

PLURISUBHARMONICITY OF ENVELOPES OF DISC FUNCTIONALS ON MANIFOLDS

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ABSTRACT. We show that a disc functional on a complex manifold has a plurisubharmonic envelope if all its pullbacks by holomorphic submersions from domains of holomorphy in affine space do and it is locally bounded above and upper semicontinuous in a certain weak sense. For naturally defined classes of disc functionals on manifolds, this result reduces a property somewhat stronger than having a plurisubharmonic envelope to the affine case. The proof uses a recent Stein neighbourhood construction of Rosay, who proved the plurisubharmonicity of the Poisson envelope on all manifolds. As a consequence, the Riesz envelope and the Lelong envelope are plurisubharmonic on all manifolds; for the former, we make use of new work of Edigarian. The basic theory of the three main classes of disc functionals is thereby extended to all manifolds.

Introduction

The theory of disc functionals was founded by Evgeny A. Poletsky in the late 1980s with the papers [P1, PS]. In its applications in pluripotential theory and elsewhere in complex analysis, we rarely consider a single disc functional in isolation. Rather, we normally have a class of disc functionals of a given type on each complex manifold, preserved by taking pullbacks by holomorphic maps. The question of plurisubharmonicity of envelopes is fundamental. An analytic disc can always be lifted by a suitable holomorphic submersion from a domain of holomorphy in affine space, and, with varying degrees of effort, so can certain configurations of discs. This suggests the possibility of reducing plurisubharmonicity of envelopes of disc functionals of a given type to the affine case, where the problem may have been studied already.

The main theorem of this paper states that a property somewhat stronger than having a plurisubharmonic envelope reduces to the affine case, meaning that a disc functional has this property if all its pullbacks by holomorphic submersions from domains of holomorphy

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in affine space do. We do not know if the property of having a plurisubharmonic envelope by itself reduces to the affine case. Still, as a consequence, plurisubharmonicity of envelopes is established for the three main classes of disc functionals, the Poisson, Riesz, and Lelong functionals, on all manifolds.

Thereby, an extra hypothesis is removed from various theorems based on the theory of disc functionals, such as the product property of relative extremal functions [EP], the product property of generalized Green functions [LS2], and results in Chapters 6 and 7 of [LS1]. For instance, our *Kontinuitätssatz*, characterizing pseudoconvexity in terms of analytic discs, is valid for all manifolds.

In our paper [LS1], we identified the construction of a certain Stein neighbourhood as the obstacle to extending the theory of disc functionals to all manifolds. The long-awaited breakthrough came with Rosay's paper [Rs] in April of 2001, in which the Poisson envelope was shown to be plurisubharmonic on all manifolds. We thank Armen Edigarian for sharing with us a draft of a simplified version of Rosay's argument [E2], on which the proof of Theorem 1.2 is partly based.

For background information on the Poisson, Riesz, and Lelong functionals and their envelopes, we refer the reader to [LS1, LS2].

Some notation: We denote by D_r the open disc $\{z \in \mathbb{C}; |z| < r\}$, $r > 0$, by $\overline{\mathbb{R}}$ the extended real line $[-\infty, +\infty]$, and by λ the *normalized* arc length measure on the unit circle \mathbb{T} (note that in our previous papers, we did not normalize λ).

1. Reduction to the affine case

Before stating and proving our reduction theorem, we recall a few definitions. Let X be a complex manifold and \mathcal{A}_X be the set of maps $f : \overline{\mathbb{D}} \rightarrow X$ which are holomorphic in a neighbourhood of the closure $\overline{\mathbb{D}}$ of the unit disc \mathbb{D} . Such maps are called (*closed*) *analytic discs* in X . For a point $p \in X$, we denote the constant disc at p by p . A *disc functional* on X is a map $H : \mathcal{A}_X \rightarrow \overline{\mathbb{R}}$. The *envelope* of H is the function $EH : X \rightarrow \overline{\mathbb{R}}$ defined by the formula

$$EH(x) = \inf \{H(f); f \in \mathcal{A}_X, f(0) = x\}, \quad x \in X.$$

If $\varphi : Y \rightarrow X$ is a holomorphic map, then the pullback φ^*H is the disc functional on Y defined by the formula $f \mapsto H(\varphi \circ f)$. Clearly, $EH \circ \varphi \leq E\varphi^*H$.

1.1. Lemma. *Let H be a disc functional on an n -dimensional complex manifold X such that the envelope $E\varphi^*H$ is upper semicontinuous for every holomorphic submersion φ from the $(n+1)$ -dimensional polydisc into X . Then EH is upper semicontinuous.*

Proof. We first observe that EH is nowhere $+\infty$. Namely, for $p \in X$, choose a coordinate polydisc U centred at p and let φ be the projection $U \times \mathbb{D} \rightarrow U$. Since $E\varphi^*H$ is upper semicontinuous by hypothesis, $E\varphi^*H(p, 0) < +\infty$, so there is an analytic disc f in $U \times \mathbb{D}$ with $f(0) = (p, 0)$ and $\varphi^*H(f) < +\infty$. Then $\varphi \circ f$ is an analytic disc in $U \subset X$ with $0 \mapsto p$, and $EH(p) \leq H(\varphi \circ f) = \varphi^*H(f) < +\infty$.

Now let $p \in X$ and take $a > EH(p)$. We need to find a neighbourhood U of p such that $EH < a$ on U . By definition of EH , there exists an analytic disc $f_0 : D_r \rightarrow X$, $r > 1$, such that $f_0(0) = p$ and $H(f_0) < a$. In the proof of Lemma 2.3 in [LS1], using the theorem of Siu on the existence of Stein neighbourhoods of Stein subvarieties, we showed that for each $t \in (1, r)$, there exists a biholomorphism Ψ from a neighbourhood of the graph $\{(z, f_0(z)); z \in D_t\}$ in $D_t \times X$ onto D_t^{n+1} , such that $\Psi(z, f_0(z)) = (z, 0)$ for all $z \in D_t$. We note that this also follows from an older result of Royden [Ry].

Consider the holomorphic submersion $\varphi = \text{pr} \circ \Psi^{-1} : D_t^{n+1} \rightarrow X$, where $\text{pr} : \mathbb{C} \times X \rightarrow X$ is the projection. The disc $\tilde{f}_0 : D_t \rightarrow D_t^{n+1}$, $z \mapsto (z, 0)$, is a lifting of f_0 by φ , so $E\varphi^*H(0) \leq \varphi^*H(\tilde{f}_0) = H(f_0) < a$. By assumption, there exists a neighbourhood W of 0 in D_t^{n+1} such that $E\varphi^*H < a$ on W . Since φ is a submersion, $U = \varphi(W)$ is an open neighbourhood of p in X . If $q \in U$, then there is $\tilde{q} \in W$ and $\tilde{f} \in \mathcal{A}_V$ with $\varphi(\tilde{q}) = q$, $\tilde{f}(0) = \tilde{q}$, and $\varphi^*H(\tilde{f}) < a$. Then $f = \varphi \circ \tilde{f} \in \mathcal{A}_X$, $f(0) = q$, and $EH(q) \leq H(f) = \varphi^*H(\tilde{f}) < a$. \square

The following theorem is the main result of the paper.

1.2. Reduction Theorem. *A disc functional H on a complex manifold X has a plurisubharmonic envelope if it satisfies the following three conditions.*

- (1) *The envelope $E\varphi^*H$ is plurisubharmonic for every holomorphic submersion φ from a domain of holomorphy in affine space into X .*
- (2) *There is an open cover of X by subsets U with a pluripolar subset $Z \subset U$ such that for every $h \in \mathcal{A}_U$ with $h(\overline{\mathbb{D}}) \not\subset Z$, the function $w \mapsto H(h(w))$ is dominated by an integrable function on \mathbb{T} .*
- (3) *If $h \in \mathcal{A}_X$, $w \in \mathbb{T}$, and $\varepsilon > 0$, then w has a neighbourhood U in \mathbb{C} such that for every sufficiently small closed arc J in \mathbb{T} containing w there is a holomorphic map $F : D_r \times U \rightarrow X$, $r > 1$, such that $F(0, \cdot) = h|_U$ and*

$$\frac{1}{\lambda(J)} \int_J H(F(\cdot, t)) d\lambda(t) \leq EH(h(w)) + \varepsilon.$$

We view (3) as a weak upper semicontinuity condition on H . Such conditions, subtle and complicated as they are, appear naturally and inevitably in the theory of disc functionals. In the absence of any measurability assumptions on H , the integral on the left-hand side of the inequality is a lower integral, i.e., the supremum of the integrals of all integrable Borel functions dominated by the integrand. The following remark is trivial, but worth stating for the record.

1.3. Remark. *A disc functional H on a complex manifold X satisfies hypothesis (3) above if for each $f \in \mathcal{A}_X$ and $\beta > H(f)$ there exists a neighbourhood V of $p = f(0)$ and a holomorphic map $F : D_r \times V \rightarrow X$, $r > 1$, such that $F(\cdot, p) = f$, $F(0, x) = x$, and $H(F(\cdot, x)) < \beta$ for all $x \in V$.*

By Lemma 2.3 in [LS1], the Poisson functional for an upper semicontinuous function and the Lelong functional satisfy the condition in the remark, but we need the weaker condition (3) to capture the Poisson functional for a plurisuperharmonic function and thereby the Riesz functional.

We view (2) as a mild and easily-verifiable boundedness condition. Note that (2) holds if H is bounded above on the set of analytic discs with image in K , whenever K is a compact subset of X . The Poisson functional for an upper semicontinuous function and the Lelong functional satisfy this stronger condition, but we need (2) to cover the Poisson functional for a plurisuperharmonic function.

Proof of Theorem 1.2. By (1) and Lemma 1.1, EH is upper semicontinuous, so it remains to show that

$$EH(h(0)) \leq \int_{\mathbb{T}} EH \circ h \, d\lambda$$

for every $h \in \mathcal{A}_X$. This follows if for every $\varepsilon > 0$ and every continuous function $v : X \rightarrow \mathbb{R}$ with $v \geq EH$, there exists $g \in \mathcal{A}_X$ such that $g(0) = h(0)$ and

$$H(g) \leq \int_{\mathbb{T}} v \circ h \, d\lambda + \varepsilon.$$

Say h is holomorphic on D_r , $1 < r < 2$.

We may assume that $h(\overline{\mathbb{D}})$ lies in any element U of an open cover of X and is not contained in a given pluripolar subset of U . Namely, suppose an upper semicontinuous function u has the sub-mean value property with respect to analytic discs that do not lie in a pluripolar set Z . Since Z has measure zero, u has the sub-mean value property for balls, so u is subharmonic. By shrinking the domain, we may assume that $Z = w^{-1}(-\infty)$, where $w < 0$, $w \neq -\infty$ is plurisubharmonic. Then $u + cw$, $c > 0$, has the sub-mean value property with respect to all analytic discs, so it is plurisubharmonic, and the regularization $\tilde{u} \leq u$ of the limit of $u + cw$ as $c \rightarrow 0$ is plurisubharmonic and equal to u off Z . Since u , \tilde{u} are subharmonic and equal off a nullset, they are equal everywhere, so u is plurisubharmonic.

By a simple compactness argument, it follows from (2) and (3) that we can find finitely many closed arcs J_1, \dots, J_m in \mathbb{T} , open discs U_j centred on \mathbb{T} , relatively compact in D_r , containing J_j with mutually disjoint closures, and holomorphic maps $F_j : D_s \times U_j \rightarrow X$, $s > 1$, such that $F_j(0, \cdot) = h|_{U_j}$ and

$$\int_{\overline{J_j}} H(F_j(\cdot, t)) \, d\lambda(t) \leq \int_{J_j} v \circ h \, d\lambda + \frac{\varepsilon}{4} \lambda(J_j),$$

$$\int_{\mathbb{T} \setminus \cup J_j} |v \circ h| \, d\lambda < \frac{\varepsilon}{4},$$

and

$$\int_{\overline{\mathbb{T} \setminus \cup J_j}} H(h(w)) \, d\lambda(w) < \frac{\varepsilon}{4}.$$

Let $U_0 = D_r$ and $F_0 : D_s \times U_0 \rightarrow X$, $(z, w) \mapsto h(w)$.

The graph $\{(z, w, F_j(z, w)); z \in D_s, w \in U_j\}$, $0 \leq j \leq m$, is a Stein submanifold of $D_s \times U_j \times X$, being biholomorphic to $D_s \times U_j$. By a now-familiar argument based on Siu's theorem, as in the proof of Lemma 2.3 in [LS1], we conclude that if we slightly shrink U_j and s , then there is a biholomorphism Ψ_j from a neighbourhood of the graph onto $D_s \times U_j \times D_s^n$ such that

$$\Psi_j(z, w, F_j(z, w)) = (z, w, 0), \quad z \in D_s, w \in U_j.$$

By again shrinking U_j and s , and choosing $\gamma > 0$ sufficiently small, we obtain a holomorphic map $\Phi_j : U_j \times D_s^{n+3} \rightarrow \mathbb{C}^4 \times X$, well defined by the formula

$$\Phi_j(\zeta) = (\zeta_1, \zeta_2, -\zeta_2, -\zeta_1) + (0, 0, \Psi_j^{-1}(\zeta_2 + \gamma\zeta_4, \zeta_1 + \gamma\zeta_3, \zeta')),$$

where $\zeta = (\zeta_1, \dots, \zeta_4, \zeta')$ and we intend to add the first term on the right to the first four components of the second term. The map Φ_j is a biholomorphism from $U_j \times D_s^{n+3}$ onto its image in $\mathbb{C}^4 \times X$, and

$$\Phi_j(w, z, 0) = (w, z, 0, 0, F_j(z, w)), \quad w \in U_j, z \in D_s,$$

so in particular,

$$\Phi_j(w, 0) = (w, 0, 0, 0, h(w)), \quad w \in U_j.$$

Let

$$K_0 = \{(w, 0, 0, 0, h(w)); w \in \overline{\mathbb{D}}\} = \Phi_0(\overline{\mathbb{D}} \times \{0\}^{n+3})$$

and

$$K_j = \{(w, z, 0, 0, F_j(z, w)); w \in J_j, z \in \overline{\mathbb{D}}\} = \Phi_j(J_j \times \overline{\mathbb{D}} \times \{0\}^{n+2}), \quad j = 1, \dots, m.$$

The crux of the proof is the existence of a Stein neighbourhood V of $K = K_0 \cup \dots \cup K_m$ in $\mathbb{C}^4 \times X$. Let us assume this for a moment and finish proving that EH has the sub-mean value property. Let $\tau : V \rightarrow \mathbb{C}^N$ be an embedding and $\sigma : W \rightarrow \tau(V)$ be a holomorphic retraction from a Stein neighbourhood W of $\tau(V)$ in \mathbb{C}^N . Finally, let φ be the holomorphic submersion $\text{pr} \circ \tau^{-1} \circ \sigma : W \rightarrow X$, where $\text{pr} : \mathbb{C}^4 \times X \rightarrow X$ is the projection.

By assumption, $\tilde{u} = E\varphi^*H$ is plurisubharmonic on W , so

$$\tilde{u}(\tilde{h}(0)) \leq \int_{\mathbb{T}} \tilde{u} \circ \tilde{h} d\lambda,$$

where $\tilde{h} : D_r \rightarrow W$ is the lifting $w \mapsto \tau(w, 0, 0, 0, h(w))$ of h by φ . Hence, there is a disc $\tilde{g} \in \mathcal{A}_W$ with $\tilde{g}(0) = \tilde{h}(0)$ such that

$$H(g) = \varphi^*H(\tilde{g}) \leq \int_{\mathbb{T}} \tilde{u} \circ \tilde{h} d\lambda + \frac{\varepsilon}{4},$$

where $g = \varphi \circ \tilde{g} \in \mathcal{A}_X$ has $g(0) = h(0)$. Now if $w \in J_j$, $1 \leq j \leq m$, then $z \mapsto \tau \circ \Phi_j(w, z, 0)$ is a lifting by φ of $F_j(\cdot, w)$ with $0 \mapsto \tilde{h}(w)$, so $\tilde{u}(\tilde{h}(w)) \leq H(F_j(\cdot, w))$. Also, if $w \in \mathbb{T} \setminus \cup J_j$, then $\tilde{u}(\tilde{h}(w)) \leq \varphi^* H(\tilde{h}(w)) = H(h(w))$. Hence,

$$\begin{aligned} \int_{\mathbb{T}} \tilde{u} \circ \tilde{h} \, d\lambda &\leq \sum_{j=1}^m \int_{\overline{J_j}} H(F_j(\cdot, w)) \, d\lambda(w) + \int_{\overline{\mathbb{T} \setminus \cup J_j}} H(h(w)) \, d\lambda(w) \\ &\leq \int_{\cup J_j} v \circ h \, d\lambda + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \int_{\mathbb{T}} v \circ h \, d\lambda + \frac{3\varepsilon}{4}. \end{aligned}$$

We now turn to the all-important Stein neighbourhood. We shall construct a continuous, strictly plurisubharmonic exhaustion function $\rho : V \rightarrow [0, 1)$ on a neighbourhood V of K in $\mathbb{C}^4 \times X$.

For $j = 1, \dots, m$, choose open discs U'_j, U''_j concentric with U_j , such that

$$J_j \subset U''_j \Subset U'_j \Subset U_j.$$

Since $\Phi_0(w, 0) = \Phi_j(w, 0)$ for all $w \in U_j$, we have

$$\Phi_0(\overline{U'_j} \times D_\varepsilon^{n+3}) \Subset \Phi_j(U_j \times D_s^{n+3}) \quad (*)$$

for $\varepsilon > 0$ small enough. Our next step is to modify Φ_0 so that $\Phi_0 : D_r \times D_\varepsilon^{n+3} \rightarrow \mathbb{C}^4 \times X$ is still a biholomorphism onto its image, with a slightly smaller r and $\varepsilon > 0$ such that (*) still holds, and $\Phi_0(w, 0)$ is still $(w, 0, 0, 0, h(w))$ for $w \in D_r$, which is what we need above, and so that the derivative of the composition $\Phi_j^{-1} \circ \Phi_0$ is close to the identity at each point $(w, 0)$ with w near $\overline{U'_j}$. This ensures that

$$\frac{1}{A} |\pi_1|^2 \leq |\pi_1 \circ \Phi_j^{-1} \circ \Phi_0|^2 \leq A |\pi_1|^2 \quad \text{near } \overline{U'_j} \times D_\varepsilon^{n+3},$$

with $A > 1$ close to 1 if $\varepsilon > 0$ is small enough. Here, $\pi_1 : \mathbb{C}^{n+4} \rightarrow \mathbb{C}^{n+3}$ chops off the first coordinate, and $|\cdot|$ is the euclidean norm. This inequality with some $A > 1$ is immediate from the Mean Value Theorem, but we need it with A close to 1. We choose $\varepsilon \in (0, s)$ small enough to give $\frac{1}{A} > 1 - \frac{1}{3}(1 - R)^2$, where $R < 1$ is the maximum of the radii of the discs U'_1, \dots, U'_m .

We can modify Φ_0 in this way by precomposing it with a suitable biholomorphism μ from one neighbourhood of $D_r \times \{0\}^{n+3}$ in $D_r \times \mathbb{C}^{n+3}$ onto another, which is the identity on $D_r \times \{0\}^{n+3}$. Differentiating the inverse of $\Phi_j^{-1} \circ \Phi_0$ at points $(w, 0)$, $w \in U_j$, gives a holomorphic, matrix-valued map on $U_1 \cup \dots \cup U_m$. We Runge-approximate it on a neighbourhood of $\overline{U'_1} \cup \dots \cup \overline{U'_m}$ by a holomorphic, matrix-valued map ν on D_r with

$\nu_{11} = 1$ and $\nu_{i1} = 0$ for $i \geq 2$, and let $\mu(x) = \nu(x_1) \cdot x$, where x is viewed as a column vector.

The subset $L = (\overline{\mathbb{D}} \times \{0\}) \cup \bigcup_{j=1}^m (J_j \times \overline{\mathbb{D}})$ of \mathbb{C}^2 is polynomially convex, so there is a smooth, plurisubharmonic function ρ_0 on \mathbb{C}^2 such that $\rho_0 = 0$ on L and $\rho_0 > 0$ on $\mathbb{C}^2 \setminus L$. Find $\delta > 0$ small enough that $\Phi_j(U_j'' \times D_\delta^{n+3})$ is relatively compact in $\Phi_0(D_r \times D_\varepsilon^{n+3})$ for $j = 1, \dots, m$. By multiplying ρ_0 by a large enough constant, we get $\rho_0 \geq 1$ on each of the sets $\partial D_r \times D_\varepsilon$, $U_j'' \times \partial D_s$, and $\partial U_j'' \times (D_s \setminus D_\delta)$. The desired function ρ will be of the form $x \mapsto \rho_0(x_1, x_2) + \rho_1(x)$.

We now define ρ_1 on the open subset

$$U = \Phi_0(D_r \times D_\varepsilon^{n+3}) \cup \bigcup_{j=1}^m \Phi_j(U_j'' \times D_s^{n+3})$$

of $\mathbb{C}^4 \times X$. We start by choosing strictly positive, continuous, subharmonic functions α , β on \mathbb{C} , such that for $j = 1, \dots, m$,

$$\alpha = \frac{\varepsilon^2}{3} \text{ on } J_j, \quad \alpha \geq 1 \text{ and } \beta \leq \frac{1}{A} \text{ near } \partial U_j'',$$

$$\beta \geq \frac{1}{A} - \eta \text{ on } U_j' \setminus U_j'', \quad \beta > \max\{\alpha, A\} \text{ on } D_r \setminus \bigcup_{j=1}^m U_j',$$

where $\eta = \frac{1}{A} - 1 + \frac{1}{3}(1-R)^2 > 0$. On each of the mutually disjoint open sets $\Phi_j(U_j'' \times D_s^{n+3})$, let

$$\rho_1(\Phi_j(x)) = \frac{1}{3}|x_1|^2 + \frac{1}{\varepsilon^2}(\alpha(x_1)|x_2|^2 + |x''|^2),$$

where $x = (x_1, x') = (x_1, x_2, x'')$. On the open set $\Phi_0((D_r \setminus \bigcup_{j=1}^m \overline{U_j}') \times D_\varepsilon^{n+3})$, let

$$\rho_1(\Phi_0(x)) = \frac{1}{3}|x_1|^2 + \frac{1}{\varepsilon^2}\beta(x_1)|x'|^2.$$

By adding the set $\Phi_0((\bigcup_{j=1}^m \overline{U_j}') \setminus U_j'') \times D_\varepsilon^{n+3})$, we obtain a partition of U , and on this set we let

$$\rho_1(\Phi_0(x)) = \frac{1}{3}|x_1|^2 + \frac{1}{\varepsilon^2} \max\{\beta(x_1)|x'|^2, \alpha(x_1)|x_2|^2 + |\pi_2 \Phi_j^{-1} \Phi_0(x)|^2\},$$

where $\pi_2 : \mathbb{C}^{n+4} \rightarrow \mathbb{C}^{n+2}$ chops off the first two coordinates. By the choice of α and β , we have

$$\beta(x_1)|x'|^2 \leq \alpha(x_1)|x_2|^2 + |\pi_2 \Phi_j^{-1} \Phi_0(x)|^2 \quad \text{for } x_1 \text{ near } \partial U_j'',$$

and

$$\beta(x_1)|x'|^2 \geq \alpha(x_1)|x_2|^2 + |\pi_2\Phi_j^{-1}\Phi_0(x)|^2 \quad \text{for } x_1 \text{ near } \partial U'_j,$$

so ρ_1 is a well-defined, positive, continuous, strictly plurisubharmonic function on U .

On K , $\rho(x) = \frac{1}{3}(|x_1|^2 + |x_2|^2)$, so $\rho(K) = [0, \frac{2}{3}]$. To complete the proof, we will verify that $\liminf \rho(x) \geq 1$ as x goes to infinity in U , i.e., as x eventually leaves each compact subset of U . This implies that ρ exhausts the neighbourhood $V = \rho^{-1}[0, 1)$ of K .

First let $\Phi_j(x) \rightarrow \infty_U$ with $x \in U''_j \times D_s^{n+3}$. We may assume that $x \notin U''_j \times D_\delta^{n+3}$ since the Φ_j -image of this set is relatively compact in $\Phi_0(D_r \times D_\varepsilon^{n+3})$. Now either $x_1 \rightarrow \partial U''_j$ with $x' \in D_s^{n+3} \setminus D_\delta^{n+3}$, or $x_2 \rightarrow \partial D_s$, or $x'' \rightarrow \partial D_s^{n+2}$, so $\liminf \rho(x)$ is larger than either $\inf \rho_0(\partial U''_j \times (D_s \setminus D_\delta) \cup U''_j \times \partial D_s)$ or s^2/ε^2 , so it is at least 1.

Next let $\Phi_0(x) \rightarrow \infty_U$ with $x \in D_r \times D_\varepsilon^{n+3}$, $x_1 \notin \bigcup_{j=1}^m U'_j$. Then either $x_1 \rightarrow \partial D_r$ or $x' \rightarrow \partial D_\varepsilon^{n+3}$, so $\liminf \rho(x)$ is larger than either $\inf \rho_0(\partial D_r \times D_\varepsilon)$ or $\inf \beta(D_r \setminus \bigcup_{j=1}^m U'_j)$, so it is at least 1.

Finally, let $\Phi_0(x) \rightarrow \infty_U$ with $x_1 \in U'_j \setminus U''_j$. Then $x' \rightarrow \partial D_\varepsilon^{n+3}$, and $\liminf \rho$ is at least $\frac{1}{3}(1 - \text{radius } U'_j)^2 + \frac{1}{A} - \eta \geq 1$. \square

Both hypotheses (2) and (3) may be reduced to the affine case in the sense that they hold for a disc functional H on a complex manifold X if they hold for every pullback φ^*H of H by a holomorphic submersion φ from a domain of holomorphy in affine space into X . To see this for (2), simply cover X by open balls and apply (2) to the pullback of H by the inclusion of each ball into X . The union of the covers thus obtained for each ball is the required cover of X . As for (3), take $h \in \mathcal{A}_X$, $w \in \mathbb{T}$, and $\varepsilon > 0$. Find $f \in \mathcal{A}_X$ such that $f(0) = h(w)$ and $H(f) \leq EH(h(w)) + \varepsilon/2$. Suppose f and h are both holomorphic on D_r , $r > 1$. The union of the embedded discs $\{(z, w, f(z)); z \in D_r\}$ and $\{(0, z, h(z)); z \in D_r\}$ is a Stein subvariety of $D_r^2 \times X$ and has a Stein neighbourhood V by Siu's theorem. Let $\tau : V \rightarrow \mathbb{C}^m$ be an embedding and $\sigma : W \rightarrow \tau(V)$ be a holomorphic retraction from a Stein neighbourhood W of the submanifold $\tau(V)$ in \mathbb{C}^m . Now $\varphi = \text{pr} \circ \tau^{-1} \circ \sigma : W \rightarrow X$ is a holomorphic submersion by which both f and h lift: $\tilde{f} : z \mapsto \tau(z, w, f(z))$ is a lifting of f and $\tilde{h} : z \mapsto \tau(0, z, h(z))$ of h . Here, $\text{pr} : \mathbb{C}^2 \times X \rightarrow X$ is the projection. To get condition (3) for H we postcompose by φ the holomorphic maps provided by (3) for φ^*H and the data \tilde{h} , w , $\varepsilon/2$, and observe that

$$E\varphi^*H(\tilde{h}(w)) + \varepsilon/2 \leq \varphi^*H(\tilde{f}) + \varepsilon/2 = H(f) + \varepsilon/2 \leq EH(h(w)) + \varepsilon$$

by the choice of f .

Suppose we have defined a class of disc functionals on each complex manifold, preserved by taking pullbacks by holomorphic submersions. Consider the property of a disc functional of satisfying (2) and (3) and having a plurisubharmonic envelope. The Reduction Theorem implies that to establish this property for all disc functionals of the given type, it suffices to prove it on domains of holomorphy in affine space. We do not know if the property of having a plurisubharmonic envelope by itself reduces to the affine case.

2. Plurisubharmonicity of the Poisson and Riesz envelopes

Let X be a complex manifold and $v : X \rightarrow \overline{\mathbb{R}}$ be a Borel function which is locally bounded from above or from below. Then the Poisson functional H_P^v on X associated to v is defined by the formula

$$H_P^v(f) = \int_{\mathbb{T}} v \circ f d\lambda, \quad f \in \mathcal{A}_X.$$

Rosay's theorem [Rs] states that the Poisson envelope EH_P^v is plurisubharmonic on X when $v : X \rightarrow [-\infty, +\infty)$ is upper semicontinuous, and then EH_P^v is the largest plurisubharmonic minorant of v . This may also be derived from the Reduction Theorem 1.2 (to whose proof Rosay's ideas are essential) and plurisubharmonicity of the Poisson envelope on domains in affine space, proved independently (with different proofs) first by Poletsky [P1] and later by Bu and Schachermayer [BS].

Now suppose $v : X \rightarrow (-\infty, +\infty]$ is plurisuperharmonic, not identically $+\infty$. Edigarian [E1] has shown that the Poisson envelope EH_P^v is plurisubharmonic when X belongs to a large class of complex manifolds (the so-called class \mathcal{P} from [LS1]), including domains in affine space. Hence, H_P^v satisfies hypothesis (1) in the Reduction Theorem 1.2 (note that the pullback $\varphi^*H_P^v$ is the Poisson functional $H_P^{v \circ \varphi}$ of the plurisuperharmonic function $v \circ \varphi$). Also, (2) holds: take $U = X$ and $Z = v^{-1}(+\infty)$.

Let us verify that H_P^v satisfies (3). Take $h \in \mathcal{A}_X$, $w \in \mathbb{T}$, and $\beta > EH_P^v(h(w))$. By Lemma 9 in [E1], there is a holomorphic map $G : D_r \times B_r \times V \rightarrow X$, where B_r is the ball of radius $r > 1$ in \mathbb{C}^n , $n = \dim X$, and V is a neighbourhood of $h(w)$ in X , such that $G(0, y, x) = x$ for $y \in B_r$ and $x \in V$, and for each $x \in V$, the average of $y \mapsto H_P^v(G(\cdot, y, x))$ on B_1 is less than β . If $J \subset U = h^{-1}(V)$ is a closed arc in \mathbb{T} , then the average over $B_1 \times J$ of $(y, w) \mapsto H_P^v(G(\cdot, y, h(w)))$ is less than β . Hence, there is $y_0 \in B_1$ (depending on J , of course) such that the average over J of $w \mapsto H_P^v(G(\cdot, y_0, h(w)))$ is less than β . We let $F : D_r \times U \rightarrow X$, $(z, w) \mapsto G(z, y_0, h(w))$, and observe that $F(0, w) = G(0, y_0, h(w)) = h(w)$ for $w \in U$.

The Reduction Theorem 1.2 now implies the following result.

2.1. Theorem. *Let $v : X \rightarrow (-\infty, +\infty]$ be a plurisuperharmonic function, not identically $+\infty$, on a complex manifold X . Then the Poisson envelope EH_P^v of v is plurisubharmonic on X and is the largest plurisubharmonic minorant of v .*

A plurisubharmonic function u on a complex manifold X defines the Riesz functional H_R^u on X by the formula

$$H_R^u(f) = \frac{1}{2\pi} \int_{\mathbb{D}} \log |\cdot| \Delta(u \circ f), \quad f \in \mathcal{A}_X,$$

where $\Delta(u \circ f)$ is considered as a positive Borel measure on \mathbb{D} . If $f \in \mathcal{A}_X$ and $u \circ f = -\infty$, then we set $H_R^u(f) = 0$. By the Riesz representation formula,

$$EH_R^u = u + EH_P^{-u},$$

so we have the following immediate consequence of Theorem 2.1 and the first part of Theorem 4.4 in [LS1].

2.2. Theorem. *The Riesz envelope EH_R^u of a plurisubharmonic function u on a complex manifold X is plurisubharmonic on X and is the largest negative plurisubharmonic function on X with Levi form no smaller than that of u .*

We would like to draw the reader's attention to the important paper [P2], in which Poletsky develops a new approach to the general theory of disc functionals and proves, among other things, the plurisubharmonicity of the Riesz envelope on domains in affine space.

3. Plurisubharmonicity of the Lelong envelope

Let α be a positive function on a complex manifold X . We make no regularity assumptions on α ; it need not even be measurable. In [LS1], we defined the Lelong functional H_L^α on X associated to α by the formula

$$H_L^\alpha(f) = \sum_{z \in \mathbb{D}} \alpha(f(z)) m_z(f) \log |z|, \quad f \in \mathcal{A}_X.$$

By omitting the multiplicity $m_z(f)$, we obtain the *reduced Lelong functional*

$$\tilde{H}_L^\alpha(f) = \sum_{z \in \mathbb{D}} \alpha(f(z)) \log |z|,$$

whose advantage over the Lelong functional is that its pullback by a holomorphic map is a functional of the same type. (For this reason, and since the envelopes are the same, as proved below, we suggest that the reduced Lelong functional be known as the Lelong functional; the original Lelong functional is perhaps best forgotten.)

3.1. Lemma. *Let α be a positive function on a domain X in \mathbb{C}^n . Then the Lelong functional H_L^α and the reduced Lelong functional \tilde{H}_L^α have the same envelope.*

Proof. Since $H_L^\alpha \leq \tilde{H}_L^\alpha$, it suffices to show that for every $f \in \mathcal{A}_X$ and $\beta > H_L^\alpha(f)$, there exists $g \in \mathcal{A}_X$ such that $g(0) = f(0)$ and $\tilde{H}_L^\alpha(g) < \beta$.

By definition of H_L^α , there exist finitely many distinct points $a_1, \dots, a_N \in \mathbb{D}$ with multiplicities $m_j = m_{a_j}(f)$ such that $\sum \alpha(f(a_j)) m_j \log |a_j| < \beta$. If this sum is $-\infty$, then there is j with $a_j = 0$ and $\alpha(f(a_j)) > 0$, so $\tilde{H}_L^\alpha(f) = -\infty < \beta$. Hence, we may assume that $a_j \neq 0$ for all j . We set $m = m_1 + \dots + m_N$ and define $b = (b_0, \dots, b_m) \in \mathbb{D}^{m+1}$ as the vector $(0, a_1, \dots, a_1, \dots, a_N, \dots, a_N)$ with the point a_j repeated m_j times. We will choose $t_j \in \mathbb{T}$ close to 1 such that $c_j = t_j b_j$ are all different, and show that there exists $g \in \mathcal{A}_X$ such that $g(c_j) = f(b_j)$ for $j = 0, \dots, m$. Then

$$\tilde{H}_L^\alpha(g) \leq \sum_{j=1}^m \alpha(g(c_j)) \log |c_j| = \sum_{j=1}^m \alpha(f(b_j)) \log |b_j| = \sum_{j=1}^N \alpha(f(a_j)) m_j \log |a_j| < \beta.$$

We write $f = p + h$, where $p : \mathbb{C} \rightarrow \mathbb{C}^n$ is the polynomial map of degree at most m that solves the interpolation problem $p(0) = f(0)$ and $p^{(k)}(a_j) = f^{(k)}(a_j)$ for $k = 0, \dots, m_j - 1$. By the Newton interpolation formula,

$$p(z) = f[b_0] + \sum_{j=1}^m f[b_0, \dots, b_j](z - b_0) \cdots (z - b_{j-1}) \quad (*)$$

and $h(z) = f[b_0, \dots, b_m, z](z - b_0) \cdots (z - b_m)$, where $f[b_i, \dots, b_{i+j}]$ are the finite differences defined for $j = 0$ as $f(b_i)$, and for $j > 0$ as $f^{(j)}(b_i)/j!$ if $b_i = b_{i+j}$ and as $(f[b_{i+1}, \dots, b_{i+j}] - f[b_i, \dots, b_{i+j-1}]) / (b_{i+j} - b_i)$ if $b_i \neq b_{i+j}$.

Let $q : \mathbb{C} \rightarrow \mathbb{C}^n$ be the polynomial map of degree at most m that solves the interpolation problem $q(c_j) = f(b_j)$ for $j = 0, \dots, m$, let $k(z) = f[b_0, \dots, b_m, z](z - c_0) \cdots (z - c_m)$, and set $g = q + k$. Then $g(c_j) = f(b_j)$ for $j = 0, \dots, m$, so if we can show that $g \rightarrow f$ uniformly on \mathbb{D} as $c \rightarrow b$, then $g \in \mathcal{A}_X$ for c sufficiently close to b and the proof is complete. Clearly, $k \rightarrow h$ uniformly on \mathbb{D} as $c \rightarrow b$. To prove that $q \rightarrow p$, we observe that the Newton formula for q for interpolating the values of g at the points c_j is (*) with p, f , and b replaced by q, g , and c , respectively. We have $g[c_i] = f[b_i]$ for all i . If $j > 0$ and $b_i = b_{i+j}$, then clearly $g[c_i, \dots, c_{i+j}] = 0$, and $f[b_i, \dots, b_{i+j}] = f^{(j)}(b_i)/j! = 0$ since b_i is one of the points a_ν and $j < m_\nu$. It now follows easily by induction on j that $g[c_i, \dots, c_{i+j}] \rightarrow f[b_i, \dots, b_{i+j}]$ for all i and j , so $q \rightarrow p$ locally uniformly on \mathbb{C} , as $c \rightarrow b$. \square

3.2. Theorem. *Let α be a positive function on a complex manifold X . Then the Lelong envelope EH_L^α and the reduced Lelong envelope $E\tilde{H}_L^\alpha$ associated to α are equal and plurisubharmonic on X , and they are the largest negative plurisubharmonic function on X with Lelong numbers at least α .*

Proof. If X is a domain in affine space, then EH_L^α is plurisubharmonic by Theorem 5.3 in [LS1], and equal to $E\tilde{H}_L^\alpha$ by Lemma 3.1. In the general case, consider the class \mathcal{F}_α of negative plurisubharmonic functions on X with Lelong numbers at least α . It is easily seen that $\sup \mathcal{F}_\alpha \leq EH_L^\alpha \leq E\tilde{H}_L^\alpha$. By Proposition 5.1 in [LS1], which is stated for H_L^α but applies equally well to \tilde{H}_L^α , the envelope $E\tilde{H}_L^\alpha$ is plurisubharmonic if and only if $E\tilde{H}_L^\alpha \in \mathcal{F}_\alpha$, and then $E\tilde{H}_L^\alpha = \sup \mathcal{F}_\alpha$, so $E\tilde{H}_L^\alpha = EH_L^\alpha$.

It therefore remains to show that $E\tilde{H}_L^\alpha$ is plurisubharmonic on our complex manifold. By the above, hypothesis (1) in the Reduction Theorem 1.2 is satisfied. Hypothesis (2) is clear, since $\tilde{H}_L^\alpha \leq 0$. Finally, by Lemma 2.3 in [LS1], \tilde{H}_L^α (as well as H_L^α) satisfies the condition in Remark 1.3, so hypothesis (3) holds. \square

By Theorem 5.3 in [LS1], on a domain in a Stein manifold, the Lelong envelope is the Poisson envelope of the so-called Lempert function k_α . In fact, we deduced the plurisubharmonicity of the Lelong envelope from this. It is easy to see that this holds on all manifolds, now that we know that $EH_P^{k_\alpha}$ and EH_L^α are plurisubharmonic: $EH_P^{k_\alpha} \leq EH_L^\alpha$ [LS1, p. 22], and since $k_\alpha \geq EH_L^\alpha$ and EH_L^α is plurisubharmonic, $EH_P^{k_\alpha} \geq EH_L^\alpha$.

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