for any compact set K in $X \setminus S$ with diameter less than that of S. Let

$$A_n = \{a^n > G_o > a^{n+2}\}, \qquad n \geq 1.$$

Let $S_\nu = \gamma_\nu(S)$, $\nu \geq 1$, with γ_ν in the covering group Γ, be mutually disjoint in $\{G_o < a\}$, such that $X = \bigcup \gamma_\nu(S')$ for some compact $S' \subset X$ containing S. Then each S_ν is contained in some A_n, and each point in $\{G_o < a\}$ lies in A_n for either one or two values of n. For convenience we will assume that all a^n, $n \geq 1$, are regular values of G_o. If a^n is critical, we can replace it by a nearby regular value without affecting the argument.

Now the (1,0)-derivative ∂G_o extends to a holomorphic 1-form η on the double A_n' of A_n. For the definition of the double of a bordered Riemann surface, see [AS] or [FK]. Let g be the genus of A_n' and k_n be the number of critical points of G_o in A_n, counted with multiplicities. By the Riemann-Roch theorem,

$$2k_n = \deg(\eta) = 2g - 2.$$

Also, the first Betti number $b_1(A_n)$ of A_n is g, so

$$k_n = b_1(A_n) - 1.$$

Each $S_\nu \subset A_n$ contains a handle. Using Mayer-Vietoris sequences, we see that these handles give independent cycles in the homology of A_n. Therefore we obtain the rough inequality

$$k_n \geq \#\{\nu : S_\nu \subset A_n\}.$$

Hence, we can form a sequence (z_ν) of critical points of G_o such that z_ν and S_ν belong to the same A_n, and the number of times each critical point occurs in the sequence is at most twice its multiplicity.

For each ν, choose a point x_ν in S_ν. Then the sum $\sum G_o(z)$ taken over all critical points of G_o, counted with multiplicity, is at least

$$\tfrac{1}{2}\sum_\nu G_o(z_\nu) \geq \tfrac{1}{2}a^2\sum_\nu G_o(x_\nu).$$

Choose $p \in X \setminus \Gamma o$. Let m be the number of $\gamma \in \Gamma$ with $\gamma(p) \in S'$. By Harnack's inequality, there is $c > 0$ such that

$$G_o(\gamma(p)) \leq c\, G_o(x_\nu)$$

if $\gamma(p) \in \gamma_\nu(S')$. Hence,

$$\tfrac{1}{2}a^2\sum_\nu G_o(x_\nu) \geq \frac{a^2}{2mc}\sum_{\gamma\in\Gamma} G_o(\gamma(p)),$$

so if X is Parreau-Widom, then

$$\sum_{\gamma\in\Gamma} G_o(\gamma(p)) < \infty.$$

But then $\sum_\gamma G_o \circ \gamma$ descends to a non-constant superharmonic function on the compact quotient surface X/Γ, which is absurd. □

To conclude this section, we present examples of covering surfaces of type (3) with a non-elementary hyperbolic covering group but no non-constant bounded holomorphic functions. Our reference for the following paragraph is [Mas].

Let D be a domain in the Riemann sphere, bounded by $2g$, $g \geq 1$, mutually disjoint simple closed curves $C_1, C_1', \ldots, C_g, C_g'$. For $i = 1, \ldots, g$, let γ_i be a Möbius transformation with $\gamma_i(C_i) = C_i'$ and $\gamma_i(D) \cap D = \varnothing$. The Schottky group Γ generated by $\gamma_1, \ldots, \gamma_g$

is free on these generators, and hence hyperbolic. It is a Kleinian group with fundamental domain D and region of discontinuity

$$\Omega = \bigcup_{\gamma \in \Gamma} \gamma(D \cup C_1 \cup \cdots \cup C_g).$$

Furthermore, Ω is connected, and the quotient Ω/Γ is a compact Riemann surface of genus g. It is a classical theorem that every compact Riemann surface of non-zero genus can be represented by a Schottky group.

Assume that $g \geq 2$. Then Ω is clearly of type (3). Since Γ is non-amenable, Ω has a non-constant bounded harmonic function [LS]. This may also be proved by showing that the complement of Ω has positive logarithmic capacity. By theorem 6.1 below, Ω actually has an infinite dimensional space of bounded harmonic functions. However, $\Omega \in O_{AD}$ [AS]. Moreover, in many cases, $E = \mathbb{C} \setminus \Omega$ has linear measure zero, which implies that Ω has no non-constant bounded holomorphic functions [AS].

Let us consider the classical case where $C_1, \ldots, C_g'$ are circles in $\mathbb{C}$ and D is unbounded. Schottky himself proved in [Sch] that if there are mutually disjoint circles in D that divide D into triply connected regions, then the limit set E has linear measure zero. This holds for instance if $C_1, \ldots, C_g'$ are all centred on the same line. Also, E has linear measure zero if the distance between any two of the circles $C_1, \ldots, C_g'$ is sufficiently large compared to their radii [Aka1, Aka2].

There are examples of classical Schottky coverings for which E has non-zero linear measure [Aka2]. I do not know if they lie in O_{AB}. It would be interesting to know which Schottky coverings lie in O_{AB}.

6. A theorem of Kifer and Toledo.

In 1988, Kifer [Kif] and Toledo [Tol] independently published proofs of the following theorem.

6.1. Theorem (Kifer, Toledo). *Let X be a Galois covering space of a compact Riemannian manifold. If X has a non-constant bounded harmonic function, then the space of bounded harmonic functions on X is infinite dimensional.*

This implies that if the covering group Γ is non-amenable, then X has an infinite dimensional space of bounded harmonic functions [LS].

It is possible to give a much simpler proof of theorem 6.1 using the Martin boundary Δ of X with the harmonic measure. From this viewpoint, the theorem states that if Δ is not an atom, then $L^\infty(\Delta)$ is infinite dimensional.

So suppose that Δ is not an atom, but that Δ contains an atom a, for if Δ has no atoms, the conclusion is clear. An atom in Δ is the union of a nullset and a point with positive mass, so we may take a to be a point in Δ. If the Γ-orbit of a is infinite, then the characteristic functions of $\{\gamma(a)\}$, $\gamma \in \Gamma$, generate an infinite dimensional subspace in $L^\infty(\Delta)$.

If the Γ-orbit of a is finite, then the characteristic function of $\{a\}$ is fixed by a normal subgroup H of finite index in Γ, so its harmonic extension descends to a non-constant bounded harmonic function on the quotient X/H. But this is absurd since X/H is compact.

Our argument actually shows that theorem 6.1 holds if X/Γ is only assumed to be parabolic. Namely, if X/Γ is parabolic and H is a normal subgroup of finite index in Γ, then X/H is also parabolic, so X/H has no non-constant bounded harmonic functions.

Toledo tells me that Sullivan asked him if an analogous theorem holds for positive harmonic functions. Our method shows that it does.

6.2. Theorem. *Let X be a Galois covering space of a compact Riemannian manifold. If X has a non-constant positive harmonic function, then $h^1(X)$ is infinite dimensional.*

Proof. Let Δ be the Martin boundary and Δ_1 be the minimal Martin boundary of X. The covering group Γ preserves Δ_1. In fact, for every isometry ϕ of X,

$$k_y \circ \phi = k_y(\phi(o))\, k_{\phi^{-1}(y)}, \qquad y \in \Delta, \tag{6.1}$$

where $o \in X$ is the fixed base point. This shows that if k_y is minimal, then so is $k_{\phi(y)}$.

The theorem states that $M(\Delta_1)$ is infinite dimensional if Δ_1 is not a point, which means that there is $p \in \Delta_1$ such that $H[\delta_p] = k_p$ is not constant. If Δ_1 is infinite, then $M(\Delta_1)$ is clearly infinite dimensional. So suppose Δ_1 is finite. Then p is fixed by a normal subgroup Γ_p of finite index in Γ. For $\gamma \in \Gamma_p$,

$$k_p \circ \gamma = k_p(\gamma o)\, k_p$$

by (6.1). Hence,

$$k_p(\gamma_1\gamma_2 o) = k_p(\gamma_1 o)\, k_p(\gamma_2 o), \qquad \gamma_1, \gamma_2 \in \Gamma_p,$$

so the map

$$\Gamma_p \to \mathbb{R}_+, \quad \gamma \mapsto k_p(\gamma o),$$

is a group homomorphism. The kernel is a normal subgroup H of Γ_p such that the quotient Γ_p/H is abelian.

Now the non-constant positive harmonic function k_p descends to X/H, but this is absurd since X/H is an abelian covering of the compact manifold X/Γ_p [LS]. $\square$

It is not clear if it suffices to assume that X/Γ, and hence X/Γ_p, is parabolic. The sphere with 4 points removed is parabolic, but has a $\mathbb{Z}^2$-covering with a non-constant positive harmonic function [LM].

Added in May 1995. Vadim Kaimanovich has pointed out to me that if the Martin boundary of a Galois covering space of a compact Riemannian manifold is not an atom, then it contains no atoms. This follows from his work in [Kai1]; see also [Kai2]. Theorem 6.1 is an immediate corollary.

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