PLURISUBHARMONIC EXTREMAL FUNCTIONS, LELONG NUMBERS AND COHERENT IDEAL SHEAVES

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Abstract. We introduce a new type of pluricomplex Green function which has a logarithmic pole along a complex subspace A of a complex manifold X. It is the largest negative plurisubharmonic function on X whose Lelong number is at least the Lelong number of $\log\max\{|f_1|,\dots,|f_m|\}$, where $f_1,\dots,f_m$ are local generators for the ideal sheaf of A. The pluricomplex Green function with a single logarithmic pole or a finite number of weighted poles is a very special case of our construction. We give several equivalent definitions of this function and study its properties, including boundary behaviour, continuity, and uniqueness. This is based on and extends our previous work on disc functionals and their envelopes.

1. Introduction

Let X be a complex manifold. For each function $\alpha : X \to [0,+\infty)$ we let

$$\mathcal{F}_\alpha = \{u \in \mathrm{PSH}(X)\,;\, u \le 0, \nu_u \ge \alpha\}$$

and

$$G_\alpha = \sup \mathcal{F}_\alpha,$$

where $\mathrm{PSH}(X)$ is the class of plurisubharmonic functions on X (including the constant function $-\infty$) and ν_u denotes the Lelong number of u. Then G_α is plurisubharmonic and $G_\alpha \in \mathcal{F}_\alpha$ [6, Prop. 5.1]. Recall that the Lelong number is a biholomorphic invariant, and if u is plurisubharmonic in a neighbourhood of 0 in $\mathbb{C}^n$, then

$$\nu_u(0) = \lim_{r\to 0} \frac{\sup\limits_{|z|=r} u(z)}{\log r}.$$

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If α is the characteristic function of a one-point set $\{a\}$, then G_α is the pluricomplex Green function G_a of X with a logarithmic pole at a, first defined by Klimek [4]. Such functions have also been studied e.g. by Demailly [1], Edigarian [2, 3], Lempert [9, 10], and Zeriahi [17]. If α has finite support, then G_α is the pluricomplex Green function of X with a logarithmic pole of weight $\alpha(a)$ at each point a of the support. Such functions were first defined by Lelong [8].

This paper is a study of the much larger class of functions $G_A = G_\alpha$, where A is a (closed) complex subspace of X, and $\alpha = \nu_A$ is the Lelong number of the plurisubharmonic function $\log\max\{|f_1|,\dots,|f_m|\}$, where $f_1,\dots,f_m$ are local generators for the ideal sheaf of A. This number is independent of the choice of generators. We call G_A the *pluricomplex Green function with a logarithmic pole along* A, or simply the *Green function with a pole along* A.

Our results here are based on our previous work [6] in the theory of disc functionals and their envelopes, which in turn builds on the pioneering work of Poletsky [11, 12]. Indeed, on domains in Stein manifolds (and in fact under certain much weaker conditions), G_A is the envelope of the Lelong functional associated to the function ν_A. In Section 2 we review the theory of disc functionals with emphasis on the three known classes of examples. We add a few new results to the theory of the Lelong functional, including a product property, generalizing a result of Edigarian [3].

In Section 3 we study the pluricomplex Green function with a logarithmic pole along a complex subspace. Our main results may be summarized as follows.

Main Theorem. *Let X be a relatively compact domain in a Stein manifold Y, and let A be the intersection with X of a complex subspace B of Y. Then G_A is locally bounded and maximal on $X \setminus A$, and $\nu_{G_A} = \nu_A$.*

If X has a strong plurisubharmonic barrier at $p \in \partial X \setminus B$, then $G_A(x) \to 0$ as $x \to p$.

If A is a divisor, then the Levi form of G_A is at least π times the current of integration over A.

If X has a strong plurisubharmonic barrier at every boundary point and B is a principal divisor, then the set of points in X at which G_A is discontinuous is pluripolar.

We relate G_A to the Poisson and Riesz functionals. This yields several alternative definitions of G_A. We show that G_A is uniquely determined by some of its key properties when A is a principal divisor. We also present a few instructive examples. Among other things, through an investigation of Green functions, we obtain a bounded pseudoconvex domain in $\mathbb{C}^2$ such that $\log\tanh$ of the Carathéodory distance to the boundary is not plurisubharmonic.

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2. Disc functionals and their envelopes

Let X be a complex manifold. A holomorphic map from the unit disc $\mathbb{D}$ to X is called an *analytic disc* in X, and if it can be extended holomorphically to some neighbourhood of $\overline{\mathbb{D}}$ then it is called a *closed analytic disc* in X. We let $\mathcal{O}(\mathbb{D}, X)$ denote the set of all analytic discs in X and $\mathcal{A}_X$ denote the set of all closed analytic discs in X. A *disc functional* on X is a map H from a subset of $\mathcal{O}(\mathbb{D}, X)$ containing $\mathcal{A}_X$ to $\mathbb{R} \cup \{\pm\infty\}$. The *envelope* of H is the function $EH : X \to \mathbb{R} \cup \{\pm\infty\}$ defined by the formula

$$EH(x) = \inf\{H(f)\,;\, f \in \mathcal{A}_X, f(0) = x\}.$$

Envelopes of disc functionals were first defined and studied by Poletsky [11, 12].

In our paper [6], we studied three classes of examples of disc functionals and proved that their envelopes are plurisubharmonic for a large collection of manifolds X. These functionals are called the Poisson, Riesz, and Lelong functionals.

If $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous, then the *Poisson functional* H_P^φ is defined on $\mathcal{A}_X$ by

$$H_P^\varphi(f) = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi \circ f \, d\lambda,$$

where λ is the arc length measure on the unit circle $\mathbb{T}$.

We define

$$\mathcal{F}_P^\varphi = \{u \in \mathrm{PSH}(X)\,;\, u \le \varphi\}.$$

Then $\sup \mathcal{F}_P^\varphi$ is plurisubharmonic, $\sup \mathcal{F}_P^\varphi \le EH_P^\varphi$, and equality holds if and only if EH_P^φ is plurisubharmonic [6, Prop. 2.1].

If v is a plurisubharmonic function on X, then the *Riesz functional* H_R^v is defined on $\mathcal{O}(\mathbb{D}, X)$ by

$$H_R^v(f) = \frac{1}{2\pi} \int_{\mathbb{D}} \log|\cdot|\, \Delta(v \circ f)$$

if $f \in \mathcal{O}(\mathbb{D}, X)$ and $v \circ f$ is not identically $-\infty$, where $\Delta(v \circ f)$ is considered as a positive Borel measure on $\mathbb{D}$. If $f \in \mathcal{O}(\mathbb{D}, X)$ and $v \circ f = -\infty$, then we set $H_R^v(f) = 0$.

We define

$$\mathcal{F}_R^v = \{u \in \mathrm{PSH}(X)\,;\, u \le 0,\, \mathcal{L}(u) \ge \mathcal{L}(v)\}.$$

Here, $\mathcal{L}(u)$ denotes the Levi form $i\partial\bar\partial u$ of u. We set $\mathcal{L}(-\infty) = 0$. If $v : X \to \mathbb{R}$ is continuous, then $\sup \mathcal{F}_R^v$ is plurisubharmonic, $\sup \mathcal{F}_R^v \le EH_R^v$, and equality holds if and only if EH_R^v is plurisubharmonic [6, Thm. 4.4].

If α is a nonnegative function on X, then the *Lelong functional* H_L^α is defined on $\mathcal{O}(\mathbb{D}, X)$ by the formula

$$H_L^\alpha(f) = \sum_{z \in \mathbb{D}} \alpha(f(z))\, m_z(f)\, \log|z|.$$

The sum, which may be uncountable, is defined as the infimum of its finite partial sums. Here, $m_z(f)$ denotes the multiplicity of f at z, defined in the following way. If f is constant, let $m_z(f)=\infty$. If f is nonconstant, let (U,ζ) be a coordinate neighbourhood on X with $\zeta(f(z))=0$. Then there exists an integer m such that $\zeta(f(w))=(w-z)^m g(w)$ where $g:V\to\mathbb{C}^n$ is a map defined in a neighbourhood V of z with $g(z)\neq 0$. The number m, which is independent of the choice of local coordinates, is the multiplicity of f at z.

We define

$$\mathcal{F}_L^\alpha=\{u\in \mathrm{PSH}(X)\,;\,u\le 0,\nu_u\ge\alpha\}.$$

Then $G_\alpha=\sup\mathcal{F}_L^\alpha$ is plurisubharmonic. If $u\in\mathcal{F}_L^\alpha$ and $f\in\mathcal{O}(\mathbb{D},X)$, then $u(f(0))\le H_L^\alpha(f)$, so $G_\alpha\le EH_L^\alpha$. The function EH_L^α is plurisubharmonic if and only if $EH_L^\alpha\in\mathcal{F}_L^\alpha$, and then $G_\alpha=EH_L^\alpha$ [6, Prop. 5.1].

Clearly,

$$EH_L^\alpha=\inf EH_L^\beta,$$

where the infimum is taken over all functions β with finite support such that $0\le\beta\le\alpha$. When EH_L^β is plurisubharmonic, it is a pluricomplex Green function with finitely many weighted poles.

Let us now take a closer look at the Lelong functional. For $f\in\mathcal{O}(\mathbb{D},X)$ we define $f^*\alpha:\mathbb{D}\to[0,+\infty)$ by the formula

$$f^*\alpha(z)=\alpha(f(z))m_z(f),\qquad z\in\mathbb{D}.$$

Then

$$\mu_f^\alpha=2\pi\sum_{z\in\mathbb{D}}f^*\alpha(z)\delta_z$$

is a well defined positive Borel measure on $\mathbb{D}$, where δ_z is the Dirac measure at z. We define

$$v_f^\alpha(z)=\int_{\mathbb{D}}G(z,\cdot)\,d\mu_f^\alpha=\sum_{w\in\mathbb{D}}f^*\alpha(w)\log\left|\frac{z-w}{1-\bar{w}z}\right|,\qquad z\in\mathbb{D},$$

where G denotes the Green function of the unit disc,

$$G(z,w)=\frac{1}{2\pi}\log\left|\frac{z-w}{1-\bar{w}z}\right|.$$

Then v_f^α is subharmonic in $\mathbb{D}$. We have $v_f^\alpha\neq-\infty$ if and only if

$$\sum_{z\in\mathbb{D}}f^*\alpha(z)(1-|z|)=\frac{1}{2\pi}\int_{\mathbb{D}}(1-|\cdot|)\,d\mu_f^\alpha<\infty.$$

If $v_f^\alpha\neq-\infty$, then μ_f^α has finite mass on compact sets, the sum which defines μ_f^α is convergent in the sense of distributions, and $\Delta v_f^\alpha=\mu_f^\alpha$.

The Lelong number $\nu_v(z)$ of a subharmonic function v on a domain in $\mathbb{C}$ at a point z is $\Delta v(\{z\})/2\pi$, i.e., the Riesz mass of v at the point z divided by 2π. We therefore have $\nu_{v_f^\alpha} = f^*\alpha$, $H_L^\alpha(f) = v_f^\alpha(0)$, and from the Riesz representation formula we see that v_f^α is the largest negative subharmonic function on $\mathbb{D}$ satisfying $\nu_v \geq f^*\alpha$.

Assume now that $f \in \mathcal{O}(\mathbb{D}, X)$ is an extremal disc for the Lelong functional, i.e., $G_\alpha(f(0)) = H_L^\alpha(f) = v_f^\alpha(0)$, and that $G_\alpha(f(0)) > -\infty$. Then the function $v = G_\alpha \circ f$ is subharmonic in $\mathbb{D}$, $v \leq 0$, and $\nu_v(z) \geq \nu_{G_\alpha}(f(z)) m_z(f) \geq f^*\alpha(z)$ for all $z \in \mathbb{D}$. Hence $v \leq v_f^\alpha$. The function v_f^α is harmonic outside the countable set where it takes the value $-\infty$. Since $v(0) = v_f^\alpha(0)$, the maximum principle implies that $v = v_f^\alpha$ in $\mathbb{D} \setminus (v_f^\alpha)^{-1}(-\infty)$, so $v = v_f^\alpha$ on $\mathbb{D}$. We have proved the following.

2.1. Proposition. *Let α be a nonnegative function on a complex manifold X. If $f \in \mathcal{O}(\mathbb{D}, X)$ is an extremal disc for the Lelong functional H_L^α in the sense that $G_\alpha(f(0)) = H_L^\alpha(f) = v_f^\alpha(0)$, and $G_\alpha(f(0)) > -\infty$, then*

$$G_\alpha(f(z)) = \sum_{w \in \mathbb{D}} \alpha(f(w)) m_w(f) \log \left| \frac{z - w}{1 - \bar{w}z} \right|, \qquad z \in \mathbb{D},$$

which implies that the function $G_\alpha \circ f$ is harmonic outside the countable set where it takes the value $-\infty$.

In [6] we proved that the envelopes of the three functionals are plurisubharmonic for a large class of manifolds with mild conditions on φ, v, and α. We define $\mathcal{P}$ as the class of complex manifolds X for which there exists a finite sequence of complex manifolds and holomorphic maps

$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \dots \xrightarrow{h_m} X_m = X, \qquad m \geq 0, \tag{2.1}$$

where X_0 is a domain in a Stein manifold and each h_i, $i = 1, \dots, m$, is either a covering (unbranched and possibly infinite) or a finite branched covering (i.e., a proper holomorphic surjection with finite fibres).

Assume now that $X \in \mathcal{P}$. Then EH_P^φ is plurisubharmonic for every upper semicontinuous function φ on X [6, Thms. 2.2 and 3.4].

The function EH_R^v is plurisubharmonic for every continuous plurisubharmonic function $v : X \to \mathbb{R}$ [6, Thm. 4.4].

The function EH_L^α is plurisubharmonic for every nonnegative function α on X for which the sequence (2.1) can be chosen such that $\alpha^{-1}[c, \infty) \setminus B$ is dense in $\alpha^{-1}[c, \infty)$ in the analytic Zariski topology on X for every $c > 0$, where

$$B = \bigcup_{i=1}^{m} (h_m \circ \dots \circ h_{i+1})(B_i),$$

and B_i denotes the (possibly empty) branch locus of h_i [6, Thm. 5.12]. Observe that this condition holds in particular if $\alpha = 0$ on B, or if X is a domain in a Stein manifold.

It turns out that EH_L^α is related to the Kobayashi pseudodistance κ_X on X. By definition κ_X is the largest pseudodistance on X smaller than or equal to δ_X, where

$$\delta_X(x, a) = \inf\{\varrho_{\mathbb{D}}(z, w)\,;\, f(z) = x, f(w) = a \text{ for some } f \in \mathcal{O}(\mathbb{D}, X)\},$$

and $\varrho_{\mathbb{D}}$ denotes the Poincaré distance in $\mathbb{D}$,

$$\varrho_{\mathbb{D}}(z, w) = \tanh^{-1}\left|\frac{z - w}{1 - \bar{w}z}\right|.$$

We define $k_X = \log\tanh\delta_X$. By composing the map f in the definition of δ_X with an automorphism which sends 0 to w and z to the a point on the positive real axis, and then replacing it by $z \mapsto f(z/r)$ with $r > 1$ and r close to 1, we see that

$$k_X(x, a) = \inf\{\log t\,;\, t \in (0, 1), f(t) = a, f(0) = x, \text{ for some } f \in \mathcal{A}_X\}.$$

We now define

$$k_X^\alpha(x) = \inf_{a\in X} \alpha(a)k_X(x, a), \qquad x \in X.$$

If X is a domain in a Stein manifold, then $G_\alpha = EH_L^\alpha = EH_P^{k_X^\alpha}$ [6, Thm. 5.3].

Let G_a be the pluricomplex Green function with a logarithmic pole at a. Then $G_a \le k_X(\cdot, a)$. If $u \in \mathcal{F}_L^\alpha$ and $a \in X$, then $\nu_u(a) \ge \alpha(a)$, so $u \le \alpha(a)G_a$, and

$$G_\alpha \le \inf_{a\in X} \alpha(a)G_a \le k_X^\alpha.$$

The above results may be summarized as follows.

2.2. Theorem. *Let α be a nonnegative function on a domain X in a Stein manifold. Then*

$$G_\alpha = EH_L^\alpha = EH_P^{k_X^\alpha} = \sup\{u \in \mathrm{PSH}(X)\,;\, u \le \inf_{a\in X} \alpha(a)G_a\}.$$

Our next result shows that G_α has no Monge-Ampère mass where it is locally bounded.

2.3. Proposition. *Let α be a nonnegative function on a complex manifold X. Then the function G_α is maximal in the open subset of X where it is locally bounded.*

Proof. Let U be a relatively compact domain in the open subset of X where G_α is locally bounded, and let v be plurisubharmonic on X such that $v \le G_\alpha$ on ∂U. Let

$$w = \begin{cases} \max\{v, G_\alpha\} & \text{on } U, \\ G_\alpha & \text{on } X \setminus U. \end{cases}$$

Then w is plurisubharmonic on X and $w \in \mathcal{F}_L^\alpha$, so $w \le G_\alpha$. Hence, $v \le G_\alpha$ on U. $\square$

Now we turn to the boundary behaviour of the supremum of the Lelong class. Let X be a domain in a complex manifold Y, and p be a boundary point of X. Recall that a plurisubharmonic function v on X is called a *strong (plurisubharmonic) barrier* at p if $\lim\limits_{x\to p} v(x) = 0$, and $\sup\limits_{X\setminus V} v < 0$ for every neighbourhood V of p in Y.

A relatively compact domain X in a complex manifold is said to be *B-regular* if every continuous function on the boundary of X extends to a continuous function on the closure of X which is plurisubharmonic on X. This notion is due to Sibony [13]. It is easily seen that a B-regular domain has a strong barrier at every boundary point, and for domains in $\mathbb{C}^n$, the converse holds. For a weaker result on an arbitrary Kähler manifold, see Lemma 3.7. In $\mathbb{C}^n$, strongly pseudoconvex domains and smoothly bounded pseudoconvex domains of finite type are B-regular, and B-regular domains are hyperconvex, but not vice versa.

2.4. Proposition. *Let α be a nonnegative function on a domain X in a complex manifold Y, and assume that there exists a strong plurisubharmonic barrier v at $p \in \partial X$. If some u in $\mathcal{F}_L^\alpha$ is bounded below on a neighbourhood of p, then G_α has limit zero at p.*

Example 3.4 shows that this result may fail if existence of a strong barrier is replaced by hyperconvexity.

Proof. Choose a neighbourhood V of p in Y such that $u > \beta \in \mathbb{R}$ in a neighbourhood of $X \cap \overline{V}$, and choose $c > 0$ such that $\sup\limits_{X\setminus V} v < \beta/c$. Then $u > cv$ in a neighbourhood of $\partial V \cap X$, so the function w defined by

$$w = \begin{cases} \max\{u, cv\} & \text{on } X \cap V, \\ u & \text{on } X \setminus V, \end{cases}$$

is plurisubharmonic on X. Since $u > \beta$ in a neighbourhood of $X \cap \overline{V}$, we have $\nu_u = 0 = \alpha$ there. Hence, $w \in \mathcal{F}_L^\alpha$, and we get

$$\liminf_{x\to p} G_\alpha(x) \ge \liminf_{x\to p} w(x) \ge \lim_{x\to p} cv(x) = 0. \quad \square$$

Edigarian [3] has proved that if X_1 and X_2 are domains in $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$ respectively, and $a = (a_1, a_2) \in X = X_1 \times X_2$, then

$$G_a(x) = \max\{G_{a_1}(x_1), G_{a_2}(x_2)\}, \qquad x = (x_1, x_2) \in X.$$

This is called the *product property* of the pluricomplex Green function. By a modification of Edigarian's proof we get the following result.

2.5. Theorem. *Let* α_1 *and* α_2 *be the characteristic functions of subsets* A_1 *and* A_2 *of complex manifolds* X_1 *and* X_2 *respectively, and let* α *denote the characteristic function of* $A = A_1 \times A_2$ *on the product manifold* $X = X_1 \times X_2$*, so*

$$\alpha(x) = \min\{\alpha_1(x_1), \alpha_2(x_2)\}, \qquad x = (x_1, x_2) \in X.$$

Then

$$EH_L^\alpha(x) = \max\{EH_L^{\alpha_1}(x_1), EH_L^{\alpha_2}(x_2)\}.$$

If $EH_L^{\alpha_1}$ *and* $EH_L^{\alpha_2}$ *are plurisubharmonic, then*

$$G_\alpha(x) = \max\{G_{\alpha_1}(x_1), G_{\alpha_2}(x_2)\}.$$

It is easy to see that the product property fails in general. Let $\alpha_1 = \chi_{\{0\}}$ and $\alpha_2 = 2\alpha_1$ on $\mathbb{D}$. Then $\alpha = \chi_{\{(0,0)\}}$ on $\mathbb{D} \times \mathbb{D}$, and

$$EH_L^\alpha(z_1, z_2) = G_{(0,0)}(z_1, z_2) = \max\{\log|z_1|, \log|z_2|\},$$

but

$$\max\{EH_L^{\alpha_1}(z_1), EH_L^{\alpha_2}(z_2)\} = \max\{\log|z_1|, 2\log|z_2|\}.$$

The latter function is not even a Lelong envelope (although it is presumably the envelope of a disc functional involving *directional* Lelong numbers). Both functions have Lelong number 1 at $(0,0)$ and 0 elsewhere.

2.6. Lemma. *Let* X *be a complex manifold,* α *be a nonnegative function on* X*, and* $\beta \in (-\infty, 0)$*. If* $EH_L^\alpha(x) < \beta$*, then there exists* $f \in \mathcal{A}_X$ *with* $f(0) = x$ *and finitely many points* $a_1, \ldots, a_l$ *in* $\mathbb{D} \setminus \{0\}$ *such that*

$$-\infty < \sum_{k=1}^{l} \alpha(f(a_k))\, m_{a_k}(f)\, \log|a_k| < \beta. \tag{2.2}$$

Proof. By the definition of the envelope, there exists $f \in \mathcal{A}_X$ with $f(0) = x$ such that $H_L^\alpha(f) < \beta$, and by the definition of H_L^α there are finitely many points $a_1, \ldots, a_l$ in $\mathbb{D}$ such that the right inequality in (2.2) holds.

If the sum equals $-\infty$ and f is nonconstant, then $a_k = 0$ for some k and $\alpha(f(0)) > 0$. Then we may assume that $l = 1$ and $a_1 = 0$. We choose $a \in \mathbb{D} \setminus \{0\}$ so close to 0 that f is holomorphic in a neighbourhood of the image of $\overline{\mathbb{D}} \to \mathbb{C}$, $z \mapsto z(z-a)$, and $m_0(f)\log|a| < \beta$. If we replace $a_1 = 0$ by a and f by $z \mapsto f(z(z-a))$, then (2.2) holds.

If f is constant, then $m_a(f) = +\infty$ for all $a \in \mathbb{D}$ and the sum in (2.2) equals $-\infty$. We choose $a \in \mathbb{D} \setminus \{0\}$ so close to zero that $\alpha(x)\log|a| < \beta$. Let U be a neighbourhood of x in X with a biholomorphism $\Psi : U \to \mathbb{D}^n$, $x \mapsto 0$. Let $r > 1$ and $\Phi = \mathrm{id} \times \Psi : D_r \times U \to$

$D_r \times \mathbb{D}^n$, where $D_r = \{z \in \mathbb{C}\,;\, |z| < r\}$. Now we set $l = 1$, $a_1 = a$ and replace f by the disc

$$z \mapsto \operatorname{pr}\big(\Phi^{-1}(z, \varepsilon z(z-a), 0, \ldots, 0)\big),$$

where $\operatorname{pr} : \mathbb{C} \times X \to X$ is the projection, and $\varepsilon > 0$ is chosen so small that $\varepsilon z(z-a) \in \mathbb{D}$ if $z \in \overline{\mathbb{D}}$. Then f is nonconstant, $f(0) = f(a) = x$, and (2.2) holds. $\square$

Proof of Theorem 2.5. We have $m_z(f) = \min\{m_z(f_1), m_z(f_2)\}$ for all $f = (f_1, f_2) \in \mathcal{A}_X$. Hence

$$H_L^{\alpha_j}(f_j) = \sum_{z\in\mathbb{D}} \alpha_j(f_j(z))\, m_z(f_j)\, \log|z| \le H_L^{\alpha}(f), \qquad j = 1, 2,$$

and

$$EH_L^{\alpha}(x) \ge \max\{EH_L^{\alpha_1}(x_1), EH_L^{\alpha_2}(x_2)\}.$$

To establish the reverse inequality, we assume that $EH_L^{\alpha_j}(x_j) < \beta \in (-\infty, 0)$ for $j = 1, 2$, and show that $EH_L^{\alpha}(x) < \beta$. By Lemma 2.6, there are $f_j \in \mathcal{A}_{X_j}$ with $f_j(0) = x_j$ and $a_{jk} \in \mathbb{D}$, $k = 1, \ldots, l_j$, $j = 1, 2$, such that

$$-\infty < \sum_{k=1}^{l_j} \alpha_j(f_j(a_{jk}))\, m_{a_{jk}}(f_j)\, \log|a_{jk}| < \beta, \qquad j = 1, 2. \tag{2.3}$$

Choose f_j such that l_j is as small as possible. Then $\alpha_j(f_j(a_{jk})) = 1$ and $a_{jk} \ne 0$ for all j and k.

Assume that $|a_{j1}| \le |a_{j2}| \le \ldots$. Set

$$\mu_{jk} = m_{a_{jk}}(f_j), \quad \mu_j = \sum_{k=1}^{l_j} \mu_{jk}, \quad b_j = \prod_{k=1}^{l_j} a_{jk}^{\mu_{jk}} \ne 0, \quad \text{and} \quad c_j = a_{jl_j}.$$

Then

$$|c_j|^{\mu_j} e^{\beta} \le |b_j| < e^{\beta}. \tag{2.4}$$

The second inequality is equivalent to (2.3). To prove the first one, suppose $|b_j| < |c_j|^{\mu_j} e^{\beta}$. Then

$$\prod_{k=1}^{m_j} \left|\frac{a_{jk}}{c_j}\right|^{\mu_{jk}} < e^{\beta},$$

where $m_j < l_j$ is the smallest number with $|a_{jk}| = |c_j|$ for $k > m_j$. Hence, (2.3) holds with f_j replaced by $z \mapsto f_j(c_j z)$, a_{jk} replaced by a_{jk}/c_j, and l_j replaced by m_j, which contradicts the fact that l_j is minimal.

We define the Blaschke products

$$B_j(z) = \prod_{k=1}^{l_j} \left(\frac{a_{jk} - z}{1 - \bar{a}_{jk} z}\right)^{\mu_{jk}}.$$

Then $B_j(0) = b_j$. We may assume that $|b_1| \geq |b_2|$. By precomposing f_1 by a suitable embedding of $\mathbb{D}$ into $\mathbb{D}$ with $0 \mapsto 0$, and thereby changing a_{1k} slightly, we may assume that $B_1(0)$ is not a critical value of B_1. By Schwarz' Lemma, we still have $|b_1| \geq |b_2|$.

We may assume that $b_1 = b_2$. Indeed, if $|b_1| > |b_2|$, choose $t \in (0,1)$ with $t^{-\mu_2}|b_2| = |b_1|$. Then $|a_{2k}/t| < 1$, since by (2.4),

$$|a_{2k}|^{\mu_2} \leq |c_2|^{\mu_2} \leq |b_2| e^{-\beta} < |b_2/b_1| = t^{\mu_2}.$$

Replacing f_2 by $z \mapsto f_2(tz)$ and a_{2k} by a_{2k}/t, we get $|b_1| = |b_2|$. Finally, replacing f_2 by $z \mapsto f_2(e^{i\theta}z)$, where $e^{i\theta\mu_2} = b_2/b_1$, and replacing a_{2k} by $e^{-i\theta}a_{2k}$, we get $b_1 = b_2$.

Exactly as in [7], we obtain (possibly infinite) Blaschke products φ_j with $0 \mapsto 0$ and

$$\sigma = B_1 \circ \varphi_1 = B_2 \circ \varphi_2.$$

Now $|\sigma(0)| < e^\beta$ and $|\sigma| = 1$ almost everywhere on $\mathbb{T}$. Choose $r \in (0,1)$ so close to 1 that

$$\log|\sigma(0)| - \frac{1}{2\pi}\int_0^{2\pi} \log|\sigma(re^{i\theta})|\, d\theta < \beta.$$

By the Riesz representation formula, the left hand side equals

$$\sum_{i=1}^n m_{z_i}(\sigma(r\cdot)) \log|z_i|,$$

where $z_1, \ldots, z_n$ are the zeros of $z \mapsto \sigma(rz)$ in $\mathbb{D}$.

Now define $f \in \mathcal{A}_X$ with $f(0) = (x_1, x_2)$ by

$$f(z) = (f_1 \circ \varphi_1(rz), f_2 \circ \varphi_2(rz)), \qquad z \in \overline{\mathbb{D}}.$$

Since $\sigma(rz_i) = 0$, we have $\varphi_j(rz_i) = a_{jk_j}$ for some k_j, and

$$m_{z_i}(\sigma(r\cdot)) = \mu_{jk_j} m_{z_i}(\varphi_j(r\cdot)) = m_{a_{jk_j}}(f_j) m_{z_i}(\varphi_j(r\cdot)) = m_{z_i}(f_j \circ \varphi_j(r\cdot)).$$

Since $\alpha(f(z_i)) = \min_j \alpha_j(f_j(a_{jk_j})) = 1$, we get

$$H_L^\alpha(f) = \sum_{z\in\mathbb{D}} \alpha(f(z)) m_z(f) \log|z| \leq \sum_{i=1}^n m_{z_i}(f) \log|z_i| = \sum_{i=1}^n m_{z_i}(\sigma(r\cdot)) \log|z_i| < \beta,$$

so $EH_L^\alpha(x) < \beta$. $\square$

3. The Green function with a pole along a complex subspace

Let X be a complex manifold and A be a (closed) complex subspace of X, which is the same thing as a coherent sheaf $\mathcal{I} = \mathcal{I}_A$ of ideals in the sheaf $\mathcal{O}_X$ of holomorphic functions on X. Suppose the stalk $\mathcal{I}_p$ of $\mathcal{I}$ at $p \in X$ is generated by germs $f_1, \dots, f_m$. The plurisubharmonic functions $\max\limits_{i=1,\dots,m} \log|f_i|$ and $\log \sum\limits_{i=1}^m |f_i|$ have the same Lelong number $\nu_A(p)$ at p. This number is independent of the choice of generators. Namely, say $g_1, \dots, g_k$ also generate $\mathcal{I}_p$. Then $f_i = \sum h_{ij} g_j$ for some $h_{ij} \in \mathcal{O}_{X,p}$, so $\sum |f_i| \leq c \sum |g_j|$ on a neighbourhood of p for some constant $c > 0$. Hence, the Lelong number of $\log \sum |f_i|$ at p is no smaller than that of $\log \sum |g_j|$. Interchanging $\{f_i\}$ and $\{g_j\}$, we see that these Lelong numbers are the same.

We might call $\nu_A(p)$ the *multiplicity* of A at p. If A is smooth at p, then $\nu_A(p) = 1$, so if A is a submanifold of X, then ν_A is the characteristic function χ_A of A. We set

$$\mathcal{F}_A = \{u \in \mathrm{PSH}(X)\,;\, u \leq 0, \nu_u \geq \nu_A\}, \qquad G_A = \sup \mathcal{F}_A, \qquad k_A = k_X^{\nu_A}.$$

We call G_A the *pluricomplex Green function with a logarithmic pole along A*.

If $\mathcal{I}$ is principal, meaning that each stalk $\mathcal{I}_p$ is a principal ideal in $\mathcal{O}_{X,p}$, and $\mathcal{I}$ is neither 0 nor $\mathcal{O}_X$, then A is a hypersurface, i.e., of pure codimension 1. If $\mathcal{I}$ is reduced, i.e., A is a subvariety of X, then the converse holds. A principal coherent ideal sheaf different from the zero sheaf is nothing but an effective divisor.

Suppose now that A is an effective divisor. Then the current $[A]$ of integration over A is a closed positive (1,1)-current on X, locally defined as $\frac{1}{\pi}\mathcal{L}(\log|h|)$, where h is a local generator for $\mathcal{I}$. If $f \in \mathcal{O}(\mathbb{D}, X)$, then there is a positive Borel measure $f^*[A]$ on $\mathbb{D}$ defined locally as $\frac{1}{2\pi}\Delta \log|h \circ f|$, unless $h \circ f = 0$, in which case we set $f^*[A] = 0$. There is a *generalized Riesz functional* H_R^A associated to A, defined by the formula

$$H_R^A(f) = \int_{\mathbb{D}} \log|\cdot|\, f^*[A], \qquad f \in \mathcal{O}(\mathbb{D}, X).$$

If A is principal, say A is the divisor of a holomorphic function h on X, then H_R^A is the Riesz functional $H_R^{\log|h|}$.

3.1. Example. Let X be the unit ball in $\mathbb{C}^n$. We have $\kappa_X = \delta_X = c_X$, where c_X is the Carathéodory distance on X, and $G_a = k_X(\cdot, a) = \log\tanh \delta_X(\cdot, a)$, $a \in X$. This is in fact true on any bounded convex domain in $\mathbb{C}^n$ by work of Lempert [9]. Hence, for every submanifold A of X,

$$G_A \leq k_A = \inf_{a \in A} G_a = \log\tanh c_X(\cdot, A) = \log\tanh \kappa_X(\cdot, A).$$

Now let A be the hypersurface defined by the equation $z_1 = 0$, and write $z \in \mathbb{C}^n$ as $z = (z_1, z')$ with $z_1 \in \mathbb{C}$ and $z' \in \mathbb{C}^{n-1}$. Then

$$G_A(z) = \log \frac{|z_1|}{\sqrt{1-|z'|^2}}, \qquad z \in X.$$

Namely, if u denotes the function on the right, then $u \in \mathcal{F}_A$, so $u \le G_A$. Also, on each disc $D_c = \{z \in X\,;\, z' = c\}$ where c is a constant, u is subharmonic with $\Delta u = 2\pi\delta_{(0,c)}$ and $u|\partial D_c = 0$, and $G_A \le 0$ is subharmonic with $\Delta G_A \ge 2\pi\delta_{(0,c)}$. Hence, by the Riesz representation formula, $G_A \le u$ on D_c.

For $a \in X$, the pluricomplex Green function on X with a logarithmic pole at a is $G_a = \log|T_a|$, where T_a is any automorphism of X with $a \mapsto 0$. For $a \ne 0$, one such automorphism is given by the formula

$$T_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle},$$

where P_a is the orthogonal projection onto the linear space spanned by a, Q_a is the orthogonal projection onto its orthogonal complement, and $s_a = \sqrt{1-|a|^2}$. If $a = (0, a')$ and $z' = a'$, then $P_a(z) = (0, z') = a$ and $Q_a(z) = (z_1, 0)$, so

$$T_a(z) = \Big(\frac{-z_1}{\sqrt{1-|z'|^2}}, 0\Big),$$

and

$$G_A(z) = G_{(0,z')}(z) = \inf_{a\in A} G_a(z).$$

This equality is in fact very exceptional, as Example 3.5 will indicate.

3.2. Proposition. *Let A be an effective divisor in a complex manifold X. If f is a holomorphic function generating the ideal sheaf of A on an open set U, then the plurisubharmonic function $G_A - \log|f|$ on $U \setminus A$ is locally bounded above on $A \cap U$, so it extends to a plurisubharmonic function on U. Hence,*

$$\mathcal{L}(G_A) \ge \pi[A].$$

Proof. Let $p \in A \cap U$. We will show that $G_A - \log|f|$ is locally bounded above at p. We may assume that p is a smooth point of the reduction of A, since plurisubharmonic functions always extend across subvarieties of codimension at least 2. By applying a local biholomorphism, we may assume that $p = 0 \in \mathbb{C}^n$, that the unit polydisc P centred at p is in U, and that the reduction of A is given by the equation $z_1 = 0$ in P. If $z = (z_1, z') \in P$, then the analytic disc $\zeta \mapsto (\zeta, z')$ maps 0 to $(0, z')$ and z_1 to z, so

$$k_X(z, (0, z')) = \log\tanh\delta_X(z, (0, z')) \le \log\tanh\varrho_{\mathbb{D}}(0, z_1) = \log|z_1|,$$

and

$$G_A(z) \le k_A(z) \le \nu_A(0,z')k_X(z,(0,z')) \le \nu_A(0,z')\log|z_1|.$$

At p, the germ of f is the product of z_1^s and a unit, where $s = \nu_A(p) = \nu_{\log|f|}(p)$ is the order of the zero of f at p, which equals the order of vanishing of f along the reduction of A at p, and this is the same at every point of $A \cap P$. Hence, for $z = (z_1, z') \in P$, we have $\nu_A(0,z') = s$, so

$$G_A(z) - \log|f(z)| \le s\log|z_1| - \log|f(z)|,$$

and this is bounded above near p. $\square$

If A is not a divisor and $f_1, \dots, f_m$, $m \ge 2$, are local generators for the ideal sheaf of A, then $G_A - \log \sum_{i=1}^{m} |f_i|$ need not be locally bounded above. This is shown by the coherent ideal sheaf on $\mathbb{D} \times \mathbb{D}$ generated by z_1 and z_2^2.

3.3. Theorem. *Let X be a relatively compact domain in a Stein manifold Y, and let A be the intersection with X of a complex subspace B of Y. Then G_A is locally bounded and maximal on $X \setminus A$,*

$$\nu_{G_A} = \nu_A,$$

and

$$\begin{aligned} G_A &= EH_L^{\nu_A} = EH_P^{k_A} = \sup\{u \in \mathrm{PSH}(X)\,;\, u \le k_A\} \\ &= \sup\{u \in \mathrm{PSH}(X)\,;\, u \le \inf_{a\in A} \nu_A(a) G_a\}. \end{aligned}$$

If X has a strong plurisubharmonic barrier at $p \in \partial X \setminus B$, then

$$\lim_{x \to p} G_A(x) = 0.$$

If A is a divisor, then

$$G_A = EH_R^A = \sup\{u \in \mathrm{PSH}(X)\,;\, u \le 0, \mathcal{L}(u) \ge \pi[A]\}.$$

The hypotheses of the theorem are satisfied when X is a smoothly bounded B-regular domain in $\mathbb{C}^n$ and A is the intersection with X of a complex subspace of a neighbourhood of $\overline{X}$, because $\overline{X}$ has a Stein neighbourhood basis [13].

Proof. Since Y is Stein, each stalk of the ideal sheaf $\mathcal{I}_B$ is generated by global sections of $\mathcal{I}_B$ by Cartan's Theorem A. Since X is relatively compact in Y, there are finitely many holomorphic functions $f_1, \dots, f_m \in \mathcal{I}_B(Y)$ which generate all the stalks $\mathcal{I}_{B,p}$, $p \in X$. We may assume that $|f_i| < 1$ on X. Let $u = \max_{i=1,\dots,m} \log|f_i|$ on X. Then $u \in \mathrm{PSH}(X)$, $u \leq 0$, and $\nu_u = \nu_A$. In particular, $u \in \mathcal{F}_A$, so $u \leq G_A$. This shows that G_A is locally bounded in $X \setminus A$, and hence maximal there by Proposition 2.3, and $\nu_{G_A} = \nu_A$.

The next four equations follow from Theorem 2.2. The statement about boundary limits follows from Proposition 2.4.

Let $f \in \mathcal{A}_X$, and suppose $f(0) \notin A$, so $f^{-1}(A)$ is finite. Let $b \in \mathbb{D}$, let h be a local generator for $\mathcal{I}_A$ on a neighbourhood U of $f(b)$, and let $V \subset \mathbb{D}$ be a neighbourhood of b such that $f(V) \subset U$. If $z \in V$ and $h(f(z)) = 0$, then $m_z(h \circ f)$ is no smaller than $m_z(f)$ times the order of the zero of h at $f(z)$, which is the Lelong number $\nu_A(f(z))$ of $\log|h|$ at $f(z)$. Hence, we have

$$f^*[A] = \frac{1}{2\pi}\Delta \log|h \circ f| = \sum_{z \in V \cap f^{-1}(A)} m_z(h \circ f)\delta_z \geq \sum_{z \in V} \nu_A(f(z)) m_z(f)\delta_z$$

on V. Integrating the function $\log|\cdot|$ over any measurable subset S of V with respect to these measures gives

$$\int_S \log|\cdot|\, f^*[A] \leq \sum_{z \in S} \nu_A(f(z)) m_z(f) \log|z|.$$

This shows that $H_R^A(f) \leq H_L^{\nu_A}(f)$, so $G_A = EH_L^{\nu_A} \geq EH_R^A$ on $X \setminus A$. On A, both envelopes equal $-\infty$.

Let

$$\mathcal{F}_R^A = \{u \in \mathrm{PSH}(X)\,;\, u \leq 0, \mathcal{L}(u) \geq \pi[A]\}.$$

By Proposition 3.2, $G_A \in \mathcal{F}_R^A$, so $G_A \leq \sup \mathcal{F}_R^A$.

Now take $u \in \mathcal{F}_R^A$ and $f \in \mathcal{A}_X$. If h is a local generator for $\mathcal{I}_A$, then $\mathcal{L}(\log|h|) = \pi[A] \leq \mathcal{L}(u)$, so $2\pi f^*[A] = \Delta \log|h \circ f| \leq \Delta(u \circ f)$. Hence, by the Riesz representation formula,

$$u(f(0)) = \frac{1}{2\pi}\int_{\mathbb{T}} u \circ f \, d\lambda + \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\, \Delta(u \circ f) \leq \int_{\mathbb{D}} \log|\cdot|\, f^*[A] = H_R^A(f).$$

This shows that $\sup \mathcal{F}_R^A \leq EH_R^A$, so

$$G_A \leq \sup \mathcal{F}_R^A \leq EH_R^A \leq EH_L^{\nu_A} = G_A. \quad \square$$

3.4. Example. Let $X = \mathbb{D} \times \mathbb{D}$ be the unit bidisc in $\mathbb{C}^2$. Then X is hyperconvex but not B-regular. Let $A = \{z_1 = 0\}$. Let $\alpha_1 = \chi_{\{0\}}$ and $\alpha_2 = \chi_{\mathbb{D}}$ on $\mathbb{D}$. Then

$$\alpha(z) = \min\{\alpha_1(z_1), \alpha_2(z_2)\} = \chi_A(z)$$

for $z=(z_1,z_2)\in X$. By Theorem 2.5, or as in Example 3.1,

$$G_A(z)=G_\alpha(z)=\max\{G_{\alpha_1}(z_1),G_{\alpha_2}(z_2)\}=\max\{\log|z_1|,-\infty\}=\log|z_1|.$$

Clearly, G_A does not go to zero at all points of $\partial X\setminus A$.

3.5. Example. Let X be the unit ball in $\mathbb{C}^n$. It is well known that for $a\in X$, the Lelong functional $H_L^{\chi_{\{a\}}}$, whose envelope is the pluricomplex Green function G_a with a logarithmic pole at a, has essentially unique extremal discs, whose images are complex geodesics in X. More precisely, for $x\in X$, $x\neq a$, there is $f\in\mathcal{A}_X$ with $f(0)=x$, unique modulo precomposition by a rotation, such that $G_a(x)=H_L^{\chi_{\{a\}}}(f)$, namely $f(z)=T(z,0,\dots,0)$, where T is an automorphism of X with $T(0)=x$ and $T^{-1}(a)\in\mathbb{D}\times\{0\}^{n-1}$.

Now let A be a submanifold of X and $x\notin A$. Then $\inf\limits_{a\in A}G_a(x)$ is actually a minimum, and $G_A(x)=\inf\limits_{a\in A}G_a(x)$ if and only if there is $b\in A$ and an extremal disc $f\in\mathcal{A}_X$ with $f(0)=x$ such that

$$\log|f^{-1}(b)|=G_b(x)=G_A(x)\le H_L^{\chi_A}(f)=\sum_{f(z)\in A}\log|z|.$$

This implies that

$$f(\mathbb{D})\cap A=\{b\}.$$

In other words, if $G_A(x)=\inf\limits_{a\in A}G_a(x)$, then the complex geodesic realizing the hyperbolic distance from x to A intersects A in only one point.

There is no shortage of counterexamples to this. When $n=2$, take for instance the smooth curve

$$A=X\cap\{(z,w)\in\mathbb{C}^2\,;\,z^2+w^2=c\},\qquad 0<|c|<1,$$

which is connected and intersects each complex geodesic through the origin in either zero or two points.

This proves the following. There is a smooth curve in $\mathbb{C}^2$ whose intersection A with the unit ball X is nonempty and connected such that:

(1) $G_A\neq\inf\limits_{a\in A}G_a$,
(2) the functions $\inf\limits_{a\in A}G_a$, k_A, $\log\tanh c_X(\cdot,A)$, and $\log\tanh\kappa_X(\cdot,A)$ (which are in fact all equal) are not plurisubharmonic on X, and
(3) $\log\tanh c_X(\cdot,A)$ is not dominated by G_A, even though $\log\tanh c_X(\cdot,a)\le G_a$ for every $a\in X$.

Furthermore, the domain $Y = X \setminus A$ has the same Carathéodory distance as X because A is removable for bounded holomorphic functions. The strongly convex part of ∂Y is infinitely distant from any point of Y, so $c_X(y, A)$ is in fact the Carathéodory distance from $y \in Y$ to ∂Y. Hence, Y is an example of a bounded pseudoconvex domain in $\mathbb{C}^n$ such that log tanh of the Carathéodory distance to the boundary is not plurisubharmonic.

Finally, we shall establish a uniqueness property and a continuity property of the pluricomplex Green function with a logarithmic pole along a complex subspace A. Although we expect these results to hold in general, at present we can only prove them when A is a principal divisor.

Recall that if X is a Stein space with $H^2(X, \mathbb{Z}) = 0$, then the second Cousin problem can be solved on X, so every divisor on X is principal.

3.6. Uniqueness Theorem. *Let X be a relatively compact domain in a complex manifold. Let A be the divisor of a holomorphic function f on X. If u is a negative plurisubharmonic function on X such that*

(1) *u is locally bounded and maximal in $X \setminus A$,*
(2) *for every $\varepsilon > 0$ there is a compact subset K of X such that $G_A \leq u + \varepsilon$ on $X \setminus K$, and*
(3) *every point in A has a neighbourhood U with a constant $C > 0$ such that*

$$\log |f| - C \leq u \leq \log |f| + C \qquad \text{on } U,$$

then $u = G_A$.

Proof. By Proposition 3.2, $G_A - \log |f|$ extends to a plurisubharmonic function $\tilde{G}_A$ on X. The function $\tilde{u} = u - \log |f|$ is plurisubharmonic on $X \setminus A$ and locally bounded on X, so it extends to a locally bounded plurisubharmonic function on X. Since u is maximal on $X \setminus A$, so is $\tilde{u}$. Since A is pluripolar, $\tilde{u}$ is maximal on X [5, Prop. 4.6.4], so by (2), $\tilde{G}_A \leq \tilde{u}$ and $G_A \leq u$. Finally, by (3), $u \in \mathcal{F}_A$, so $u = G_A$. $\square$

The following lemma solves the Dirichlet problem for the Monge-Ampère operator on a Kähler manifold, without continuity of the solution. On a Stein manifold, the solution is continuous by Lemma 3.8.

3.7. Lemma. *Let X be a relatively compact domain in a Kähler manifold (e.g. a Stein manifold) with a strong plurisubharmonic barrier at every boundary point. Let $\varphi : \partial X \to \mathbb{R}$ be a continuous function. Then there is a unique maximal plurisubharmonic function u on X such that*

$$\lim_{x \to p} u(x) = \varphi(p) \qquad \text{for every } p \in \partial X,$$

namely $u = \sup \mathcal{F}$, where

$$\mathcal{F} = \{v \in \mathrm{PSH}(X) \,;\, v^*|\partial X \leq \varphi\}.$$

Here, v^* denotes the upper semicontinuous function $p \mapsto \limsup_{x \to p} v(x)$ on $\overline{X}$.

The Kähler condition provides a link between pluripotential theory and real potential theory. It implies that the Laplacian is the trace of the Levi form [16, p. 90], so plurisubharmonic functions are subharmonic with respect to the associated Riemannian metric. Here, this has the important consequence that the class $\{v \in \mathcal{F}\,;\, v \geq \min \varphi\}$ is compact. We do not know if this is true without the Kähler condition.

Proof. By hypothesis, X is regular with respect to the Laplacian, so there is a continuous function h on $\overline{X}$, harmonic on X, with $h|\partial X = \varphi$. Then $\mathcal{F}$ coincides with the class $\{v \in \mathrm{PSH}(X)\,;\, v \leq h\}$. Hence, $u = \sup \mathcal{F} \leq h$, so $u^* \leq h$ by continuity of h. Since u^* is plurisubharmonic, $u^* \in \mathcal{F}$, so $u^* = u$. This shows that $u \in \mathcal{F}$.

Let U be a relatively compact domain in X, and let v be plurisubharmonic on X such that $v \leq u$ on ∂U. Let

$$w = \begin{cases} \max\{u, v\} & \text{on } U, \\ u & \text{on } X \setminus U. \end{cases}$$

Then w is plurisubharmonic on X and $w^*|\partial X = u^*|\partial X \leq \varphi$, so $w \leq u$. Hence, $v \leq u$ on U. This shows that u is maximal, and uniqueness follows.

Now let $p \in \partial X$ and $\beta < \varphi(p)$. Let w be a strong barrier at p. Choose a neighbourhood V of p such that $\varphi > \beta$ in $\partial X \cap V$, and choose $c > 0$ such that $\beta + c \sup_{X \setminus V} w < \min \varphi$. Then $v = \beta + cw \in \mathcal{F}$, so $v \leq u$, and

$$\beta = \lim_{x \to p} v(x) \leq \liminf_{x \to p} u(x) \leq \varphi(p).$$

Since β is arbitrary, this shows that $\lim_{x \to p} u(x) = \varphi(p)$. $\square$

The next lemma was proved by Walsh [15] for domains in $\mathbb{C}^n$, and generalized to Banach spaces by Lelong [8]. Our proof is based on Lelong's argument.

In a metric space (Y, d), we let $B(x, \varrho)$ denote the open ball with centre x and radius ϱ, and for any $A \subset Y$ we let $A_\varrho = \{x \in A\,;\, d(x, \partial A) > \varrho\}$.

3.8. Lemma. *Let X be a relatively compact domain in a manifold Y. Assume that there exists a metric defining the topology on Y such that for every $v \in \mathrm{PSH}(X)$ and $r > 0$, there exists a continuous plurisubharmonic function $v_r : X_r \to \mathbb{R}$ with*

$$v(x) \leq v_r(x) \leq \sup_{\overline{B}(x,r)} v, \qquad x \in X_r.$$

This holds in particular if Y is Stein. Let $h : \overline{X} \to \mathbb{R}$ be continuous, and

$$u = \sup\{v \in \mathrm{PSH}(X)\,;\, v \leq h\}.$$

If u^* is continuous on ∂X, then u is continuous on X.

Proof. Since h is continuous on the compact set $\overline{X}$, the function u is plurisubharmonic and real-valued. It suffices to prove that for every $a \in X$ and $\varepsilon > 0$, there exists a neighbourhood U of a such that $u(a) - \varepsilon < u$ on U.

Since h is uniformly continuous on the compact set $\overline{X}$, and u^* is continuous on the compact set ∂X, there is $\delta > 0$ such that $a \in X_\delta$,

$$|h(x) - h(y)| < \tfrac{1}{2}\varepsilon \quad \text{for all } (x, y) \in \overline{X} \times \overline{X} \text{ with } d(x, y) < \delta, \text{ and} \tag{3.1}$$

$$|u^*(x) - u^*(y)| < \tfrac{1}{4}\varepsilon \quad \text{for all } (x, y) \in \partial X \times \overline{X} \text{ with } d(x, y) < 3\delta. \tag{3.2}$$

By assumption, there exists a continuous plurisubharmonic $v : X_\delta \to \mathbb{R}$ such that $u(x) \leq v(x) \leq \sup_{\overline{B}(x,\delta)} u$ for $x \in X_\delta$. We define w on X by

$$w = \begin{cases} \max\{v - \tfrac{1}{2}\varepsilon, u\} & \text{on } X_\delta, \\ u & \text{on } X \setminus X_\delta. \end{cases}$$

If $x \in X_\delta \setminus X_{2\delta}$, then $\delta < d(x, \partial X) \leq 2\delta$. There exists $b \in \partial X$ such that $d(b, x) \leq 2\delta$, so for every $y \in B(x, \delta)$ we have $d(b, y) < 3\delta$, and (3.2) implies that

$$|u(x) - u(y)| < |u^*(b) - u(x)| + |u^*(b) - u(y)| < \tfrac{1}{2}\varepsilon,$$

so

$$v(x) - \tfrac{1}{2}\varepsilon \leq \sup_{\overline{B}(x,\delta)} u - \tfrac{1}{2}\varepsilon < u(x).$$

Hence, $w = u$ on $X \setminus X_{2\delta}$, so $w \in \mathrm{PSH}(X)$. By (3.1),

$$v(x) - \tfrac{1}{2}\varepsilon \leq \sup_{\overline{B}(x,\delta)} u - \tfrac{1}{2}\varepsilon \leq \sup_{B(x,\delta)} h - \tfrac{1}{2}\varepsilon < h(x), \qquad x \in X_\delta,$$

so $w \leq h$. Hence, $v - \tfrac{1}{2}\varepsilon \leq w \leq u$ on X_δ. There exists a neighbourhood U of a in X_δ such that $v(a) < v(x) + \tfrac{1}{2}\varepsilon$ for all $x \in U$, and then

$$u(a) - \varepsilon \leq v(a) - \varepsilon < v(x) - \tfrac{1}{2}\varepsilon \leq u(x), \qquad x \in U.$$

Assume now that Y is Stein. We may assume that Y is a closed submanifold of $\mathbb{C}^N$. Let $\mathbb{C}^N$ have the euclidean metric and Y have the induced metric. By Siu [14, Main Thm. and Cor. 1], there is a Stein neighbourhood V of Y in $\mathbb{C}^N$ and a holomorphic retraction $\sigma : V \to Y$. Let $v \in \mathrm{PSH}(X)$ and $r > 0$. Then $\tilde{v} = v \circ \sigma$ is plurisubharmonic on $W = \sigma^{-1}(X)$. Since X is relatively compact, there is $\varrho \in (0, r)$ such that $\sigma(\overline{B}(x, \varrho)) \subset B(x, r)$ for all $x \in X$. We choose a nonnegative radially symmetric $\chi \in C_0^\infty(\mathbb{C}^N)$ with Lebesgue integral 1 and support in $B(0, \varrho)$. Then the convolution $\tilde{v} * \chi$ defines a smooth plurisubharmonic function on W_ϱ. Now take $v_r = \tilde{v} * \chi | X_r$. $\square$

Being plurisubharmonic, G_A is quasi-continuous [5, Thm. 3.5.5], but a quasi-continuous function may be discontinuous everywhere.

3.9. Theorem. *Let X be a relatively compact domain in a Stein manifold with a strong plurisubharmonic barrier at every boundary point. Let A be the divisor of a holomorphic function f on X which extends to a continuous function on $\overline{X}$. Then the set of points in X at which G_A is discontinuous is pluripolar.*

We are unable to prove that G_A is continuous, nor do we have counterexamples. Continuity of the single-pole Green function on a bounded hyperconvex domain in a Stein manifold was proved by Demailly [1]. Continuity of the Green function with finitely many weighted poles on a bounded hyperconvex domain in a Banach space was proved by Lelong [8].

Proof. By Proposition 3.2, $G_A - \log|f|$ extends to a plurisubharmonic function $\tilde{G}_A$ on X. By Lemmas 3.7 and 3.8, there are continuous maximal plurisubharmonic functions v_j, $j \in \mathbb{N}$, on X such that

$$\lim_{x \to p} v_j(x) = \min\{-\log|f(p)|, j\} \qquad \text{for every } p \in \partial X.$$

Then $v_j + \log|f| \in \mathcal{F}_A$, so $v_j \leq \tilde{G}_A$, and the increasing sequence (v_j) is locally bounded above. Hence, (v_j) converges pointwise to a lower semicontinuous function $v : X \to \mathbb{R}$, whose upper semicontinuous regularization v^* is plurisubharmonic on X, and $v^* \leq \tilde{G}_A$. The subset N of X where $v < v^*$ is pluripolar. Furthermore, since the v_j are maximal, so is v^* [5, Thm. 3.6.1].

For $p \in \partial X$, we have

$$\liminf_{x \to p} v(x) \geq \liminf_{x \to p} v_j(x) = \min\{-\log|f(p)|, j\}$$

for each j, so letting $j \to \infty$ we get

$$\limsup_{x \to p} \tilde{G}_A(x) \leq -\log|f(p)| \leq \liminf_{x \to p} v(x) \leq \liminf_{x \to p} v^*(x).$$

Since v^* is maximal, this shows that $\tilde{G}_A \leq v^*$, so $\tilde{G}_A = v^*$.

For $p \in X$,

$$v(p) = \liminf_{x \to p} v(x) \leq \liminf_{x \to p} \tilde{G}_A(x) \leq \limsup_{x \to p} \tilde{G}_A(x) = \tilde{G}_A(p) = v^*(p),$$

so $\tilde{G}_A$ is continuous at p if $p \notin N$. Hence, G_A is continuous at all points outside the pluripolar set $N \setminus A$. (Note that this is stronger than saying that the restriction of G_A to the complement of a pluripolar set is continuous.) $\square$

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