PLURISUBHARMONIC FUNCTIONS AND ANALYTIC DISCS ON MANIFOLDS

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25 May 1998

Abstract. Let X be a complex manifold and $\mathcal{A}_X$ be the family of maps $\overline{\mathbb{D}} \to X$ which are holomorphic in a neighbourhood of the closure of the unit disc $\mathbb{D}$. Such maps are called *(closed) analytic discs* in X. A *disc functional* on X is a map $H : \mathcal{A}_X \to \mathbb{R} \cup \{-\infty\}$. The *envelope* of H is the function $EH : X \to \mathbb{R} \cup \{-\infty\}$, $x \mapsto \inf\{H(f)\,;\, f \in \mathcal{A}_X, f(0) = x\}$. Through work of Evgeny Poletsky, it has transpired that certain disc functionals on domains in $\mathbb{C}^n$ have plurisubharmonic envelopes.

There are essentially only three known classes of disc functionals with plurisubharmonic envelopes. The Poisson functional associated to an upper semi-continuous function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ takes $f \in \mathcal{A}_X$ to $\frac{1}{2\pi}\int_{\mathbb{T}} \varphi \circ f \, d\lambda$, where λ is the arc length measure on the unit circle $\mathbb{T}$. The Riesz functional associated to a plurisubharmonic function v on X takes f to $\frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\,\Delta(v \circ f)$, where $\Delta(v \circ f)$ is considered as a positive Borel measure on $\mathbb{D}$, equal to zero if $v \circ f = -\infty$. The Lelong functional associated to a non-negative function α on X takes f to $\sum_{z \in \mathbb{D}} \alpha(f(z))\, m_z(f) \log|z|$, where $m_z(f)$ denotes the multiplicity of f at z.

Define $\mathcal{P}$ as the class of complex manifolds X for which there exists a finite sequence of complex manifolds and holomorphic maps $X_0 \xrightarrow{h_1} \dots \xrightarrow{h_m} X_m = X$, $m \geq 0$, where X_0 is a domain in a Stein manifold and each h_i is either a covering (unbranched and possibly infinite) or a finite branched covering (a proper holomorphic surjection with finite fibres). The class $\mathcal{P}$ is closed under taking products and passing to subdomains. Besides domains in Stein manifolds, $\mathcal{P}$ contains for instance all Riemann surfaces and all covering spaces of projective manifolds.

The main result of the paper is that if X is a manifold in $\mathcal{P}$, then the Poisson functional, the Riesz functional associated to a continuous v, and the Lelong functional associated to a generic α have plurisubharmonic envelopes. In each case, the envelope is the supremum of a naturally defined class of plurisubharmonic functions.

2000 *Mathematics Subject Classification*. Primary: 32F05; secondary: 32C10, 14E20.

The first-named author was supported in part by the Natural Sciences and Engineering Research Council of Canada, and by a VPRSC grant from the University of Western Ontario.

Typeset by $\mathcal{AMS}$-TEX

1. Introduction

Let X be a complex manifold and $\mathcal{A}_X$ be the family of maps $f : \overline{\mathbb{D}} \to X$ which are holomorphic in a neighbourhood of the closure $\overline{\mathbb{D}}$ of the unit disc $\mathbb{D}$. Such maps are called *(closed) analytic discs* in X. A *disc functional* on X is a map $H : \mathcal{A}_X \to \mathbb{R} \cup \{-\infty\}$. The *envelope* of H is the function $EH : X \to \mathbb{R} \cup \{-\infty\}$ defined by the formula

$$EH(x) = \inf\{H(f)\,;\, f \in \mathcal{A}_X, f(0) = x\}, \qquad x \in X.$$

Through work of Poletsky, it has transpired that certain disc functionals on domains in $\mathbb{C}^n$ have *plurisubharmonic* envelopes. In this paper, we will prove that three classes of disc functionals have plurisubharmonic envelopes for a large collection of manifolds. This result can be viewed as a new method for constructing plurisubharmonic functions on manifolds.

The three functionals we shall consider are the following.

Let $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous function. Define the functional $H_1 = H_1^\varphi$ by the formula

$$H_1(f) = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi \circ f \, d\lambda, \qquad f \in \mathcal{A}_X,$$

where λ is the arc length measure on the unit circle $\mathbb{T}$. We call H_1 the *Poisson functional*.

Let v be a plurisubharmonic function on X. We define the functional $H_2 = H_2^v$ as follows. If $f \in \mathcal{A}_X$ and $v \circ f$ is not identically $-\infty$, then

$$H_2(f) = \frac{1}{2\pi} \int_{\mathbb{D}} \log|\cdot|\, \Delta(v \circ f),$$

where $\Delta(v \circ f)$ is considered as a positive Borel measure on $\mathbb{D}$. If $f \in \mathcal{A}_X$ and $v \circ f = -\infty$, then we set $H_2(f) = 0$. We call H_2 the *Riesz functional*.

Let α be a non-negative function on X, and define the functional $H_3 = H_3^\alpha$ by the formula

$$H_3(f) = \sum_{z \in \mathbb{D}} \alpha(f(z))\, m_z(f)\, \log|z|, \qquad f \in \mathcal{A}_X.$$

The sum, which may be uncountable, is defined as the infimum of its finite partial sums. Here, $m_z(f)$ denotes the multiplicity of f at z, defined in the following way. If f is constant, set $m_z(f) = \infty$. If f is non-constant, let (U, ζ) be a coordinate neighbourhood on X with $\zeta(f(z)) = 0$. Then there exists an integer m such that

$$\zeta(f(w)) = (w - z)^m g(w)$$

where $g : V \to \mathbb{C}^n$ is a map defined in a neighbourhood V of z with $g(z) \neq 0$. The number m, which is independent of the choice of local coordinates, is the multiplicity of f at z. We call H_3 the *Lelong functional*.

Define $\mathcal{P}$ as the class of complex manifolds X for which there exists a finite sequence of complex manifolds and holomorphic maps

$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \dots \xrightarrow{h_m} X_m = X, \qquad m \geq 0,$$

where X_0 is a domain in a Stein manifold and each h_i, $i = 1, \dots, m$, is either a covering (unbranched and possibly infinite) or a finite branched covering (i.e., a proper holomorphic surjection with finite fibres). The class $\mathcal{P}$ is closed under taking products and passing to subdomains. Besides domains in Stein manifolds, $\mathcal{P}$ contains for instance all Riemann surfaces and all covering spaces of projective manifolds. By a projective manifold we mean a complex submanifold (or a smooth algebraic subvariety) of complex projective space $\mathbb{P}^k$ of some dimension k.

We say that a non-negative function α on a manifold X in $\mathcal{P}$ is *admissible* if there exists a sequence of maps as above such that $\alpha^{-1}[c, \infty) \setminus B$ is dense in $\alpha^{-1}[c, \infty)$ in the analytic Zariski toplogy on X for every $c > 0$, where

$$B = \bigcup_{i=1}^{m} (h_m \circ \dots \circ h_{i+1})(B_i),$$

and B_i denotes the (possibly empty) branch locus of h_i. This holds in particular if $\alpha = 0$ on B. Clearly, if X is a domain in a Stein manifold, then every non-negative function on X is admissible. We will show that if X is a covering space over a projective manifold, then every non-negative function which vanishes outside a countable set is admissible.

Our main results may be summarized as follows.

Main Theorem. *Let X be a manifold in $\mathcal{P}$. If φ is an upper semi-continuous function on X, then EH_1^φ is plurisubharmonic, and*

$$EH_1^\varphi = \sup\{u \in \mathrm{PSH}(X)\,;\, u \leq \varphi\}.$$

If v is a continuous plurisubharmonic function on X, then EH_2^v is plurisubharmonic, and

$$EH_2^v = \sup\{u \in \mathrm{PSH}(X)\,;\, u \leq 0,\, \mathcal{L}(u) \geq \mathcal{L}(v)\}.$$

If α is an admissible non-negative function on X, then EH_3^α is plurisubharmonic, and

$$EH_3^\alpha = \sup\{u \in \mathrm{PSH}(X)\,;\, u \leq 0, \nu_u \geq \alpha\}.$$

Here, $\mathrm{PSH}(X)$ denotes the cone of all plurisubharmonic functions on X. In our terminology, the constant function $-\infty$ is considered plurisubharmonic. A continuous plurisubharmonic function is assumed to take values in $\mathbb{R}$. If $u \in \mathrm{PSH}(X)$, then we denote by $\mathcal{L}(u)$ the Levi form $i\partial\overline{\partial}u$ of u, which is a closed, positive (1,1)-current on X, and we denote the Lelong number of u at $x \in X$ by $\nu_u(x)$. Recall that the Lelong number is a biholomorphic invariant, and if u is plurisubharmonic in a neighbourhood of 0 in $\mathbb{C}^n$, then

$$\nu_u(0) = \lim_{r \to 0} \frac{\sup_{|z|=r} u(z)}{\log r}.$$

We agree that $\mathcal{L}(-\infty) = 0$, and $\nu_{-\infty} = +\infty$. We denote the Euclidean norm in $\mathbb{C}^n$ by $|\cdot|$. We let D_r denote the open disc in $\mathbb{C}$ with centre 0 and radius r. If $A \subset \mathbb{C}$, then we set $A^* = A \setminus \{0\}$. We consider all manifolds to be connected by definition.

If u is plurisubharmonic in a neighbourhood of 0 in $\mathbb{C}^n$, then $\nu_u(0) \geq 1$ if and only if $u - \log|\cdot|$ is bounded above near 0. Hence, $\sup\{u \in \mathrm{PSH}(X)\,;\, u \leq 0, \nu_u \geq \alpha\}$ is the pluricomplex Green function of X with a pole at p when $\alpha = 1$ at p and $\alpha = 0$ elsewhere. More generally, when α has discrete support, the supremum is a pluricomplex Green function with weighted poles.

The supremum of a class of plurisubharmonic functions is not always plurisubharmonic, for it need not be upper semi-continuous. However, as we will show later, the suprema in the Main Theorem are upper semi-continuous and hence plurisubharmonic for every manifold X, every upper semi-continuous function φ on X, every plurisubharmonic function v on X, and every non-negative function α on X.

Suppose Φ is a map which associates to a disc $f \in \mathcal{A}_X$ a pair (μ_f, ν_f), where μ_f is a positive Borel measure on $\mathbb{D}$, and ν_f is a real Borel measure on $\mathbb{T}$ with finite positive part. Then Φ defines a disc functional H on X by the formula $H(f) = v_f(0)$, $f \in \mathcal{A}_X$, where v_f is the subharmonic function on $\mathbb{D}$ given by the Riesz representation formula

$$v_f(z) = \int_{\mathbb{D}} G(z, \cdot)\, d\mu_f + \int_{\mathbb{T}} P(z, \cdot)\, d\nu_f,$$

where G denotes the Green function and P denotes the Poisson kernel for $\mathbb{D}$,

$$G(z,\zeta) = \frac{1}{2\pi} \log \left| \frac{z-\zeta}{1-z\bar{\zeta}} \right| \qquad \text{and} \qquad P(z,\zeta) = \frac{1-|z|^2}{2\pi|z-\zeta|^2}.$$

Each of our three functionals is given by a map Φ in this way. For H_1, we have $\mu_f = 0$ and ν_f is the arc length measure multiplied by the restriction of $\varphi \circ f$ to $\mathbb{T}$. For H_2, we have $\mu_f = \Delta(v \circ f)$ and $\nu_f = 0$. Finally, for H_3, we have

$$\mu_f = 2\pi \sum_{z \in \mathbb{D}} \alpha(f(z))\, m_z(f)\, \delta_z$$

and $\nu_f = 0$, where δ_z is the Dirac measure at the point z. It is easy to show that μ_f is a well defined Borel measure on $\mathbb{D}$.

Abstracting from these three examples, Poletsky [1991, 1993] introduced the notion of a *holomorphic current*. (For earlier work, see Poletsky and Shabat [1989, Section 2.9]). He proved that if X is a domain in $\mathbb{C}^n$ and Φ is a holomorphic current with certain additional properties, then the functional H defined by Φ as above has a plurisubharmonic envelope. Poletsky's theorem is quite involved, both in its statement and its proof, so in our search for generalizations to disc functionals on manifolds, we decided to concentrate on the existing examples: the Poisson, Riesz, and Lelong functionals. It is a challenging problem to find a simply and abstractly defined class of disc functionals with plurisubharmonic envelopes that contains these three examples.

The paper is organized as follows. In Section 2, we prove in detail that the Poisson functional has plurisubharmonic envelopes on domains in Stein manifolds. We isolate a weaker sufficient condition for the theorem to hold on a complex manifold. In Section 3, we show that the Poisson functional has plurisubharmonic envelopes on manifolds in the class $\mathcal{P}$, and we discuss the scope of $\mathcal{P}$. In Section 4, we study the Riesz functional, and in Section 5 the Lelong functional. In Section 6, we consider the case of compact manifolds. Section 7 contains some final remarks.

Acknowledgements. We would like to thank Evgeny A. Poletsky for helpful conversations. In writing this paper, we have relied heavily on his pioneering work. We would also like to thank Donu Arapura, Frédéric Campana, Norm Levenberg, and Ahmed Zeriahi for valuable discussions.

2. The Poisson functional

Let X be a complex manifold. In this section, we let H be the Poisson functional H_1^φ defined by the formula

$$H_1^\varphi(f) = \frac{1}{2\pi}\int_{\mathbb{T}} \varphi\circ f\, d\lambda, \qquad f\in\mathcal{A}_X,$$

where $\varphi : X\to\mathbb{R}\cup\{-\infty\}$ is an upper semi-continuous function. Considering discs whose image is a point, we see that $EH\leq\varphi$.

We first observe that if EH is plurisubharmonic, then it is given as the supremum of a naturally defined class of plurisubharmonic functions.

2.1. Proposition. *Let*

$$\mathcal{F} = \{v\in \mathrm{PSH}(X)\,;\, v\leq\varphi\}.$$

Then $\sup\mathcal{F}\in\mathrm{PSH}(X)$ *and* $\sup\mathcal{F}\leq EH$. *Furthermore,* $EH\in\mathrm{PSH}(X)$ *if and only if* $EH\in\mathcal{F}$*, and then* $EH=\sup\mathcal{F}$*.*

Let us recall that if X is a complex manifold, then the L^1_{loc} topology on $\mathrm{PSH}(X)\setminus\{-\infty\}$ is the same as the weak topology induced from the space $\mathcal{D}'(X)$ of distributions on X, and this topology is induced by a complete metric on $\mathrm{PSH}(X)\setminus\{-\infty\}$. See Hörmander [1994, Theorem 3.2.12]. If $\mathcal{F}\subset\mathrm{PSH}(X)\setminus\{-\infty\}$ is compact, then $\sup\mathcal{F}$ is upper semi-continuous and hence plurisubharmonic. See Sigurdsson [1991, Proposition 2.1]. Therefore, the supremum of a subset of $\mathrm{PSH}(X)$ is plurisubharmonic if it is $-\infty$ or it can be expressed as the supremum of a compact subset of $\mathrm{PSH}(X) \setminus \{-\infty\}$.

Proof. Suppose $\mathcal{F} \neq \{-\infty\}$. Let $v_0 \in \mathcal{F}$, $v_0 \neq -\infty$, and consider the class $\mathcal{F}_0 = \{v \in \mathcal{F}\,;\, v \geq v_0\}$. A sequence in $\mathcal{F}_0$ has locally uniform upper bounds and does not tend to $-\infty$ locally uniformly, so it has a subsequence converging to $w \in \mathrm{PSH}(X)$. In fact, w is the upper semi-continuous regularization of the limes superior of the subsequence. See Hörmander, loc. cit. Since φ is upper semi-continuous, we have $w \leq \varphi$, so $w \in \mathcal{F}_0$. This shows that $\mathcal{F}_0$ is compact, so $\sup \mathcal{F}_0 = \sup \mathcal{F}$ is plurisubharmonic.

Let $v \in \mathcal{F}$ and $f \in \mathcal{A}_X$. Then

$$v(f(0)) \leq \frac{1}{2\pi} \int_{\mathbb{T}} v \circ f \, d\lambda \leq \frac{1}{2\pi} \int_{\mathbb{T}} \varphi \circ f \, d\lambda = H(f).$$

This shows that $\sup \mathcal{F} \leq EH$. $\square$

The following theorem is the main result of this section.

2.2. Theorem. *If X is a domain in a Stein manifold, and $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is an upper semi-continuous function, then EH_1^φ is plurisubharmonic.*

To prove the theorem it suffices to show that the envelope $u = EH$ is upper semi-continuous, and that

$$u(h(0)) \leq \frac{1}{2\pi} \int_{\mathbb{T}} u \circ h \, d\lambda, \tag{2.1}$$

for every $h \in \mathcal{A}_X$.

Before we go into the details of the proof, let us give a brief outline of it. By the monotone convergence theorem, $H_1^{\varphi_j}(f) \searrow H_1^\varphi(f)$ as $j \to \infty$ for every sequence of upper semi-continuous functions $\varphi_j \searrow \varphi$, so $EH_1^{\varphi_j} \searrow EH_1^\varphi$. Since the limit of a decreasing sequence of plurisubharmonic functions is plurisubharmonic, and since there exists a decreasing sequence of continuous functions tending to φ, in the proof we may assume that φ is continuous.

First we prove the existence of holomorphic variations of analytic discs. More precisely, if $f_0 \in \mathcal{A}_X$ and $x_0 = f_0(0)$, then there exists $r > 1$, a neighbourhood V of x_0, and $f \in \mathcal{O}(D_r \times V, X)$, such that $f(z, x_0) = f_0(z)$ for $z \in D_r$, and $f(0, x) = x$ for $x \in V$. It then follows that $x \mapsto H(f(\cdot, x))$ is continuous. This shows that if $x_0 \in X$, then for every $\beta > u(x_0)$ there is a continuous function v on a neighbourhood U of x_0, such that $u \leq v < \beta$ on U. Hence, u is upper semi-continuous.

To prove (2.1) it suffices to show that for every $\varepsilon > 0$ and $v \in C(X, \mathbb{R})$ with $v \geq u$, there exists $g \in \mathcal{A}_X$ such that $g(0) = h(0)$ and

$$H(g) \leq \frac{1}{2\pi} \int_{\mathbb{T}} v \circ h \, d\lambda + \varepsilon. \tag{2.2}$$

The construction of g is performed in three steps. First we show that there exist $r > 1$ and $F \in C^\infty(D_r \times \mathbb{T}, X)$, such that $F(\cdot, w) \in \mathcal{A}_X$, $F(0,w) = h(w)$ for all $w \in \mathbb{T}$, and

$$\int_0^{2\pi} H(F(\cdot, e^{i\theta}))\, d\theta \leq \int_{\mathbb{T}} v \circ h \, d\lambda + \varepsilon. \tag{2.3}$$

Next we show that there exist $s \in (1, r)$ and $G \in \mathcal{O}(D_s \times D_s, X)$, such that $G(0, w) = h(w)$ for all $w \in D_s$, and

$$\int_0^{2\pi} H(G(\cdot, e^{i\theta}))\, d\theta \leq \int_0^{2\pi} H(F(\cdot, e^{i\theta}))\, d\theta + \varepsilon. \tag{2.4}$$

Finally, we show that there is $\theta_0 \in [0, 2\pi]$ such that if g is defined by the formula $g(z) = G(e^{i\theta_0} z, z)$, then

$$H(g) \leq \frac{1}{2\pi} \int_0^{2\pi} H(G(\cdot, e^{i\theta}))\, d\theta. \tag{2.5}$$

By combining the inequalities (2.3-5), we get (2.2), and (2.1) follows.

For the proof of Theorem 2.2 we need several lemmas. In all of them we assume that X is a complex manifold, not necessarily a domain in a Stein manifold, and that u is the envelope of H. The first lemma states that we have a holomorphic variation of discs in X.

2.3. Lemma. *Let $f_0 \in \mathcal{A}_X$. Then there exists an open neighbourhood V of $x_0 = f_0(0)$ in X, $r > 1$, and $f \in \mathcal{O}(D_r \times V, X)$, such that*

(i) *$f(z, x_0) = f_0(z)$ for all $z \in D_r$, and*

(ii) *$f(0, x) = x$ for all $x \in V$.*

Moreover, if f_0 is non-constant, then for every finite set $M \subset \mathbb{D} \setminus \{0\}$ we can find f such that $f(a, x) = f_0(a)$ and $m_a(f(\cdot, x)) = m_a(f_0)$ for all $a \in M$ and all $x \in V$. If f_0 is constant, then for every finite set $M \subset \mathbb{D} \setminus \{0\}$ and every $N > 0$, we can find f such that $f(a, x) = f_0(a)$ and $m_a(f(\cdot, x)) \geq N$ for all $a \in M$ and all $x \in V$.

Proof. Choose $r_0 > 1$ such that $f_0 \in \mathcal{O}(D_{r_0}, X)$. For every $t \in (1, r_0]$, the graph $S_t = \{(z, f_0(z))\,;\, z \in D_t\}$ is a submanifold of $D_t \times X$. It is isomorphic to D_t, and hence Stein. By a theorem of Siu [1976, Main Theorem and Corollary 1], there exists a Stein neighbourhood W of S_{r_0} in $D_{r_0} \times X$, and a biholomorphic map of W onto a neighbourhood of the zero section of the normal bundle of S_{r_0} which identifies S_{r_0} with the zero section. Since the normal bundle of S_{r_0} is trivial, it is biholomorphic to $S_{r_0} \times \mathbb{C}^n$, where n is the dimension of X. Now we take $t \in (1, r_0)$, and conclude that there exists a neighbourhood U of S_t in $D_{r_0} \times X$ and a biholomorphic map $\Phi : U \to D_t \times \mathbb{D}^n$ such that $\Phi(z, f_0(z)) = (z, 0)$ for all $z \in D_t$. The map f is then defined by the formula

$$f(z, x) = \operatorname{pr}\big(\Phi^{-1}((z, 0) + \chi(z)\Phi(0, x))\big) \tag{2.6}$$

where $\mathrm{pr} : \mathbb{C} \times X \to X$ is the natural projection and χ is a polynomial.

If $M = \varnothing$, then we set $\chi = 1$. If $M \neq \varnothing$, then for $a \in M$ we set $n_a = m_a(f_0)$ if f_0 is non-constant, but choose $n_a \geq N$ if f_0 is constant. We then define χ by $\chi(z) = \prod_{a \in M}(1 - z/a)^{n_a}$. If $r \in (1, t)$, then there exists a neighbourhood V of x_0 such that $(z, 0) + \chi(z)\Phi(0, x) \in D_t \times \mathbb{D}^n$ for all $z \in D_r$ and $x \in V$. If we define f by (2.6), then it is apparent that all the conditions are satisfied. $\square$

2.4. Lemma. *Let $x_0 \in X$, $\beta \in \mathbb{R}$, and assume that $u(x_0) < \beta$. Then there exists a neighbourhood V of x_0 in X, $r > 1$, and $f \in \mathcal{O}(D_r \times V, X)$, such that $f(0, x) = x$ and $u(x) \leq H(f(\cdot, x)) < \beta$ for all $x \in V$.*

Proof. By the definition of u, there exists $f_0 \in \mathcal{O}(D_{r_0}, X)$, $r_0 > 1$, such that $f_0(0) = x_0$ and $H(f_0) < \beta$. We set $M = \varnothing$ and choose f satisfying the conditions in Lemma 2.3. As mentioned above, we may assume that $\varphi \in C(X, \mathbb{R})$. Then the function $V \to \mathbb{R}$, $x \mapsto H(f(\cdot, x))$, is continuous. We have $H(f(\cdot, x_0)) = H(f_0) < \beta$, so replacing V by a smaller neighbourhood of x_0, we conclude that $u(x) \leq H(f(\cdot, x)) < \beta$ for all $x \in V$. $\square$

2.5. Lemma. *Let $h \in \mathcal{A}_X$, $\varepsilon > 0$, and $v \in C(X, \mathbb{R})$ with $v \geq u$. Then there exist $r > 1$ and $F \in C^\infty(D_r \times \mathbb{T}, X)$, such that $F(\cdot, w) \in \mathcal{A}_X$, $F(0, w) = h(w)$ for all $w \in \mathbb{T}$, and* (2.3) *holds.*

Proof. Let $w_0 \in \mathbb{T}$, and set $x_0 = h(w_0)$. By Lemma 2.4, there exists $r_0 > 1$, an open neighbourhood V_0 of x_0, and $f \in \mathcal{O}(D_{r_0} \times V_0, X)$, such that $f(0, x) = x$ and $H(f(\cdot, x)) < v(x) + \varepsilon/8\pi$ for all $x \in V_0$. We can take an open arc $I_0 \subset \mathbb{T}$ containing w_0 such that $h(w) \in V_0$ for all $w \in I_0$, and define $F_0 : D_{r_0} \times I_0 \to X$ by $F_0(z, w) = f(z, h(w))$. By replacing r_0 by a smaller number greater than 1 and I_0 by a smaller open arc containing w_0, we may assume that $F_0(D_{r_0} \times I_0)$ is relatively compact in X.

A simple compactness argument now shows that there exists a cover of $\mathbb{T}$ by open arcs $\{I_j\}_{j=1}^N$, $r_j > 1$, and $F_j \in C^\infty(D_{r_j} \times I_j, X)$, such that $F_j(\cdot, w) \in \mathcal{A}_X$, $F_j(0, w) = h(w)$, $F_j(D_{r_j} \times I_j)$ is relatively compact in X, and $H(F_j(\cdot, w)) < v(h(w)) + \varepsilon/8\pi$ for all $w \in I_j$. We set $r = \min_j r_j$.

Let M be a compact subset of X containing the image of all the functions F_j, and let $C > \max\{0, \sup H(f)\} + \sup_M |v|$, where the first supremum is taken over all $f \in \mathcal{A}_X$ with $f(D_t) \subset M$ for some $t > 1$.

There exists a subset A in $\{1, \ldots, N\}$ and *disjoint* closed arcs $J_j \subset I_j$, $j \in A$, such that $\lambda(\mathbb{T} \setminus \cup J_j) < \varepsilon/(4C)$. By possibly removing some arcs I_j from the cover of $\mathbb{T}$, we may assume that $A = \{1, \ldots, N\}$. We choose *disjoint* open arcs K_j, $J_j \subset K_j \subset I_j$, and a function $\varrho \in C^\infty(\mathbb{T})$, such that $0 \leq \varrho \leq 1$, $\varrho(w) = 1$ if $w \in \cup J_j$, and $\varrho(w) = 0$ if $w \in \mathbb{T} \setminus \cup K_j$, and finally define $F : D_r \times \mathbb{T} \to X$ by

$$F(z, w) = \begin{cases} F_j(\varrho(w)z, w), & z \in D_r,\ w \in K_j, \\ h(w), & z \in D_r,\ w \in \mathbb{T} \setminus \cup K_j. \end{cases}$$

The choice of ϱ ensures that $F \in C^\infty(D_r \times \mathbb{T}, X)$, $F(\cdot, w) \in \mathcal{A}_X$, and $F(0,w) = h(w)$. If we combine the inequalities we already have, then we get

$$\int_0^{2\pi} H(F(\cdot, e^{i\theta}))\, d\theta \le \sum_j \int_{J_j} H(F_j(\cdot, w))\, d\lambda(w) + \frac{\varepsilon}{4}$$
$$\le \sum_j \int_{J_j} v \circ h \, d\lambda + \frac{\varepsilon}{2} \le \int_{\mathbb{T}} v \circ h \, d\lambda + \varepsilon,$$

and we have proved (2.3). □

2.6. Lemma. *Let $r > 1$, $h \in \mathcal{O}(D_r, X)$, and $F \in C^\infty(D_r \times \mathbb{T}, X)$, such that $F(\cdot, w) \in \mathcal{O}(D_r, X)$, and $F(0, w) = h(w)$ for all $w \in \mathbb{T}$. Furthermore, assume that there exists an open neighbourhood of*

$$M_r = \{(z, w, F(z,w))\,;\, z \in D_r, w \in \mathbb{T}\} \cup \{(0, w, h(w))\,;\, w \in D_r\} \tag{2.7}$$

in $D_r \times D_r \times X$, which is biholomorphic to a domain in a Stein manifold. Then there exists $s \in (1, r)$, a natural number j_0, and a sequence $F_j \in \mathcal{O}(D_s \times A_j, X)$, $j \ge j_0$, where A_j is an open annulus containing $\mathbb{T}$, such that

(i) *$F_j \to F$ uniformly on $D_s \times \mathbb{T}$ as $j \to \infty$,*
(ii) *there is an integer $k_j \ge j$ such that the map $(z, w) \mapsto F_j(zw^{k_j}, w)$ can be extended to a map $G_j \in \mathcal{O}(D_{s_j} \times D_{s_j}, X)$, where $s_j \in (1, s)$, and*
(iii) *$G_j(0, w) = h(w)$ for all $w \in D_{s_j}$.*

Proof. Every Stein manifold is biholomorphic to a submanifold of $\mathbb{C}^\nu$ for some ν, so by assumption there exists a biholomorphic map $\Phi : U \to V$ from a neighbourhood U of M_r onto a domain V in some submanifold Y of $\mathbb{C}^\nu$. By Siu [1976, Main Theorem and Corollary 1], there is a Stein neighbourhood Z of Y in $\mathbb{C}^\nu$, and a holomorphic retraction $\sigma : Z \to Y$. We set $\tilde{V} = \sigma^{-1}(V)$. Then $\tilde{V}$ is open in $\mathbb{C}^\nu$. We define $\tilde{F} \in C^\infty(D_r \times \mathbb{T}, \mathbb{C}^\nu)$ by $\tilde{F}(z, w) = \Phi(z, w, F(z, w))$, $\tilde{h} \in \mathcal{O}(D_r, \mathbb{C}^\nu)$ by $\tilde{h}(w) = \Phi(0, w, h(w))$, and for any $j \in \mathbb{N}$ we define $\tilde{F}_j \in \mathcal{O}(D_r \times D_r^*, \mathbb{C}^\nu)$ by

$$\tilde{F}_j(z, w) = \tilde{h}(w) + \sum_{k=-j}^{j} \left(\frac{1}{2\pi} \int_0^{2\pi} \big(\tilde{F}(z, e^{i\theta}) - \tilde{h}(e^{i\theta})\big) e^{-ik\theta}\, d\theta \right) w^k. \tag{2.8}$$

Since the function $\theta \mapsto \tilde{F}(z, e^{i\theta}) - \tilde{h}(e^{i\theta})$ is infinitely differentiable with period 2π, its Fourier series converges uniformly on $\mathbb{R}$. Hence the series in (2.8) converges uniformly on $\{z\} \times \mathbb{T}$ for every $z \in D_r$ as $j \to \infty$. The convergence is uniform on $D_t \times \mathbb{T}$, $t \in (1, r)$. In fact, an integration by parts of the integral in (2.8) shows that it can be estimated by

$$k^{-2} \max_{z \in D_t, \theta \in \mathbb{R}} \left| \frac{\partial^2 \big(\tilde{F}(z, e^{i\theta}) - \tilde{h}(e^{i\theta})\big)}{\partial \theta^2} \right|, \qquad k \ne 0,$$

so this follows from Weierstrass' theorem.

Now we let $t \in (1,r)$. Since $\tilde{F}(z,w) \in V$ for all $(z,w) \in D_r \times \mathbb{T}$, and $\tilde{F}_j \to \tilde{F}$ uniformly on $D_t \times \mathbb{T}$, we can choose j_0 so large that $\tilde{F}_j(z,w) \in \tilde{V}$ for all $(z,w) \in D_t \times \mathbb{T}$ and $j \geq j_0$. If $s \in (1,t)$, then by continuity we can choose an open annulus A_j containing $\mathbb{T}$ such that $\tilde{F}_j(z,w) \in \tilde{V}$ for all $(z,w) \in D_s \times A_j$. We define $F_j \in \mathcal{O}(D_s \times A_j, X)$ by $F_j = \mathrm{pr} \circ \Phi^{-1} \circ \sigma \circ \tilde{F}_j$, where $\mathrm{pr} : \mathbb{C}^2 \times X \to X$ is the natural projection. Then (i) holds.

For every $z \in D_r$, the map $w \mapsto \tilde{F}_j(z,w) - \tilde{h}(w)$ has a pole of order at most j at the origin, and for every $w \in D_r^*$, the map $z \mapsto \tilde{F}_j(z,w) - \tilde{h}(w)$ has a zero at the origin. Hence $(z,w) \mapsto \tilde{F}_j(zw^k, w)$ can be extended to a holomorphic map $\overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}^\nu$ for every $k \geq j$.

Since $\tilde{F}_j(0,w) = \tilde{h}(w) \in V$ for all $w \in D_r^*$, there exists $\delta > 0$ such that $\tilde{F}_j(zw^k, w) \in \tilde{V}$ for all integers $k \geq j$ and $(z,w) \in D_\delta \times \overline{\mathbb{D}}$. Since $\tilde{F}_j(z,w) \in \tilde{V}$ for all $(z,w) \in \overline{\mathbb{D}} \times \mathbb{T}$, we can choose $\varrho_j \in (0,1)$ such that $\tilde{F}_j(z,w) \in \tilde{V}$ for all $(z,w) \in \overline{\mathbb{D}} \times (\overline{\mathbb{D}} \setminus D_{\varrho_j})$, so we conclude that $\tilde{F}_j(zw^k, w) \in \tilde{V}$ for all $(z,w) \in \overline{\mathbb{D}} \times (\overline{\mathbb{D}} \setminus D_{\varrho_j})$ and all integers $k \geq j$. Now we take k_j so large that $|zw^{k_j}| < \delta$ for all $(z,w) \in \overline{\mathbb{D}} \times D_{\varrho_j}$. Then $\tilde{F}_j(zw^{k_j}, w) \in \tilde{V}$ for all $(z,w) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. We finally choose $s_j \in (1,s)$ such that $\tilde{F}_j(zw^{k_j}, w) \in \tilde{V}$ for all $(z,w) \in D_{s_j} \times D_{s_j}$, and define G_j by $G_j(z,w) = (\mathrm{pr} \circ \Phi^{-1} \circ \sigma \circ \tilde{F}_j)(zw^{k_j}, w)$. Then (ii) and (iii) hold. $\square$

The condition in Lemma 2.6 is obviously satisfied if X is a domain in a Stein manifold.

2.7. Lemma. *Let h and F satisfy the conditions in Lemma 2.6. Then for every $\varepsilon > 0$, there exist $s \in (1,r)$ and $G \in \mathcal{O}(D_s \times D_s, X)$, such that $G(0,w) = h(w)$ for all $w \in D_s$, and* (2.4) *holds.*

Proof. Let F_j and G_j be sequences satisfying the conditions in Lemma 2.6. Assume that $\varphi \in C(X,\mathbb{R})$. Then for sufficiently large j we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \varphi(F_j(e^{it}, e^{i\theta}))\,dtd\theta &\leq \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \varphi(F(e^{it}, e^{i\theta}))\,dtd\theta + \varepsilon \\ &= \int_0^{2\pi} H(F(\cdot, e^{i\theta}))\,d\theta + \varepsilon. \end{aligned} \tag{2.9}$$

Now we fix a value j for which (2.9) holds, set $s = s_j$, and define $G \in \mathcal{O}(D_s \times D_s, X)$ by $G(z,w) = G_j(z,w)$. Then it is clear that $G(0,w) = h(w)$ for all $w \in D_s$, and we have

$$\begin{aligned}\int_0^{2\pi} H(G(\cdot, e^{i\theta}))\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \varphi(F_j(e^{i(t+k_j\theta)}, e^{i\theta}))\,dtd\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \varphi(F_j(e^{it}, e^{i\theta}))\,dtd\theta. \end{aligned} \tag{2.10}$$

The inequality (2.4) now follows by combining (2.9) and (2.10). $\square$

2.8. Lemma. *Let $s > 1$, and $G \in \mathcal{O}(D_s \times D_s, X)$. Then there exists $g \in \mathcal{O}(D_s, X)$ such that $g(0) = G(0,0)$ and (2.5) holds.*

Proof. The right hand side of (2.5) is equal to

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(G(e^{it}, e^{i\theta}))\,dt\,d\theta = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(G(e^{i\theta}e^{it}, e^{it}))\,dt\,d\theta.$$

Here we have considered the map $\chi : \mathbb{R}^2 \to \mathbb{R}$, $(t,\theta) \mapsto \varphi(G(e^{it}, e^{i\theta}))$, and made the change of variables $(t,\theta) \mapsto (\theta + t, t)$, which has Jacobian -1. There is $\theta_0 \in [0, 2\pi]$ such that

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(G(e^{it}, e^{i\theta}))\,dt d\theta \geq \frac{1}{2\pi}\int_0^{2\pi}\varphi(G(e^{i\theta_0}e^{it}, e^{it}))\,dt.$$

If we set $g(z) = G(e^{i\theta_0}z, z)$, then $g(0) = G(0,0)$ and (2.5) holds. □

Proof of Theorem 2.2. We may assume that $\varphi \in C(X, \mathbb{R})$. By Lemma 2.4, u is upper semi-continuous. We need to prove that for every $h \in \mathcal{A}_X$, the inequality (2.1) holds. As we noted after the statement of the theorem, for every $\varepsilon > 0$ and $v \in C(X, \mathbb{R})$ with $v \geq u$, we need to construct $g \in \mathcal{A}_X$ such that $g(0) = h(0)$ and (2.2) holds. Choose r, s, F, G, and g, such that all the conditions in Lemmas 2.5-8 are satisfied. If we combine the inequalities (2.3-5), then we get (2.2), and (2.1) follows. □

The application of Lemma 2.6 is the only place in the proof where we need X to be a domain in a Stein manifold.

3. Extensions to manifolds

If X and Y are complex manifolds, $h : X \to Y$ is a holomorphic map, and H is a functional on $\mathcal{A}_Y$, then a pullback functional h^*H on $\mathcal{A}_X$ is naturally defined by the formula

$$h^*H(f) = H(h \circ f), \qquad f \in \mathcal{A}_X.$$

Note that

$$E[h^*H] \geq EH \circ h.$$

We have

$$h^*H_1^{\varphi} = H_1^{\varphi\circ h}.$$

Suppose h is a covering. Then for any $p \in X$ there is a bijective correspondence between $f \in \mathcal{A}_X$ with $f(0) = p$ and $g \in \mathcal{A}_Y$ with $g(0) = h(p)$ such that $g = h \circ f$. Hence,

$$E[h^*H] = EH \circ h,$$

so EH is plurisubharmonic if and only if $E[h^*H]$ is.

We have established the following.

3.1. Proposition. *Let X and Y be complex manifolds such that there exists a holomorphic covering $X \to Y$. If $EH_1^\varphi \in \mathrm{PSH}(X)$ for every upper semi-continuous function φ on X, then $EH_1^\varphi \in \mathrm{PSH}(Y)$ for every upper semi-continuous function φ on Y.*

To prove the analogous result for finite branched coverings, we need the first part of the following lemma. The second part will be used in Section 4.

3.2. Lemma. *Let $h : X \to Y$ be a k-sheeted finite branched covering. Let u be a plurisubharmonic function on X. Then the function h_*u, defined by the formula*

$$h_*u(y) = \frac{1}{k} \sum_{x \in h^{-1}(y)} m_x(h)u(x),$$

is plurisubharmonic on Y, and

$$k\mathcal{L}(h_*u) = h_*\mathcal{L}(u),$$

where $h_*\mathcal{L}(u)$ denotes the direct image under h of the closed positive (1,1)-current $\mathcal{L}(u)$.

Proof. Let $B \subset Y$ be the branch locus of h. Then $h : X \setminus h^{-1}(B) \to Y \setminus B$ is a finite unbranched covering, so h_*u is plurisubharmonic on $Y \setminus B$. The restriction $h_*u|Y \setminus B$ extends to a plurisubharmonic function v on Y with

$$v(y) = \limsup_{z \to y, z \notin B} h_*u(z), \qquad y \in B.$$

If u is continuous, then h_*u is continuous, so $v = h_*u$, and h_*u is plurisubharmonic. In the general case, let $p \in B$ and U be an open coordinate ball containing p. We have a finite map $h : h^{-1}(U) \to U$, so $h^{-1}(U)$ is Stein, and the main approximation theorem for plurisubharmonic functions holds on $h^{-1}(U)$. Let V be a relatively compact open ball in U with $p \in V$. Then there are smooth plurisubharmonic functions u_n on $h^{-1}(V)$ such that $u_n \searrow u$. Since h_*u_n are plurisubharmonic and $h_*u_n \searrow h_*u$, we conclude that h_*u is plurisubharmonic on V.

To prove the second part of the lemma, assume $u \not\equiv -\infty$, and let η be a smooth $(n-1, n-1)$-form on Y with compact support, $n = \dim Y$. Then

$$\begin{aligned} k\mathcal{L}(h_*u)(\eta) &= ki \int_Y (h_*u)\partial\overline{\partial}\eta = ki \int_{Y \setminus B} (h_*u)\partial\overline{\partial}\eta \\ &= i \int_{X \setminus h^{-1}(B)} u\, h^*(\partial\overline{\partial}\eta) = i \int_X u\, \partial\overline{\partial}(h^*\eta) = \mathcal{L}(u)(h^*\eta) = h_*\mathcal{L}(u)(\eta), \end{aligned}$$

so $k\mathcal{L}(h_*u) = h_*\mathcal{L}(u)$. $\square$

3.3. Proposition. *Let X and Y be complex manifolds such that there exists a finite branched covering $h : X \to Y$. If $EH_1^\varphi \in \mathrm{PSH}(X)$ for every upper semi-continuous function φ on X, then $EH_1^\varphi \in \mathrm{PSH}(Y)$ for every upper semi-continuous function φ on Y.*

Proof. Let $\varphi : Y \to \mathbb{R} \cup \{-\infty\}$ be upper semi-continuous. Let $w = EH_1^{\varphi\circ h}$. By assumption, w is plurisubharmonic on X, and $EH_1^\varphi \circ h \leq w$. Now $w \leq \varphi \circ h$, so $h_* w \leq \varphi$, and $h_* w \leq EH_1^\varphi$. Hence,

$$(h_* w) \circ h \leq EH_1^\varphi \circ h \leq w.$$

Now $(h_* w) \circ h \leq w$ implies that $(h_* w) \circ h = w$, so $EH_1^\varphi \circ h = w$, and $EH_1^\varphi = h_* w$ is plurisubharmonic. $\square$

In the introduction, we defined $\mathcal{P}$ as the class of complex manifolds X for which there exists a finite sequence of complex manifolds and holomorphic maps

$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \dots \xrightarrow{h_m} X_m = X, \qquad m \geq 0,$$

where X_0 is a domain in a Stein manifold and each h_i, $i = 1, \dots, m$, is either a covering or a finite branched covering. Theorem 2.2 and Propositions 3.1 and 3.3 now imply the following result.

3.4. Theorem. *The Poisson functional has plurisubharmonic envelopes on manifolds in $\mathcal{P}$.*

In the remainder of this section we will study the scope of the class $\mathcal{P}$. First of all, it is clear that if X is in $\mathcal{P}$, and $X \to Y$ is either a covering or a finite branched covering, then Y is in $\mathcal{P}$.

3.5. Proposition. *Let Y be a domain in a complex manifold X. If X is in $\mathcal{P}$, then Y is in $\mathcal{P}$.*

Proof. First, let us note that if $h : X' \to X$ is a holomorphic covering, Y is a domain in X and Y' is a connected component of $h^{-1}(Y)$, then $h|Y' \to Y$ is a holomorphic covering. Likewise, if $h : X' \to X$ is a finite branched covering, Y is a domain in X and Y' is a connected component of $h^{-1}(Y)$, then $h|Y' \to Y$ is a finite branched covering.

Now let

$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \dots \xrightarrow{h_m} X_m = X$$

be a sequence as in the definition of $\mathcal{P}$. If $m = 0$, then X is a domain in a Stein manifold, so Y is too, so Y is in $\mathcal{P}$. Suppose $m \geq 1$. Define a sequence

$$Y_0 \xrightarrow{k_1} Y_1 \xrightarrow{k_2} \dots \xrightarrow{k_m} Y_m = Y$$

by induction as follows. For $i = m, \dots, 1$, let $Y_{i-1} \subset X_{i-1}$ be a connected component of $h_i^{-1}(Y_i)$ and let $k_i = h_i|Y_{i-1}$. Then Y_0 is a domain in a Stein manifold, and we see that Y is in $\mathcal{P}$. $\square$

3.6. Proposition. *Let X and Y be manifolds in $\mathcal{P}$. Then the product $X \times Y$ is in $\mathcal{P}$.*

Proof. First let us note that if $h : X \to Y$ is a holomorphic covering, and Z is a manifold, then $h \times \mathrm{id} : X \times Z \to Y \times Z$ is a holomorphic covering. Likewise, if $h : X \to Y$ is a finite branched covering, and Z is a manifold, then $h \times \mathrm{id} : X \times Z \to Y \times Z$ is a finite branched covering. Also recall that the product of Stein manifolds is Stein.

Now let

$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \dots \xrightarrow{h_m} X_m = X, \qquad Y_0 \xrightarrow{k_1} Y_1 \xrightarrow{k_2} \dots \xrightarrow{k_m} Y_m = Y$$

be sequences as in the definition of $\mathcal{P}$. We may assume that they are of the same length, because such sequences can always be extended by identity maps. Now replace each map $X_i \to X_{i+1}$ by the composition $X_i \to X_{i+1} = X_{i+1}$, and each map $Y_i \to Y_{i+1}$ by the composition $Y_i = Y_i \to Y_{i+1}$. Then the sequence

$$X_0 \times Y_0 \xrightarrow{h_1 \times k_1} \dots \xrightarrow{h_m \times k_m} X \times Y$$

shows that $X \times Y$ is in $\mathcal{P}$, because each map in the sequence is of the form $h \times \mathrm{id}$ or $\mathrm{id} \times k$, where h or k is a holomorphic covering or a finite branched covering. $\square$

3.7. Proposition. *$\mathcal{P}$ contains all Riemann surfaces.*

Proof. All Riemann surfaces except $\mathbb{P}^1$ are covered by a Stein manifold (namely $\mathbb{C}$ or $\mathbb{D}$). A non-constant meromorphic function on, say, a torus gives a finite branched covering to $\mathbb{P}^1$. $\square$

A slightly weaker version of the following theorem was given in Lárusson [1995, Proposition 4.1]. The proof here is different, although the key idea still comes from Gromov [1991, 0.3.A.(e)]. We will not need the full strength of the theorem until Section 5.

3.8. Theorem. *Let M be a projective manifold and S be a countable subset of M. Then there exists a projective manifold N and a finite branched covering $h : N \to M$, such that*

(1) *the universal covering space of N is Stein, and*
(2) *the branch locus of h does not intersect S.*

For the proof, we need the following special case of Theorem 2 in Kleiman [1974].

3.9. Theorem (Kleiman). *Let X and Y be projective manifolds, and $f : X \to \mathbb{P}^n$ and $g : Y \to \mathbb{P}^n$ be holomorphic maps. For γ in the automorphism group $\Gamma = \mathrm{PGL}(n+1, \mathbb{C})$ of $\mathbb{P}^n$, let γX denote X considered as a space over $\mathbb{P}^n$ via the map γf. Then for every γ in a dense Zariski-open subset of Γ, the fibre product $(\gamma X) \times_{\mathbb{P}^n} Y$ is smooth.*

Recall that in concrete terms, the fibre product $(\gamma X) \times_{\mathbb{P}^n} Y$ is the subvariety of points (x, y) in $X \times Y$ with $\gamma f(x) = g(y)$.

Proof of Theorem 3.8. The proof will involve the maps in the following diagram.

$$\begin{array}{ccccc} Y & \xrightarrow{q} & N & \xrightarrow{h} & M \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \gamma f} \\ \tilde{Q} & \xrightarrow[p]{} & Q & \xrightarrow[g]{} & \mathbb{P}^n \end{array}$$

Let $n = \dim M$, and let Q be an n-dimensional projective manifold. There are finite branched coverings $f : M \to \mathbb{P}^n$ and $g : Q \to \mathbb{P}^n$. For $\gamma \in \Gamma$, the projections from $(\gamma M) \times_{\mathbb{P}^n} Q$ onto M and Q are open since γf and g are open. By Kleiman's theorem, $(\gamma M) \times_{\mathbb{P}^n} Q$ is smooth for the generic γ. Let N be a connected component of $(\gamma M) \times_{\mathbb{P}^n} Q$ for such γ, and let h and k be the projections from N to M and Q respectively. Then h and k are open, and hence surjective, so they are finite branched coverings.

The branch locus of h is $(\gamma f)^{-1}(B)$, where B is the branch locus of g, so (2) holds if and only if $f(S) \cap \gamma^{-1}B = \varnothing$, which is true for γ in a Hausdorff-dense subset of Γ by a Baire category argument.

Now let $p : \tilde{Q} \to Q$ be the universal covering of Q. Let Y be a connected component of the fibre product $N \times_Q \tilde{Q}$, with projections $q : Y \to N$ and $j : Y \to \tilde{Q}$. Since p is a submersion, Y is smooth. If $x \in N$, and the neighbourhood U of $k(x)$ is evenly covered by p, then the neighbourhood $k^{-1}(U)$ of x is evenly covered by q. Hence, q is a covering. Since N is compact, j is proper. Also, j is open since k is, so j is surjective. Hence j is a finite branched covering.

Now choose Q so that $\tilde{Q}$ is Stein. For instance, we could take Q to be a product of n compact hyperbolic Riemann surfaces; then $\tilde{Q}$ is a polydisk. Since j is a finite branched covering, Y is Stein. Since Y is covered by the universal covering space $\tilde{N}$ of N, we conclude that $\tilde{N}$ is Stein. $\square$

The existence of a finite branched covering $N \to Q$ implies that N inherits various properties from Q. For instance, if Q is of general type, then so is N. Also, if Q is Kobayashi hyperbolic, then so is N. Both of these properties are satisfied if Q is chosen to be a product of hyperbolic Riemann surfaces.

Let us note an interesting consequence of the proof of Theorem 3.8. If $M_1, \dots, M_k$ are equidimensional projective manifolds, then there exists a projective manifold N with a finite branched covering $N \to M_i$ for each i.

3.10. Corollary. *A covering space of a projective manifold is in $\mathcal{P}$.*

Proof. Let M be a projective manifold and $X \to M$ be a covering. By Theorem 3.8, there is a projective manifold N with a finite branched covering $N \to M$, such that the universal covering space $\tilde{N}$ of N is Stein. Let Y be a connected component of the fibre product $N \times_M X$, with projections $q : Y \to N$ and $k : Y \to X$. As in the proof of

Theorem 3.8, we see that Y is smooth, q is a covering, and k is a finite branched covering. Now Y is covered by $\tilde{N}$, and the diagram $\tilde{N} \to Y \to X$ shows that X is in $\mathcal{P}$. $\square$

The corollary implies that all projective manifolds lie in $\mathcal{P}$. There are many examples of non-projective compact manifolds in $\mathcal{P}$. All tori lie in $\mathcal{P}$. So does a Hopf manifold, because its universal covering space is $\mathbb{C}^n \setminus \{0\}$, which is a domain in the Stein manifold $\mathbb{C}^n$, but it is not projective or even Kähler. An Inoue surface X is in $\mathcal{P}$ because its universal covering space is $\mathbb{C} \times \mathbb{D}$, but X does not contain any curves.

Compact complex manifolds X whose universal covering space is a domain Ω in $\mathbb{P}^n$ with non-empty complement of $(2n-2)$-dimensional Hausdorff measure zero are studied in Lárusson [1998]. Blanchard manifolds and Nori's higher dimensional Schottky coverings are examples of such manifolds. As shown in Lárusson [1998], X is not of class $\mathcal{C}$; in particular, X is neither Kähler nor Moishezon. Since $\mathbb{P}^n$ is in $\mathcal{P}$, so is Ω. Hence, X is in $\mathcal{P}$.

Recall that a simply connected compact complex surface with trivial canonical bundle is called a K3 surface. There are K3 surfaces X that contain no curves. Such X do not lie in $\mathcal{P}$, because if $Y \to X$ is a covering or a finite branched covering, then it is an isomorphism, and X is of course not isomorphic to a domain in a Stein manifold.

4. The Riesz functional

As before, we let X be a complex manifold. Recall that a plurisubharmonic function v on X defines the Riesz functional H_2^v on X by the formula

$$H_2^v(f) = \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\Delta(v \circ f), \qquad f \in \mathcal{A}_X,$$

where $\Delta(v\circ f)$ is considered as a positive Borel measure on $\mathbb{D}$. If $f \in \mathcal{A}_X$ and $v \circ f = -\infty$, then we set $H_2^v(f) = 0$.

The Riesz functional is closely related to the Poisson functional. By the Riesz representation formula, we have

$$\begin{aligned} H_2^v(f) &= \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\Delta(v \circ f) = \int_{\mathbb{D}} G(0,\cdot)\Delta(v \circ f) \\ &= v(f(0)) - \int_{\mathbb{T}} P(0,\cdot)(v \circ f)\, d\lambda \\ &= v(f(0)) - \frac{1}{2\pi}\int_{\mathbb{T}} v \circ f\, d\lambda \end{aligned} \tag{4.1}$$

for $f \in \mathcal{A}_X$, where G is the Green function and P is the Poisson kernel for $\mathbb{D}$, so

$$H_2^v(f) = v(f(0)) + H_1^{-v}(f),$$

and

$$EH_2^v = v + EH_1^{-v}.$$

This formula, along with Theorem 3.4, implies the following result.

4.1. Proposition. *Let v be a plurisubharmonic function on a complex manifold X. If EH_1^{-v} is plurisubharmonic, then EH_2^v is plurisubharmonic.*

If X is in the class $\mathcal{P}$, and v is continuous, then EH_2^v is plurisubharmonic.

We note that if $h : X \to Y$ is a holomorphic map, and $v \in \mathrm{PSH}(Y)$, then

$$h^* H_2^v = H_2^{v \circ h}.$$

By the remarks at the beginning of Section 3, this implies that if h is a covering, then

$$EH_2^{v \circ h} = EH_2^v \circ h.$$

Hence, if $h : X \to Y$ is a holomorphic covering, and $EH_2^v \in \mathrm{PSH}(X)$ for every $v \in \mathrm{PSH}(X)$, then $EH_2^v \in \mathrm{PSH}(Y)$ for every $v \in \mathrm{PSH}(Y)$.

Assume now that v is continuous. We will show that if X is in the class $\mathcal{P}$, then EH_2^v is the supremum of a naturally defined class of plurisubharmonic functions.

First of all, let us note that formula (4.1) shows that Lemma 2.4 holds for H_2^v. So does Lemma 2.5, because it uses only Lemma 2.4, and the fact that H is bounded above on every set of discs with images in a fixed compact set. A straightforward modification of the proof shows that Lemma 2.7 holds for H_2^v. Next we need a strengthening of Lemma 2.8.

4.2. Lemma. *Let $h \in \mathcal{A}_X$, $s > 1$, $G \in \mathcal{O}(D_s \times D_s, X)$ and assume that $G(0, w) = h(w)$ for all $w \in D_s$. Then there exists $g \in \mathcal{O}(D_s, X)$ such that $g(0) = G(0, 0)$ and*

$$H_2^v(g) \leq H_2^v(h) + \frac{1}{2\pi} \int_0^{2\pi} H_2^v(G(\cdot, e^{i\theta}))\, d\theta. \tag{4.2}$$

Proof. The right hand side of (4.2) is equal to

$$\begin{aligned} v(h(0)) &- \frac{1}{2\pi} \int_0^{2\pi} v(h(e^{i\theta}))\, d\theta \\ &+ \frac{1}{2\pi} \int_0^{2\pi} \left(v(G(0, e^{i\theta})) - \frac{1}{2\pi} \int_0^{2\pi} v(G(e^{it}, e^{i\theta}))\, dt \right) d\theta \\ &= v(G(0,0)) - \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} v(G(e^{i\theta} e^{it}, e^{it}))\, dt d\theta. \end{aligned}$$

Here we have made the same change of variables as in the proof of Lemma 2.8. Now there exists $\theta_0 \in [0, 2\pi]$, such that

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} v(G(e^{i\theta} e^{it}, e^{it}))\, dt d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} v(G(e^{i\theta_0} e^{it}, e^{it}))\, dt.$$

If we set $g(z) = G(e^{i\theta_0} z, z)$ then $g(0) = G(0, 0)$ and (4.2) holds. □

Proceeding as in the proof of Theorem 2.2, but using Lemma 4.2 instead of Lemma 2.8, we can now prove the following result. We remark that continuity of v is only needed to obtain Lemmas 2.4 and 2.7.

4.3. Proposition. *Let v be a continuous plurisubharmonic function on a domain X in a Stein manifold. Then*

$$EH_2^v(f(0)) \le H_2^v(f) + \frac{1}{2\pi}\int_{\mathbb{T}} EH_2^v \circ f\, d\lambda, \qquad f \in \mathcal{A}_X. \tag{4.3}$$

Having proved this proposition for domains in Stein manifolds, we can easily extend it to manifolds which are covered by such domains by lifting discs.

For a plurisubharmonic function v on a complex manifold X, we define

$$\mathcal{F}_v = \{w \in \mathrm{PSH}(X)\,;\, w \le 0,\, \mathcal{L}(w) \ge \mathcal{L}(v)\},$$

where $\mathcal{L}(v)$ denotes the Levi form $i\partial\overline{\partial}v$ of v, which is a closed positive (1,1)-current on X. We agree that $\mathcal{L}(-\infty) = 0$.

We note that $\mathcal{F}_v$ may be empty, for instance on manifolds, such as $\mathbb{C}^n$, that have no non-constant negative plurisubharmonic functions.

4.4. Theorem. *Let X be a complex manifold and $v \in \mathrm{PSH}(X)$. Then $\sup \mathcal{F}_v \in \mathrm{PSH}(X)$. If $w \in \mathcal{F}_v$ and $f \in \mathcal{A}_X$, then $w(f(0)) \le H_2^v(f)$. Hence, $\sup \mathcal{F}_v \le EH_2^v$.*

Furthermore, if **either**

(1) *X is in the class $\mathcal{P}$, and v is continuous,* **or**

(2) *$EH_2^v \in \mathrm{PSH}(X)$ and (4.3) holds,*

then $EH_2^v \in \mathcal{F}_v$, unless $EH_2^v = -\infty$. In any case, $EH_2^v = \sup \mathcal{F}_v$.

Proof. Let $H = H_2^v$, $\mathcal{F} = \mathcal{F}_v$, and $u = EH$. As in the proof of Proposition 2.1, if $\mathcal{F} \ne \{-\infty\}$, then we take $w_0 \in \mathcal{F}\setminus\{-\infty\}$ and consider the class $\mathcal{F}_0 = \{w \in \mathcal{F}\,;\, w \ge w_0\}$. Now weak convergence implies convergence of Levi forms in the sense of currents, so $\mathcal{F}_0$ is compact, and $\sup \mathcal{F}_0 = \sup \mathcal{F} \in \mathrm{PSH}(X)$.

Let $w \in \mathcal{F}$ and $f \in \mathcal{A}_X$. Assume that $v \circ f \ne -\infty$ and $w \circ f \ne -\infty$; otherwise, $w(f(0)) \le H(f)$ is clear. Define a function s on X as $w - v$ on $X \setminus v^{-1}(-\infty)$ and as $-\infty$ on $v^{-1}(-\infty)$. Then s is locally integrable and $\mathcal{L}(s) \ge 0$, so the function $\tilde{s}$ defined locally as $\lim_{\varepsilon\to 0} s * \chi_\varepsilon$, where χ_ε are smoothing kernels, is a well defined plurisubharmonic function on X, and $\tilde{s} \circ f$ is subharmonic on $\mathbb{D}$. Since $v, w \in \mathrm{PSH}(X)$, on $X \setminus v^{-1}(-\infty)$ we have

$$\tilde{s} = \lim s * \chi_\varepsilon = \lim w * \chi_\varepsilon - \lim v * \chi_\varepsilon = w - v,$$

so $\tilde{s} \circ f = (w - v) \circ f$ almost everywhere on $\mathbb{D}$, and $\Delta(w \circ f) \ge \Delta(v \circ f)$. Hence, by the Riesz representation formula,

$$\begin{aligned} w(f(0)) &= \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\Delta(w \circ f) + \frac{1}{2\pi}\int_{\mathbb{T}} w \circ f\, d\lambda \\ &\le \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\Delta(w \circ f) \le \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\Delta(v \circ f) = H(f). \end{aligned}$$

Now suppose (2) holds. Define a function s on X as $u-v$ on $X \setminus v^{-1}(-\infty)$ and as $-\infty$ on $v^{-1}(-\infty)$. Let $f \in \mathcal{A}_X$. If $v(f(0)) \neq -\infty$, then by (4.3) and the Riesz representation formula,

$$\begin{aligned} s(f(0)) &= u(f(0)) - v(f(0)) \\ &\leq H_2^v(f) + \frac{1}{2\pi}\int_{\mathbb{T}} u \circ f \, d\lambda - H_2^v(f) - \frac{1}{2\pi}\int_{\mathbb{T}} v \circ f \, d\lambda \leq \frac{1}{2\pi}\int_{\mathbb{T}} s \circ f \, d\lambda, \end{aligned}$$

where, for the last inequality, we observe that $(v \circ f)^{-1}(-\infty) \cap \mathbb{T}$ is null with respect to λ because $\int_{\mathbb{T}} v \circ f \, d\lambda > -\infty$, so $s \circ f = (u-v) \circ f$ almost everywhere on $\mathbb{T}$.

This shows that s satisfies the sub-mean value property

$$s(f(0)) \leq \frac{1}{2\pi}\int_{\mathbb{T}} s \circ f \, d\lambda \qquad \text{for all } f \in \mathcal{A}_X.$$

If $v = -\infty$, then $u = 0 \in \mathcal{F}$. Let us therefore assume that $u, v \neq -\infty$. Then s is a locally integrable function on X. Since s satisfies the sub-mean value property, the function $\tilde{s}$ defined locally as $\lim_{\varepsilon \to 0} s * \chi_\varepsilon$, where χ_ε are smoothing kernels, is a well defined plurisubharmonic function on X. Since $u, v \in \mathrm{PSH}(X)$, on $X \setminus v^{-1}(-\infty)$ we have

$$\tilde{s} = \lim s * \chi_\varepsilon = \lim u * \chi_\varepsilon - \lim v * \chi_\varepsilon = u - v.$$

Hence, $\mathcal{L}(u-v) = \mathcal{L}(\tilde{s}) \geq 0$, so $u \in \mathcal{F}$.

We have shown that if (2) holds, then $u \in \mathcal{F}$ unless $u = -\infty$. By Propositions 4.1 and 4.3, (2) holds if v is continuous and X is a domain in a Stein manifold. To show that $u \in \mathcal{F}$ unless $u = -\infty$ if (1) is satisfied, we need only prove the following claim.

Claim. Let X and Y be complex manifolds, and $h : X \to Y$ be a holomorphic map which is either a covering or a finite branched covering. Let $v \in \mathrm{PSH}(Y)$, and suppose that $EH_2^v \in \mathrm{PSH}(Y)$ and $EH_2^{v\circ h} \in \mathrm{PSH}(X)$. If $EH_2^{v\circ h} \in \mathcal{F}_{v\circ h}$, then $EH_2^v \in \mathcal{F}_v$.

Proof of claim. First suppose h is a covering. Then $EH_2^{v\circ h} = E[h^* H_2^v] = EH_2^v \circ h$. By assumption, $\mathcal{L}(EH_2^v \circ h) \geq \mathcal{L}(v \circ h)$. Since h is a local biholomorphism, this implies immediately that $\mathcal{L}(EH_2^v) \geq \mathcal{L}(v)$, so $EH_2^v \in \mathcal{F}_v$.

Now suppose h is a k-sheeted finite branched covering. Then, by Lemma 3.2,

$$\mathcal{L}(h_* EH_2^{v\circ h}) = \frac{1}{k} h_* \mathcal{L}(EH_2^{v\circ h}) \geq \frac{1}{k} h_* \mathcal{L}(v \circ h) = \mathcal{L}(h_*(v \circ h)) = \mathcal{L}(v),$$

so $h_* EH_2^{v\circ h} \in \mathcal{F}_v$, and

$$(h_* EH_2^{v\circ h}) \circ h \leq EH_2^v \circ h \leq E[h^* H_2^v] = EH_2^{v\circ h}.$$

This implies that $EH_2^{v\circ h} = EH_2^v \circ h$, so $EH_2^v = h_* EH_2^{v\circ h}$, and, by the above,

$$\mathcal{L}(EH_2^v) = \mathcal{L}(h_* EH_2^{v\circ h}) \geq \mathcal{L}(v),$$

so $EH_2^v \in \mathcal{F}_v$. □

5. The Lelong functional

Let β be a non-negative function on $\mathbb{D}$. Then

$$\mu = 2\pi \sum_{b\in\mathbb{D}} \beta(b)\delta_b$$

is a well defined positive Borel measure on $\mathbb{D}$. Let

$$v(z) = \int_{\mathbb{D}} G(z,\cdot)d\mu = \sum_{b\in\mathbb{D}} \beta(b)\log\left|\frac{z-b}{1-\bar{b}z}\right|.$$

Then v is a subharmonic function on $\mathbb{D}$. We have $v \not\equiv -\infty$ if and only if

$$\sum_{b\in\mathbb{D}} \beta(b)(1-|b|) = \frac{1}{2\pi}\int_{\mathbb{D}} (1-|\cdot|)d\mu < \infty;$$

see Hörmander [1994, Chapter III]. Suppose $v \not\equiv -\infty$. Then μ has finite mass on compact sets, so β is zero outside a countable set, and the sum that defines μ converges in the sense of distributions. Also, $\Delta v = \mu$, so $\nu_v = \mu(\{\cdot\}) = \beta$. In fact, v is the largest negative subharmonic function on $\mathbb{D}$ with Lelong numbers at least β.

If X is a complex manifold, $u \in \mathrm{PSH}(X)$, $f \in \mathcal{A}_X$, and $b \in \mathbb{D}$, then

$$\nu_{u\circ f}(b) \geq \nu_u(f(b))m_b(f).$$

In view of this, if α is a non-negative function on X, we define $f^*\alpha : \mathbb{D} \to [0,\infty)$ by the formula

$$f^*\alpha(b) = \alpha(f(b))m_b(f).$$

Let X be a complex manifold. Recall that a non-negative function α on X defines the Lelong functional H_3^α by the formula

$$H_3^\alpha(f) = \sum_{b\in\mathbb{D}} f^*\alpha(b)\log|b|, \qquad f \in \mathcal{A}_X.$$

We have

$$H_3^\alpha(f) = v_f^\alpha(0),$$

where v_f^α is the largest negative subharmonic function on $\mathbb{D}$ with Lelong numbers at least $f^*\alpha$. If $v_f^\alpha \not\equiv -\infty$, then

$$H_3^\alpha(f) = v_f^\alpha(0) = \int_{\mathbb{D}} \log|\cdot|\Delta v_f^\alpha.$$

Let

$$\mathcal{F}_\alpha = \{w \in \mathrm{PSH}(X)\,;\, w \leq 0, \nu_w \geq \alpha\}.$$

5.1. Proposition. *Let X be a complex manifold and let $\alpha : X \to [0,\infty)$. Then $\sup \mathcal{F}_\alpha$ is plurisubharmonic. If $w \in \mathcal{F}_\alpha$ and $f \in \mathcal{A}_X$, then $w(f(0)) \le H_3^\alpha(f)$. Hence, $\sup \mathcal{F}_\alpha \le EH_3^\alpha$. Finally, EH_3^α is plurisubharmonic if and only if $EH_3^\alpha \in \mathcal{F}_\alpha$, and then $EH_3^\alpha = \sup \mathcal{F}_\alpha$.*

Proof. As in the proof of Proposition 2.1, if $\mathcal{F}_\alpha \ne \{-\infty\}$, then we take $w_0 \in \mathcal{F}_\alpha \setminus \{-\infty\}$ and consider the class $\mathcal{F}_0 = \{w \in \mathcal{F}_\alpha \,;\, w \ge w_0\}$. For $p \in X$, the map $u \mapsto \nu_u(p)$ is an upper semi-continuous function on $\mathrm{PSH}(X) \setminus \{-\infty\}$. Hence, $\mathcal{F}_0$ is compact, so $\sup \mathcal{F}_0 = \sup \mathcal{F}_\alpha$ is plurisubharmonic.

Let $w \in \mathcal{F}_\alpha$ and $f \in \mathcal{A}_X$. Then $w \circ f$ is subharmonic on $\mathbb{D}$, and $\nu_{w\circ f} \ge f^*\nu_w \ge f^*\alpha$, so $w \circ f \le v_f^\alpha$. Hence, $w(f(0)) \le v_f^\alpha(0) = H_3^\alpha(f)$.

Suppose EH_3^α is plurisubharmonic. Let (U,ζ) be a coordinate neighbourhood centred at $p \in X$. We may assume that $\zeta(U) = \{z \in \mathbb{C}^n \,;\, |z| < 2\}$. For $x \in U$ with $0 < |\zeta(x)| < 1$, define $f \in \mathcal{A}_X$ by the formula

$$f(z) = \zeta^{-1}\big(\zeta(x) + z\frac{\zeta(x)}{|\zeta(x)|}\big), \qquad z \in \overline{\mathbb{D}}.$$

Then $f(0) = x$ and $f(-|\zeta(x)|) = p$, so

$$EH_3^\alpha(x) \le H_3^\alpha(f) \le \alpha(p)\log|\zeta(x)|,$$

and $\nu_{EH_3^\alpha}(p) \ge \alpha(p)$. Hence, $EH_3^\alpha \in \mathcal{F}_\alpha$. ☐

Let us note a few simple consequences of the proposition.

Let $v \le 0$ be plurisubharmonic on X, and set $\alpha = \nu_v$. Then $v \in \mathcal{F}_\alpha$, so $v \le u = EH_3^\alpha$. Suppose u is plurisubharmonic. Then this implies that $\nu_u \le \nu_v$, but since $u \in \mathcal{F}_\alpha$, we also have $\nu_u \ge \alpha$. Hence, $\nu_u = \nu_v$.

Recall that by a theorem of Siu [1974], if $u \in \mathrm{PSH}(X)$, then $\nu_u^{-1}[c,\infty)$ is a subvariety of X (i.e., a closed analytic subset of X) for all $c > 0$. See also Kiselman [1979] and Demailly [1987]. If $u = EH_3^\alpha$ is plurisubharmonic, then the proposition implies that $\nu_u \ge \alpha$, so if $u \ne -\infty$, then $\alpha^{-1}[c,\infty)$ is contained in a proper subvariety of X for each $c > 0$.

Let α be a non-negative function on X. For $c \ge 0$, let Z_c be the Zariski closure of $\alpha^{-1}[c,\infty)$, i.e., the smallest subvariety of X containing $\alpha^{-1}[c,\infty)$. Define a non-negative function $\hat\alpha$ on X as c on $Z_c \setminus \bigcup_{t>c} Z_t$ for each $c \ge 0$. Then $\hat\alpha^{-1}[c,\infty) = Z_c$ for each $c \ge 0$, and $\hat\alpha$ is the smallest function on X with $\hat\alpha \ge \alpha$, such that $\hat\alpha^{-1}[c,\infty)$ is a subvariety of X for all $c > 0$.

If $w \in \mathrm{PSH}(X)$ and $\nu_w \ge \alpha$, then $\nu_w \ge \hat\alpha$ by Siu's theorem. Hence, $\mathcal{F}_\alpha = \mathcal{F}_{\hat\alpha}$, so if EH_3^α is plurisubharmonic, then by the proposition,

$$EH_3^{\hat\alpha} \le EH_3^\alpha = \sup \mathcal{F}_\alpha = \sup \mathcal{F}_{\hat\alpha} \le EH_3^{\hat\alpha},$$

so $EH_3^{\hat{\alpha}} = EH_3^{\alpha}$, and $EH_3^{\hat{\alpha}}$ is plurisubharmonic.

Like the Riesz functional, the Lelong functional is related to the Poisson functional, but this relationship will take considerable work to establish.

We define a function $k_X^\alpha : X \to \mathbb{R} \cup \{-\infty\}$ by the formula

$$k_X^\alpha(x) = \inf\{\alpha(f(z)) \log|z| \,;\, f \in \mathcal{A}_X, f(0) = x, z \in \mathbb{D}\}. \tag{5.1}$$

Observe that the definition does not change if we restrict z to the interval $(0, 1)$. In the case where $\operatorname{supp}\alpha = \{a\}$ and $\alpha(a) = 1$, the function k_X^α is identical to the function $k_X(a, \cdot)$ defined by Edigarian [1996] by the formula

$$k_X(a, x) = \inf\{\log t \,;\, t \in (0, 1), f(t) = a, f(0) = x, \text{ for some } f \in \mathcal{A}_X\}.$$

The function k_X^α is related to the Kobayashi pseudodistance κ_X on X. By definition, κ_X is the largest pseudodistance on X smaller than δ_X, where

$$\delta_X(x, a) = \inf\{\varrho_{\mathbb{D}}(z, w) \,;\, f(z) = x, f(w) = a \text{ for some } f \in \mathcal{O}(\mathbb{D}, X)\},$$

and $\varrho_{\mathbb{D}}$ denotes the Poincaré distance in $\mathbb{D}$,

$$\varrho_{\mathbb{D}}(z, w) = \tanh^{-1}\left|\frac{z - w}{1 - \bar{w}z}\right|.$$

By composing the map f in the definition of δ_X with an automorphism which sends 0 to z and then replacing it by $z \mapsto f(z/r)$ with $r > 1$ and r close to 1, we see that $k_X(a, x) = \log\tanh\delta_X(x, a)$ for all $x \in X$. Consequently,

$$k_X^\alpha(x) = \inf_{a \in X} \alpha(a) \log\tanh\delta_X(x, a).$$

5.2. Proposition. *Let X be a complex manifold, let $\alpha : X \to [0, \infty)$, and define k_X^α by (5.1). Then $k_X^\alpha \le 0$, k_X^α is upper semi-continuous, and for every $p \in X$ there exists a coordinate neighbourhood (U, ζ) centred at p such that*

$$k_X^\alpha(x) \le \alpha(p) \log|\zeta(x)|, \qquad x \in U. \tag{5.2}$$

Proof. It is obvious that $k_X^\alpha \le 0$. Take $x_0 \in X$, $\beta \in \mathbb{R}$, and assume that $k_X^\alpha(x_0) < \beta$. Then there exist $f_0 \in \mathcal{A}_X$ and $t_0 \in (0, 1)$, such that $f(0) = x_0$ and $\alpha(f(t_0)) \log t_0 < \beta$. By Lemma 2.3, there exists an open neighbourhood V of x_0, $r > 1$, and $f \in \mathcal{O}(D_r \times V, X)$, such that $f(z, x_0) = f_0(z)$ for all $z \in D_r$, $f(0, x) = x$, and $f(t_0, x) = f_0(t_0)$ for all $x \in V$. Then $k_X^\alpha(x) \le \alpha(f(t_0, x)) \log t_0 < \beta$ for all $x \in V$, and we have proved that k_X^α is upper semi-continuous.

Now we take $p \in X$ and choose a coordinate neighbourhood (U, ζ) centred at p with $\zeta(U) = \{z \in \mathbb{C}^n\,;\, |z| < 2\}$. For $x \in U$ with $0 < |\zeta(x)| < 1$, we construct an analytic disc $f \in \mathcal{A}_X$ with $f(0) = x$ and $f(-|\zeta(x)|) = p$ in the same way as in the proof of Proposition 5.1. Then $k_X^\alpha(x) \le \alpha(p) \log|\zeta(x)|$, so (5.2) holds. □

Now $EH_1^{k_X^\alpha} \le k_X^\alpha$, so if $EH_1^{k_X^\alpha} \in \mathrm{PSH}(X)$, then (5.2) implies that $EH_1^{k_X^\alpha} \in \mathcal{F}_\alpha$. Hence, Proposition 5.1 shows that $EH_1^{k_X^\alpha} \le EH_3^\alpha$.

Edigarian [1996] has proved that if X is a domain in $\mathbb{C}^n$, $\operatorname{supp} \alpha = \{a\}$, and $\alpha(a) = 1$, then $g_X(\cdot, a) = EH_3^\alpha = EH_1^{k_X^\alpha}$, where $g_X(\cdot, a)$ denotes the pluricomplex Green function on X with a logarithmic pole at a. We will now generalize this result to an arbitrary non-negative function α on a domain in a Stein manifold.

5.3. Theorem. *Let X be a domain in a Stein manifold, let α be a non-negative function on X, and define k_X^α by (5.1). Then $EH_3^\alpha = EH_1^{k_X^\alpha}$, i.e., for every $x \in X$ we have*

$$EH_3^\alpha(x) = \inf\{\sum_{z \in \mathbb{D}} f^*\alpha(z) \log|z|\,;\, f \in \mathcal{A}_X, f(0) = x\}$$
$$= \inf\{\frac{1}{2\pi}\int_{\mathbb{T}} k_X^\alpha \circ f\, d\lambda\,;\, f \in \mathcal{A}_X, f(0) = x\} = EH_1^{k_X^\alpha}(x).$$

Hence, EH_3^α is plurisubharmonic.

As we have already noted, we only need to show that $EH_3^\alpha \le EH_1^{k_X^\alpha}$. Since k_X^α is upper semi-continuous, this inequality follows if we can prove that for every $h \in \mathcal{A}_X$, $\varepsilon > 0$, and $v \in C(X, \mathbb{R})$ with $v \ge k_X^\alpha$, there exists $g \in \mathcal{A}_X$ such that $g(0) = h(0)$ and

$$H_3^\alpha(g) \le \frac{1}{2\pi}\int_{\mathbb{T}} v \circ h\, d\lambda + \varepsilon. \tag{5.3}$$

The construction of g is similar to that in Section 2, but somewhat more complicated. First we prove a variant of Lemma 2.5, in which we construct $F \in C^\infty(D_r \times \mathbb{T}, X)$ and finitely many functions $\sigma_\nu \in C^\infty(\mathbb{T})$, such that $F(0, w) = h(w)$ for all $w \in \mathbb{T}$, $F(\sigma_\nu(w), w) = a_\nu$ for w on a certain arc J_ν, and

$$\sum_{\nu=1}^N \alpha(a_\nu) \int_{\mathbb{T}} \log|\sigma_\nu|\, d\lambda \le \int_{\mathbb{T}} v \circ h\, d\lambda + \varepsilon. \tag{5.4}$$

Next we use a similar approximation method as in Lemmas 2.6 and 2.7 to construct $G \in \mathcal{O}(D_t \times D_t, X)$ and $\tau_\nu \in \mathcal{O}(D_t \setminus \overline{D}_{1/t})$, such that $G(0, w) = h(w)$ for all $w \in D_t$, $G(\tau_\nu(w), w) = a_\nu$ for all $w \in J_\nu$, and

$$\sum_{\nu=1}^N \alpha(a_\nu) \int_{\mathbb{T}} \log|\tau_\nu|\, d\lambda \le \sum_{\nu=1}^N \alpha(a_\nu) \int_{\mathbb{T}} \log|\sigma_\nu|\, d\lambda + \varepsilon. \tag{5.5}$$

We prove that there exist $\xi \in \mathbb{T}$, a natural number k, $c > 0$, and $\varrho \in (0,1)$, such that $f \in \mathcal{A}_X$ and $\Phi \in \mathcal{O}(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$ defined by the formulas

$$f(z) = G(\xi z^k, z) \qquad \text{and} \qquad \Phi(z,w) = w\frac{\varrho z + e^{-c/k}}{1 + e^{-c/k}\varrho z} \tag{5.6}$$

satisfy

$$\int_0^{2\pi} H_3^\alpha(f \circ \Phi(\cdot, e^{i\theta}))\, d\theta \le \sum_{\nu=1}^N \alpha(a_\nu) \int_{\mathbb{T}} \log|\tau_\nu|\, d\lambda + \varepsilon. \tag{5.7}$$

If $v_f^\alpha(0) = -\infty$, then we simply take $g = f$. Otherwise, it turns out that one can find $g \in \mathcal{A}_X$ of the form $g(z) = f \circ \Phi(e^{i\theta_0} z, z)$, such that

$$H_3^\alpha(g) \le \frac{1}{2\pi} \int_0^{2\pi} H_3^\alpha(f \circ \Phi(\cdot, e^{i\theta}))\, d\theta. \tag{5.8}$$

We get (5.3) by combining the inequalities (5.4,5,7,8).

For the proof we need some lemmas. In all of them we assume that X is a domain in a Stein manifold, $h \in \mathcal{A}_X$, $\varepsilon > 0$, and $v \in C(X, \mathbb{R})$ with $v \ge k_X^\alpha$. In each of the proofs we also assume that the conditions in the previous lemmas are satisfied. First we prove a variant of Lemma 2.5 with k_X^α in the role of u.

5.4. Lemma. *There exist $r > 1$, $s \in (1, r)$, $F \in C^\infty(D_r \times \mathbb{T}, X)$, a natural number N, and for $\nu = 1, \dots, N$, $a_\nu \in X$, $\sigma_\nu \in C^\infty(\mathbb{T}, \mathbb{C}^*)$, and disjoint closed arcs $J_\nu \subset \mathbb{T}$, such that*

- (i) $F(\cdot, w) \in \mathcal{A}_X$ *and* $F(0, w) = h(w)$ *for all* $w \in \mathbb{T}$,
- (ii) $F(\sigma_\nu(w), w) = a_\nu$ *for all* $w \in \mathbb{T}$ *such that* $|\sigma_\nu(w)| < s$ *and then* $|\sigma_\mu(w)| > s$ *for all* $\mu \neq \nu$,
- (iii) $|\sigma_\nu(w)| < 1$ *for all* $w \in J_\nu$ *and* $\lambda(\mathbb{T} \setminus \bigcup_{\nu=1}^N J_\nu) < \varepsilon$,
- (iv) $\sigma_1(w), \dots, \sigma_N(w)$ *are distinct for any* $w \in \mathbb{T}$,
- (v) $2\pi N \max_\nu \alpha(a_\nu) \max_{\mathbb{T}} \log|\sigma_\nu| < \varepsilon/2$, *and*
- (vi) (5.4) *holds.*

Proof. Let $w_0 \in \mathbb{T}$, set $x_0 = h(w_0)$, and choose $f_0 \in \mathcal{A}_X$ such that $f_0(0) = x_0$, $f(t_0) = a_0$, and $\alpha(a_0)\log|t_0| < v(x_0) + \varepsilon/8\pi$ for some $t_0 \in \mathbb{D}^*$. By Lemma 2.3, there exists $r_0 > 1$, an open neighbourhood V_0 of x_0, and $f \in \mathcal{O}(D_{r_0} \times V_0, X)$, such that $f(z, x_0) = f_0(z)$ for all $z \in D_{r_0}$, $f(0, x) = x$, and $f(t_0, x) = f_0(t_0) = a_0$ for all $x \in V_0$. By replacing V_0 by a smaller neighbourhood of x_0 we also get $\alpha(a_0)\log|t_0| < v(x) + \varepsilon/8\pi$ for all $x \in V_0$. We can take an open arc $I_0 \subset \mathbb{T}$ containing w_0 such that $h(w) \in V_0$ for all $w \in I_0$, and define $F_0 : D_{r_0} \times I_0 \to X$ by $F_0(z, w) = f(z, h(w))$. By replacing r_0 by a smaller number in $(1, \infty)$ and I_0 by a smaller open arc containing w_0, we may assume that $F_0(D_{r_0} \times I_0)$ is relatively compact in X.

A simple compactness argument now shows that there exists a cover of $\mathbb{T}$ by open arcs $\{I_\nu\}_{\nu=1}^N$, $r_\nu > 1$, $F_\nu \in C^\infty(D_{r_\nu} \times I_\nu, X)$, $t_\nu \in \mathbb{D}^*$, and $a_\nu \in X$, such that $F_\nu(\cdot, w) \in \mathcal{A}_X$, $F_\nu(0,w) = h(w)$, $F_\nu(t_\nu, w) = a_\nu$, the set $F_\nu(D_{r_\nu} \times I_\nu)$ is relatively compact in X, and $\alpha(a_\nu)\log|t_\nu| < v(h(w)) + \varepsilon/8\pi$ for all $w \in I_\nu$. By replacing F_ν by a composition with a rotation in the first variable, we may assume that the points t_ν lie on distinct rays from the origin.

We choose $1 < s < s_0 < r = \min_\nu r_\nu$, such that $2\pi N \max_\nu \alpha(a_\nu)\log s_0 < \varepsilon/4$, a compact subset M of X containing the image of all the functions F_ν, and a constant

$$C > 1 + 2\pi N \max_\nu \alpha(a_\nu)|\log|t_\nu|| + \sup_M |v|.$$

We choose J_ν and K_ν in the same way as in the proof of Lemma 2.5, and then define the functions ϱ and F as there. Then (i) holds.

We can always choose the function ϱ such that $\varrho > 0$ on $\bigcup_{\nu=1}^N K_\nu$. Furthermore, if $K_\nu = \{e^{i\theta}\,;\, \theta \in (\alpha_\nu, \beta_\nu)\}$ and $J_\nu = \{e^{i\theta}\,;\, \theta \in [\gamma_\nu, \delta_\nu]\}$, where $\alpha_\nu < \gamma_\nu < \delta_\nu < \beta_\nu$, then we can choose ϱ increasing on (α_ν, γ_ν) and decreasing on (δ_ν, β_ν). Then $J'_\nu = \{w \in K_\nu\,;\, |t_\nu|/\varrho(w) \le s\}$ is a closed arc and $J_\nu \subset J'_\nu \subset K_\nu$. We can choose $\sigma_\nu \in C^\infty(\mathbb{T})$ with image on the ray from 0 through t_ν, such that $\sigma_\nu(w) = t_\nu/\varrho(w)$ for $w \in J'_\nu$, $s < |\sigma_\nu(w)| \le s_0$ for $w \in K_\nu \setminus J'_\nu$, and $|\sigma_\nu(w)| = s_0$ for $w \in \mathbb{T} \setminus K_\nu$. Then (ii)-(v) are satisfied.

To prove (vi), we combine our inequalities in a similar way as in the proof of Lemma 2.5:

$$\begin{aligned}\sum_{\nu=1}^N \alpha(a_\nu)\int_{\mathbb{T}} \log|\sigma_\nu|\,d\lambda &\le \sum_{\nu=1}^N \alpha(a_\nu)\int_{J_\nu} \log|t_\nu|\,d\lambda + \frac{\varepsilon}{4}\\ &\le \sum_{\nu=1}^N \int_{J_\nu} v\circ h\,d\lambda + \frac{\varepsilon}{2} \le \int_{\mathbb{T}} v\circ h\,d\lambda + \varepsilon. \qquad \square\end{aligned}$$

Now we have come to the approximation property, which is analogous to Lemma 2.6.

5.5. Lemma. *There exists a natural number j_0, and for every $j \ge j_0$ a number $s_j \in (1,s)$, $G_j \in \mathcal{O}(D_{s_j} \times D_{s_j}, X)$, and $\tau_{\nu j} \in \mathcal{O}(D_{s_j} \setminus \overline{D}_{1/s_j})$, $\nu = 1, \ldots, N$, such that*

(i) *$G_j(0,w) = h(w)$ for all $w \in D_{s_j}$,*
(ii) *$|\tau_{\nu j}| \to |\sigma_\nu|$ uniformly on $\mathbb{T}$,*
(iii) *$G_j(\tau_{\nu j}(w), w) = a_\nu$ for all $w \in D_{s_j} \setminus \overline{D}_{1/s_j}$ such that $|\tau_{\nu j}(w)| < s_j$, and*
(iv) *$|\tau_{\nu j}(w)| < 1$ for all $w \in J_\nu$.*

Proof. With the same reasoning as in the proof of Lemma 2.6, we conclude that there exists a biholomorphic map $\Phi : X \to V$, where V is a domain in a submanifold Y of $\mathbb{C}^n$, and that there exists a Stein neighbourhood Z of Y in $\mathbb{C}^n$ with a holomorphic retraction $\sigma : Z \to Y$. We set $\tilde{V} = \sigma^{-1}(V)$. Then $\tilde{V}$ is open in $\mathbb{C}^n$. We set $\tilde{F} = \Phi \circ F \in C^\infty(D_r \times \mathbb{T}, \mathbb{C}^n)$, $\tilde{h} = \Phi \circ h \in \mathcal{O}(D_r, \mathbb{C}^n)$, and $\tilde{a}_\nu = \Phi(a_\nu)$.

Since $0, \sigma_1(w), \ldots, \sigma_N(w)$ are distinct for any $w \in \mathbb{T}$, there exists a unique polynomial $P(\cdot, w)$ of one complex variable with values in $\mathbb{C}^n$ and degree at most N, which solves the interpolation problem

$$P(0, w) = \tilde{h}(w), \qquad P(\sigma_\nu(w), w) = \tilde{a}_\nu, \qquad \nu = 1, \ldots, N, \tag{5.9}$$

and we can express $P(z, w)$ by Lagrange's interpolation formula

$$P(z, w) = \tilde{h}(w) \prod_{\ell=1}^{N} \frac{z - \sigma_\ell(w)}{-\sigma_\ell(w)} + \sum_{\mu=1}^{N} \frac{z \tilde{a}_\mu}{\sigma_\mu(w)} \prod_{\substack{\ell=1 \\ \ell \neq \mu}}^{N} \frac{z - \sigma_\ell(w)}{\sigma_\mu(w) - \sigma_\ell(w)}.$$

Furthermore, we can write

$$\tilde{F}(z, w) = P(z, w) + (z - \sigma_1(w)) \cdots (z - \sigma_N(w)) \tilde{F}_0(z, w).$$

We see directly that $\tilde{F}_0(0, w) = 0$, and that $\tilde{F}_0$ is a C^∞ map and holomorphic in the first variable in a neighbourhood of every point $(z, w) \in D_s \times \mathbb{T}$ for which $|\sigma_\nu(w)| > s$ for all ν. If $|\sigma_\nu(w_0)| < s$ for some ν, then $|\sigma_\mu(w_0)| > s$ for all $\mu \neq \nu$, and there exists a neighbourhood U_0 of w_0 in $\mathbb{T}$ such that $|\sigma_\nu(w)| < s$, $|\sigma_\mu(w)| > s$, and $\tilde{F}(\sigma_\nu(w), w) = \tilde{a}_\nu$ for all $\mu \neq \nu$ and $w \in U_0$. Since P is the solution of the interpolation problem (5.9), we can write

$$P(z, w) = \tilde{F}(\sigma_\nu(w), w) + (z - \sigma_\nu(w)) P_0(z, w),$$

where $P_0 \in C^\infty(\mathbb{C} \times U_0, \mathbb{C}^n)$ is a polynomial in z. This shows that for all $(z, w) \in D_s \times U_0$ with $z \neq \sigma_\nu(w)$,

$$\tilde{F}_0(z, w) = \left(\frac{\tilde{F}(z, w) - \tilde{F}(\sigma_\nu(w), w)}{z - \sigma_\nu(w)} - P_0(z, w) \right) \prod_{\substack{\ell=1 \\ \ell \neq \nu}}^{N} \frac{1}{z - \sigma_\ell(w)}.$$

Since $\tilde{F}$ is in $C^\infty(D_r \times \mathbb{T}, \mathbb{C}^n)$ and is holomorphic in the first variable, this shows that $\tilde{F}_0 \in C^\infty(D_s \times \mathbb{T}, \mathbb{C}^n)$, and that $\tilde{F}_0$ is holomorphic in the first variable. Now we let $\tilde{F}_{0j}$ and $\sigma_{\nu j}$ be the j-th partial sums of the Fourier series of $\tilde{F}_0$ and σ_ν respectively, i.e.,

$$\tilde{F}_{0j}(z, w) = \sum_{k=-j}^{j} \left(\frac{1}{2\pi} \int_0^{2\pi} \tilde{F}_0(z, e^{i\theta}) e^{-ik\theta} \, d\theta \right) w^k$$

and

$$\sigma_{\nu j}(w) = \sum_{k=-j}^{j} \left(\frac{1}{2\pi} \int_0^{2\pi} \sigma_\nu(e^{i\theta}) e^{-ik\theta}\, d\theta \right) w^k.$$

In the same way as in the proof of Lemma 2.6, we conclude that $\tilde{F}_{0j} \to \tilde{F}_0$ uniformly on $D_t \times \mathbb{T}$ for every $t \in (1, s)$. Since $\sigma_\nu \in C^\infty(\mathbb{T})$, it also follows that $\sigma_{\nu j} \to \sigma_\nu$ uniformly on $\mathbb{T}$. We now set

$$P_j(z,w) = \tilde{h}(w) \prod_{\ell=1}^{N} \frac{z - \sigma_{\ell j}(w)}{-\sigma_{\ell j}(w)} + \sum_{\mu=1}^{N} \frac{z \tilde{a}_\mu}{\sigma_{\mu j}(w)} \prod_{\substack{\ell=1 \\ \ell \neq \mu}}^{N} \frac{z - \sigma_{\ell j}(w)}{\sigma_{\mu j}(w) - \sigma_{\ell j}(w)}.$$

and

$$\tilde{F}_j(z,w) = P_j(z,w) + (z - \sigma_{1j}(w)) \cdots (z - \sigma_{Nj}(w)) \tilde{F}_{0j}(z,w).$$

The map $\tilde{F}_j$ is meromorphic on $D_s \times D_s$ with values in $\mathbb{C}^n$, with a pole of order at most j at the origin, with no poles on $D_s \times \mathbb{T}$ for large j, and $\tilde{F}_j \to \tilde{F}$ uniformly on $D_t \times \mathbb{T}$ for all $t \in (1, s)$. Furthermore, $\sigma_{\nu j} \in \mathcal{O}(\mathbb{C}^*)$ with a pole of order at most j at 0.

Now we choose j_0 such that the numbers $\sigma_{1j}(w), \ldots, \sigma_{Nj}(w)$ are distinct and non-zero for all $j \geq j_0$ and $w \in \mathbb{T}$. Then the functions $\sigma_{\mu j}$, $\mu = 1, \ldots, N$, and $\sigma_{\mu j} - \sigma_{\ell j}$, $\mu, \ell = 1, \ldots, N$, $\mu \neq \ell$, have finitely many zeros in $\mathbb{D}$. Let $b_1, \ldots, b_{n_j}$ be an enumeration of all the zeros of these functions counted with multiplicities, and let $\mathcal{N}_j$ be the set consisting of them. We define the Blaschke product B_j by

$$B_j(w) = \prod_{k=1}^{n_j} \frac{w - b_k}{1 - \bar{b}_k w}.$$

We have $\tilde{F}_j \in \mathcal{O}(D_r \times (\overline{\mathbb{D}}^* \setminus \mathcal{N}_j), \mathbb{C}^n)$. For every $w \in \overline{\mathbb{D}}^* \setminus \mathcal{N}_j$, the function $z \mapsto \tilde{F}_{0j}(z,w)$ has a zero at the origin, and for every $z \in D_r$ the function $w \mapsto \tilde{F}_{0j}(z,w)$ has a pole of order at most j at the origin. For every $w \in \mathbb{C}^* \setminus \mathcal{N}_j$ with $|\sigma_{\nu j}(w)| < r$, we have

$$\tilde{F}_j(\sigma_{\nu j}(w), w) = \tilde{a}_\nu.$$

We now observe that for $k \geq j$, the map $\tilde{F}_{jk}$ defined by

$$\tilde{F}_{jk}(z,w) = \tilde{F}_j(zw^k B_j(w), w)$$

is holomorphic in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$, and the function $\sigma_{\nu jk}$ defined by

$$\sigma_{\nu jk}(w) = \frac{\sigma_{\nu j}(w)}{w^k B_j(w)}$$

is meromorphic in $\mathbb{C}^*$ and without zeros in $\mathbb{D}^*$. Moreover, $\tilde{F}_{jk}(\sigma_{\nu jk}(w), w) = \tilde{a}_\nu$ for every $w \in D_r^*$ with $|\sigma_{\nu j}(w)| < r$. With exactly the same reasoning as in the proof of Lemma 2.6, we conclude that there exists $k_j \geq j$ and $s_j \in (1, s)$, such that $\tilde{G}_j \in \mathcal{O}(D_{s_j} \times D_{s_j}, \tilde{V})$, where $\tilde{G}_j$ is defined by the formula

$$\tilde{G}_j(z, w) = \tilde{F}_{jk_j}(z, w) = \tilde{F}_j(zw^{k_j} B_j(w), w).$$

If we set $G_j = \Phi^{-1} \circ \sigma \circ \tilde{G}_j$, $\tau_{\nu j} = \sigma_{\nu j k_j}$, and replace s_j by a smaller number such that $\tau_{\nu j}$ is holomorphic in the annulus $D_{s_j} \setminus \overline{D}_{1/s_j}$, then (i)-(iv) are satisfied. $\square$

5.6. Lemma. *There exists $t \in (1, s)$, a map $G \in \mathcal{O}(D_t \times D_t, X)$, and functions $\tau_\nu \in \mathcal{O}(D_t \setminus \overline{D}_{1/t})$, such that*

(i) $G(0, w) = h(w)$ *for all* $w \in D_t$,
(ii) $\tau_\nu(w) \neq 0$ *for all* $w \in D_t \setminus \overline{D}_{1/t}$ *and* $|\tau_\nu| < 1$ *on* J_ν,
(iii) $G(\tau_\nu(w), w) = a_\nu$ *for all* $w \in D_t \setminus \overline{D}_{1/t}$ *such that* $|\tau_\nu(w)| < t$,
(iv) $2\pi N \max_\nu \alpha(a_\nu) \max_{\mathbb{T}} \log|\tau_\nu| < \varepsilon/2$, *and*
(v) (5.5) *holds.*

Proof. This follows directly from Lemmas 5.4 and 5.5, if we take j sufficiently large, and set $G = G_j$, $t = s_j$, and $\tau_\nu = \tau_{\nu j}$. $\square$

5.7. Lemma. *There exist $\xi \in \mathbb{T}$, a natural number k, $c > 0$, and $\varrho \in (0, 1)$, such that $f \in \mathcal{A}_X$ and $\Phi \in \mathcal{O}(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$ defined by* (5.6) *satisfy* (5.7)*. Furthermore, for k sufficiently large, the derivatives of the functions $z \mapsto \Phi(z, w)$ and $z \mapsto \Phi(wz, z)$ are non-zero at every point $z \in \overline{\mathbb{D}}$ for each $w \in \mathbb{T}$.*

Proof. Since $\tau_\nu(w) \neq 0$ for all $w \in D_t \setminus \overline{D}_{1/t}$, we can choose $c > 0$ so large that

$$\log\left|\frac{\eta e^{-c} - \tau_\nu(w)}{1 - \overline{\tau_\nu(w)}\eta e^{-c}}\right| < \log|\tau_\nu(w)| + \frac{\varepsilon}{2M} \tag{5.10}$$

for all $\eta \in \overline{\mathbb{D}}$, $w \in \mathbb{T}$, and $\nu = 1, \ldots, N$, where $M > \sum_\nu \alpha(a_\nu)$. We define the function $\psi \in \mathcal{O}(\mathbb{C} \setminus \{-1\})$ by

$$\psi(z) = \exp\left(c\frac{z-1}{z+1}\right).$$

Observe that $z \mapsto (z-1)/(z+1)$ maps $\mathbb{D}$ onto the left half plane, and $\mathbb{T} \setminus \{-1\}$ onto the imaginary axis. Hence ψ maps $\mathbb{D}$ onto $\mathbb{D}^*$, and $\mathbb{T} \setminus \{-1\}$ onto $\mathbb{T}$. For every $\eta \in \mathbb{T}$ and every $w \in J_\nu$, we define $\varphi_\nu(\,\cdot\,; \eta, w) \in \mathcal{O}(\mathbb{C} \setminus \{-1\})$ by

$$\varphi_\nu(z; \eta, w) = \frac{\eta\psi(z) - \tau_\nu(w)}{1 - \overline{\tau_\nu(w)}\eta\psi(z)}.$$

Since $|\tau_\nu(w)| < 1$ for all $w \in J_\nu$, $\psi(\mathbb{D}) = \mathbb{D}^*$, and $\psi(\mathbb{T}\setminus\{-1\}) = \mathbb{T}$, we see that $\varphi_\nu(\cdot\,;\eta,w)$ is an inner function, and since it is continuous on $\overline{\mathbb{D}}\setminus\{-1\}$, we have $|\varphi_\nu(z;\eta,w)| = 1$ for all $z \in \mathbb{T}\setminus\{-1\}$, and

$$\lim_{t\to -1+} \varphi_\nu(t;\eta,w) = -\tau_\nu(w) \neq 0.$$

Hence, the function $\varphi_\nu(\cdot\,;\eta,w)$ has no radial limit equal to zero. Now Frostman's theorem implies that $\varphi_\nu(\cdot\,;\eta,w)$ is a Blaschke product. See Noshiro [1960, p. 33].

Observe that every zero of $\varphi_\nu(\cdot\,;\eta,w)$ is a zero of the function $z \mapsto \eta\psi(z) - \tau_\nu(w)$. Its derivative $\eta\psi'(z)$ is non-zero at every point $z \in \mathbb{D}$, so all the zeros are simple. Let us now take $(z_0,\eta_0,w_0) \in \mathbb{D}\times\mathbb{T}\times J_\nu$, and assume that $\varphi_\nu(z_0;\eta_0,w_0) = 0$. Take an open disk D_0 such that z_0 is the only zero of $\varphi_\nu(\cdot\,;\eta_0,w_0)$ in $\overline{D}_0$. Then there exists an open neighbourhood U_0 of (η_0,w_0) in $\mathbb{C}^2$ such that $(z,\eta,w) \mapsto \varphi_\nu(z;\eta,w)$ is holomorphic in $D_0\times U_0$, the function λ given by

$$\lambda(\eta,w) = \frac{1}{2\pi i}\int_{\partial D_0} \frac{z\eta\psi'(z)}{\eta\psi(z) - \tau_\nu(w)}\,dz$$

is holomorphic in U_0, $\lambda(\eta_0,w_0) = z_0$, and $\varphi_\nu(\lambda(\eta,w);\eta,w) = 0$ for all $(\eta,w) \in U_0$.

We let $\{z_{0\nu l}\}_{l=1}^\infty$ be the zeros of $\varphi_\nu(\cdot\,;\eta_0,w_0)$. Since $\varphi_\nu(\cdot\,;\eta_0,w_0)$ is a Blaschke product, we have

$$|\varphi_\nu(0;\eta_0,w_0)| = \left|\frac{\eta_0 e^{-c} - \tau_\nu(w_0)}{1 - \overline{\tau_\nu(w_0)}\eta_0 e^{-c}}\right| = \prod_{l=1}^\infty |z_{0\nu l}|,$$

and from (5.10) it follows that we can find a natural number L_0 and a real number $\varrho_0 \in (0,1)$ such that

$$\sum_{l=1}^{L_0} \log(|z_{0\nu l}|/\varrho_0) < \log|\tau_\nu(w_0)| + \frac{\varepsilon}{2M}.$$

Now each of the zeros $z_{0\nu l}$ defines a holomorphic function $z_{\nu l}$, such that $z_{\nu l}(\eta_0,w_0) = z_{0\nu l}$, $\varphi_\nu(z_{\nu l}(\eta,w);\eta,w) = 0$, and

$$\sum_{l=1}^{L_0} \log(|z_{\nu l}(\eta,w)|/\varrho_0) < \log|\tau_\nu(w)| + \frac{\varepsilon}{2M}$$

for all (η,w) in some neighbourhood of (η_0,w_0). By a simple compactness argument, we now conclude that there exist an integer L and $\varrho \in (0,1)$, such that for every $(\eta,w) \in \mathbb{T}\times J_\nu$ we can find zeros $\lambda_{\nu 1}(\eta,w),\dots,\lambda_{\nu L}(\eta,w)$ of $\varphi_\nu(\cdot\,;\eta,w)$, satisfying

$$\sum_{l=1}^{L} \log(|\lambda_{\nu l}(\eta,w)|/\varrho) < \log|\tau_\nu(w)| + \frac{\varepsilon}{2M}. \tag{5.11}$$

Now we define the functions $\psi_k \in \mathcal{O}(\mathbb{C} \setminus \{-e^{c/k}\})$ by

$$\psi_k(z) = \frac{z + e^{-c/k}}{1 + e^{-c/k} z}.$$

By writing

$$\psi_k(z) = 1 + (1 - e^{-c/k}) \frac{z - 1}{1 + e^{-c/k} z},$$

we see that $\psi_k \to 1$ uniformly on compact subsets of $\mathbb{D}$, and if we let log denote the principal branch of the logarithm, then we see that

$$\lim_{k\to\infty} k \log \psi_k(z) = \lim_{k\to\infty} k(1 - e^{-c/k}) \frac{z-1}{1 + e^{-c/k} z} = c\frac{z-1}{z+1},$$

and the convergence is uniform on compact subsets of $\mathbb{D}$. This implies that $\psi_k^k \to \psi$ uniformly on compact subsets of $\mathbb{D}$. We choose $t_0 \in (1, 1/\varrho)$. Since $|\tau_\nu| < 1$ on J_ν, we can choose a neighbourhood U_ν of J_ν in D_t such that $|\tau_\nu| < 1$ on U_ν, and since $\psi_k \to 1$ uniformly on compact subsets of $\mathbb{D}$, we can choose k_0 such that $w\psi_k(\varrho z) \in U_\nu$ for all $k \geq k_0$, $w \in J_\nu$, and $z \in D_{t_0}$. Condition (iii) in Lemma 5.6 now implies that

$$G(\tau_\nu(w\psi_k(\varrho z)), w\psi_k(\varrho z)) = a_\nu, \qquad k \geq k_0, w \in J_\nu, z \in D_{t_0}. \tag{5.12}$$

We observe that $\{\lambda_{\nu l}(\eta, w)/\varrho\}_{l=1}^L$ are zeros of the function

$$z \mapsto \eta\psi(\varrho z) - \tau_\nu(w),$$

and that this function is the uniform limit on D_{t_0} of the sequence of functions

$$z \mapsto \eta\psi_k(\varrho z)^k - \tau_\nu(w\psi_k(\varrho z)), \qquad k \geq k_0.$$

By Hurwitz' theorem we conclude from (5.11) that for k large enough there are zeros $\lambda_{\nu l}^k(\eta, w)$ of this function so close to $\lambda_{\nu l}(\eta, w)/\varrho$ that

$$\sum_{l=1}^{L} \log |\lambda_{\nu l}^k(\eta, w)| < \log |\tau_\nu(w)| + \frac{\varepsilon}{2M}. \tag{5.13}$$

Now (5.12) implies that for $z = \lambda_{\nu 1}^k(\eta, w), \dots, \lambda_{\nu L}^k(\eta, w)$, and $(\eta, w) \in \mathbb{T} \times J_\nu$, we have

$$G(\eta\psi_k(\varrho z)^k, w\psi_k(\varrho z)) = a_\nu. \tag{5.14}$$

If we set $Q = \bigcup(\mathbb{T} \times J_\nu)$, then (5.13-14) implies that

$$H_3^\alpha(G(\eta\psi_k(\varrho\,\boldsymbol{\cdot})^k, w\psi_k(\varrho\,\boldsymbol{\cdot}))) < \sum_{\nu=1}^{N} \alpha(a_\nu) \log |\tau_\nu(w)| + \varepsilon/2, \qquad (\eta, w) \in Q. \tag{5.15}$$

Now let S denote the image of Q under the map $\mathbb{T}^2 \to \mathbb{T}^2$, $(\eta, w) \mapsto (\eta w^{-k}, w)$. Since the absolute value of the Jacobian of this map is 1, we have $\lambda(S) = \lambda(Q) \geq 2\pi(2\pi - \varepsilon)$, and we conclude that there exists $\xi \in \mathbb{T}$ such that $\lambda(R) \geq 2\pi - \varepsilon$, where $R = \{w \in \mathbb{T}\,;\, (\xi, w) \in S\}$. By (5.15) we have

$$H_3^\alpha(G(\xi(w\psi_k(\varrho\,\cdot))^k, w\psi_k(\varrho\,\cdot))) \leq \sum_{\nu=1}^N \alpha(a_\nu) \log|\tau_\nu(w)| + \varepsilon/2, \qquad w \in R,$$

and property (iv) in Lemma 5.6 implies that

$$\int_0^{2\pi} H_3^\alpha(G(\xi(e^{i\theta}\psi_k(\varrho\,\cdot))^k, e^{i\theta}\psi_k(\varrho\,\cdot)))\, d\theta \leq \sum_{\nu=1}^N \alpha(a_\nu) \int_{\mathbb{T}} \log|\tau_\nu|\, d\lambda + \varepsilon.$$

A priori, we interpret the integral on the left as an upper integral, i.e., the infimum of the integrals of all Borel functions that dominate the integrand. Now we define f and Φ by (5.6), observe that $\Phi(z, w) = w\psi_k(\rho z)$, and conclude that (5.7) holds. The last statement of the lemma is obvious from the fact that $\psi_k \to 1$ locally uniformly on $\mathbb{D}$. $\square$

5.8. Lemma. *Let $f \in \mathcal{A}_X$ and $\Phi \in \mathcal{O}(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$ with $\Phi(\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$. Assume that $v_f^\alpha(0) > -\infty$, and that the derivatives of the functions $z \mapsto \Phi(z, w)$ and $z \mapsto \Phi(wz, z)$ are non-zero at every point $z \in \overline{\mathbb{D}}$ for each $w \in \mathbb{T}$. Then there exists $g \in \mathcal{A}_X$ defined by $g(z) = f \circ \Phi(e^{i\theta_0}z, z)$, $z \in \overline{\mathbb{D}}$, for some $\theta_0 \in [0, 2\pi]$, such that* (5.8) *holds.*

Proof. Let us first observe that

$$\Delta v_{f\circ\varphi}^\alpha = \Delta(v_f^\alpha \circ \varphi) \qquad \text{in } \mathbb{D}, \tag{5.16}$$

for every $\varphi \in \mathcal{O}(\overline{\mathbb{D}})$ with $\varphi(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$ and $\varphi'(z) \neq 0$ for every $z \in \mathbb{D}$. In fact, since $v_f^\alpha \not\equiv -\infty$, we have

$$\Delta v_f^\alpha = 2\pi \sum_{b\in\mathbb{D}} f^*\alpha(b)\delta_b,$$

where the sum is countable and convergent in the sense of distributions. By assumption, $\varphi(\mathbb{D})$ is open, so $v_f^\alpha \circ \varphi \not\equiv -\infty$, and the pullback $\varphi^* u$ of any distribution u on $\mathbb{D}$ is well defined. Therefore,

$$\Delta(v_f^\alpha \circ \varphi) = \Delta\varphi^* v_f^\alpha = \varphi^*(\Delta v_f^\alpha)|\varphi'|^2 = 2\pi \sum_{b\in\mathbb{D}} f^*\alpha(b)\varphi^*\delta_b\, |\varphi'|^2.$$

If $b \in \mathbb{D}$ and $\varphi(c) = b$ for some $c \in \mathbb{D}$, then there is a neighbourhood U of c such that $\varphi^{-1}(b) \cap U = \{c\}$, and $\varphi|U$ is biholomorphic onto a neighbourhood of b. Hence, $\varphi^*\delta_b = |\varphi'(c)|^{-2}\delta_c$ in U. This implies that

$$\Delta(v_f^\alpha \circ \varphi) = 2\pi \sum_{c\in\mathbb{D}} (f\circ\varphi)^*\alpha(c)\, \delta_c = \Delta v_{f\circ\varphi}^\alpha.$$

From (5.16) we get

$$H_3^\alpha(f\circ\varphi)=\int_{\mathbb{D}}\log|\cdot|\,\Delta(v_f^\alpha\circ\varphi). \tag{5.17}$$

Now we need the following lemma.

5.9. Lemma. (See Poletsky [1993, Lemma 3.2].) *Let $r>1$, and $w\in \mathrm{PSH}(D_r\times D_r)$. For $\theta\in\mathbb{R}$, we define a subharmonic function w_θ on D_r by the formula $w_\theta(z)=w(e^{i\theta}z,z)$. Then there exists $\theta_0\in[0,2\pi]$ such that*

$$\int_{\mathbb{D}}\log|\cdot|\,\Delta w_{\theta_0}\le\frac{1}{2\pi}\int_0^{2\pi}\left(\int_{\mathbb{D}}\log|\cdot|\,\Delta w(\cdot,e^{i\theta})\right)d\theta. \tag{5.18}$$

Proof. If $w=-\infty$, then both sides of (5.18) are zero, so we may assume that $w\not\equiv-\infty$. The Riesz representation formula applied to w_θ gives

$$2\pi w(0,0)=\int_{\mathbb{D}}\log|\cdot|\,\Delta w_\theta+\int_0^{2\pi}w(e^{i(\theta+t)},e^{it})\,dt.$$

Since $w\not\equiv-\infty$, there exists $\theta_0\in[0,2\pi]$ with $w_{\theta_0}\not\equiv-\infty$, for otherwise w would be $-\infty$ on a 3-real-dimensional set in $D_r\times D_r$, so $\int_{\mathbb{T}}w_{\theta_0}\,d\lambda\neq-\infty$. If $w(0,0)=-\infty$, this shows that the left hand side of (5.18) is $-\infty$.

Now let us assume that $w(0,0)\neq-\infty$. By integrating with respect to θ, we get

$$(2\pi)^2w(0,0)=\int_0^{2\pi}\left(\int_{\mathbb{D}}\log|\cdot|\,\Delta w_\theta\right)d\theta+\int_0^{2\pi}\int_0^{2\pi}w(e^{i\theta},e^{it})\,dt\,d\theta.$$

Applying the Riesz representation formula to $z\mapsto w(0,z)$, we get

$$2\pi w(0,0)=\int_{\mathbb{D}}\log|\cdot|\,\Delta w(0,\cdot)+\int_0^{2\pi}w(0,e^{i\theta})\,d\theta.$$

By assumption, $w(0,\cdot)\neq-\infty$ a.e. on $\mathbb{T}$, so the Riesz representation formula applied to $z\mapsto w(z,e^{i\theta})$ shows that

$$\int_{\mathbb{D}}\log|\cdot|\Delta w(\cdot,e^{i\theta})=2\pi w(0,e^{i\theta})-\int_0^{2\pi}w(e^{it},e^{i\theta})\,dt\qquad\text{for a.e. }\theta.$$

The terms on the right are semi-continuous, so $\theta\mapsto\int_{\mathbb{D}}\log|\cdot|\Delta w(\cdot,e^{i\theta})$ is Lebesgue-measurable. Also,

$$(2\pi)^2w(0,0)\le\int_0^{2\pi}\left(\int_{\mathbb{D}}\log|\cdot|\,\Delta w(\cdot,e^{i\theta})\right)d\theta+\int_0^{2\pi}\int_0^{2\pi}w(e^{it},e^{i\theta})\,dt\,d\theta.$$

Hence,

$$\int_0^{2\pi} \left(\int_{\mathbb{D}} \log|\cdot|\,\Delta w_\theta \right) d\theta \le \int_0^{2\pi} \left(\int_{\mathbb{D}} \log|\cdot|\,\Delta w(\cdot, e^{i\theta}) \right) d\theta,$$

and (5.18) follows. □

End of proof of Lemma 5.8. Set $w = v_f^\alpha \circ \Phi$, choose θ_0 such that (5.18) holds, set $g(z) = f \circ \Phi(e^{i\theta_0}z, z)$, and $\varphi(z) = \Phi(e^{i\theta_0}z, z)$. Then (5.16-18) give

$$\begin{aligned} H_3^\alpha(g) &= \int_{\mathbb{D}} \log|\cdot|\,\Delta(v_f^\alpha \circ \varphi) = \int_{\mathbb{D}} \log|\cdot|\,\Delta w_{\theta_0} \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbb{D}} \log|\cdot|\,\Delta w(\cdot, e^{i\theta}) \right) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbb{D}} \log|\cdot|\,\Delta(v_f^\alpha \circ \Phi(\cdot, e^{i\theta})) \right) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} H_3^\alpha(f \circ \Phi(\cdot, e^{i\theta}))\, d\theta. \end{aligned}$$

In particular, this computation shows that the last integrand is a Lebesgue-measurable function of θ. □

Proof of Theorem 5.3. We have already seen that $EH_3^\alpha \ge EH_1^{k_X^\alpha}$. As we noted after the statement of Theorem 5.3, the inequality $EH_3^\alpha \le EH_1^{k_X^\alpha}$ will follow if we can prove that for every $h \in \mathcal{A}_X$, $\varepsilon > 0$, and $v \in C(X, \mathbb{R})$ with $v \ge k_X^\alpha$, there exists $g \in \mathcal{A}_X$ such that $g(0) = h(0)$ and (5.3) holds. We choose F, a_ν, σ_ν, G, τ_ν, f, and Φ as in Lemmas 5.4-9. If $v_f^\alpha(0) = -\infty$, then we set $g = f$. Then $g(0) = G(0,0) = h(0)$, and (5.3) holds trivially. If $v_f^\alpha(0) > -\infty$, then we choose g satisfying the conditions in Lemma 5.8, and (5.3) follows by combining (5.4,5,7,8). Now Theorem 2.2 and Proposition 5.2 imply directly that EH_3^α is plurisubharmonic. □

In the remainder of this section, we will show that, with some restrictions, the Lelong functional has plurisubharmonic envelopes on manifolds in $\mathcal{P}$.

We note that if $h : X \to Y$ is a holomorphic covering, and $\beta : Y \to [0, \infty)$, then

$$h^* H_3^\beta = H_3^{\beta \circ h} \qquad \text{so} \qquad EH_3^{\beta \circ h} = EH_3^\beta \circ h.$$

This implies the following result.

5.10. Proposition. *Let X and Y be complex manifolds such that there is a holomorphic covering $X \to Y$. If $EH_3^\alpha \in \mathrm{PSH}(X)$ for all non-negative functions α on X, then $EH_3^\beta \in \mathrm{PSH}(Y)$ for all non-negative functions β on Y.*

For finite branched coverings we have the following result. Unfortunately, we are unable to deal with the case when the non-negative function is non-zero at a branch point.

5.11. Proposition. *Let $h : X \to Y$ be a finite branched covering. Let β be a non-negative function on Y which is zero on the branch locus of h. If $EH_3^{\beta\circ h}$ is plurisubharmonic on X, then*

$$EH_3^{\beta\circ h} = EH_3^\beta \circ h \qquad so \qquad EH_3^\beta = h_* EH_3^{\beta\circ h},$$

and EH_3^β is plurisubharmonic on Y.

Proof. If $f \in \mathcal{A}_X$, then $H_3^{\beta\circ h}(f) = H_3^\beta(h \circ f)$, so $EH_3^\beta \circ h \le EH_3^{\beta\circ h}$. By Proposition 5.1, $EH_3^{\beta\circ h} \in \mathcal{F}_{\beta\circ h}$. Since h is unbranched over points where $\beta > 0$, and the Lelong number is additive, this implies that $h_* EH_3^{\beta\circ h} \in \mathcal{F}_\beta$. Hence,

$$h_* EH_3^{\beta\circ h} \circ h \le EH_3^\beta \circ h \le EH_3^{\beta\circ h},$$

so $EH_3^{\beta\circ h} = EH_3^\beta \circ h$. $\square$

5.12. Theorem. *Let α be a non-negative function on a manifold X in $\mathcal{P}$. Suppose there exists a sequence*

$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \dots \xrightarrow{h_m} X_m = X, \qquad m \ge 0,$$

of complex manifolds and holomorphic maps, where X_0 is a domain in a Stein manifold and each h_i, $i = 1, \dots, m$, is either a covering or a finite branched covering, such that

$$\alpha^{-1}[c, \infty) \setminus B \text{ is Zariski-dense in } \alpha^{-1}[c, \infty) \text{ for each } c > 0, \tag{5.19}$$

where

$$B = \bigcup_{i=1}^m (h_m \circ \dots \circ h_{i+1})(B_i),$$

and B_i denotes the (possibly empty) branch locus of h_i.

Then EH_3^α is plurisubharmonic.

Clearly, (5.19) is true if $\alpha = 0$ on B. If $\alpha = \hat\alpha$, i.e., $\alpha^{-1}[c, \infty)$ is a subvariety of X for each $c > 0$, then $\alpha^{-1}(0, \infty)$ is a countable union of subvarieties of X, and α satisfies (5.19) if and only if B contains no irreducible component of $\alpha^{-1}(0, \infty)$.

Proof. Let $\beta = \chi\alpha$, where χ denotes the characteristic function of $X \setminus B$. Then $\hat\beta = \hat\alpha$ by assumption. Since β vanishes on B, Theorem 5.3 and Propositions 5.10 and 5.11 imply that EH_3^β is plurisubharmonic. Hence, by the remarks following Proposition 5.1, $EH_3^\beta = EH_3^{\hat\beta}$. Also, $\beta \le \alpha \le \hat\alpha = \hat\beta$, so $EH_3^\alpha = EH_3^\beta$ is plurisubharmonic. $\square$

On covering spaces of projective manifolds we can ignore the branch loci if α vanishes outside a countable set.

5.13. Theorem. *Let X be a covering space of a projective manifold. If $\alpha : X \to [0, \infty)$ vanishes outside a countable set, then EH_3^α is plurisubharmonic.*

Proof. Let $p : X \to M$ be a holomorphic covering onto a projective manifold M. Let $S = p\alpha^{-1}(0, \infty)$. Then S is a countable subset of M. Let $h : N \to M$ be the finite branched covering provided by Theorem 3.8 applied to M and S. Proceeding as in the proof of Corollary 3.10, we obtain a finite branched covering $Y \to X$, whose branch locus B is the preimage under p of the branch locus of h, so $\alpha = 0$ on B. Furthermore, Y is covered by a Stein manifold. The conclusion now follows from Theorem 5.12. □

6. Compact manifolds

We will now consider envelopes of disc functionals on a compact complex manifold X. Here, the problem takes on a different character, because all plurisubharmonic functions on X are constant, so methods for constructing them are not of interest. Instead, results on plurisubharmonicity of envelopes provide information about existence of analytic discs in X. We will consider only the Poisson functional and the Lelong functional, since the Riesz functional is identically zero on a compact manifold.

6.1. Proposition. *The following are equivalent for a compact complex manifold X.*

(1) *The Poisson functional has plurisubharmonic envelopes on X.*
(2) *For every $p \in X$, $U \neq \varnothing$ open in X, and $\varepsilon > 0$, there is $f \in \mathcal{A}_X$ such that $f(0) = p$ and*

$$\lambda(\mathbb{T} \setminus f^{-1}(U)) < \varepsilon.$$

Proof. (1) ⇒ (2): Let $p \in X$ and $U \neq \varnothing$ be open in X. Let φ equal 0 on U and 1 on $X \setminus U$. Then φ is upper semi-continuous on X, $u = EH_1^\varphi$ is plurisubharmonic on X, and $u = 0$ on U, so $u = 0$ on X. Hence, for every $\varepsilon > 0$ there is $f \in \mathcal{A}_X$ with $f(0) = p$ and

$$\varepsilon > \int_{\mathbb{T}} \varphi \circ f \, d\lambda = \lambda(\mathbb{T} \setminus f^{-1}(U)).$$

(2) ⇒ (1): Let φ be upper semi-continuous on X. Let $a \in \mathbb{R} \cup \{-\infty\}$ be the infimum of φ. To show that $EH_1^\varphi = a$, just apply (2) to the non-empty open sets $U = \{\varphi < c\}$, where $c \searrow a$, and note that by compactness, φ is bounded above. □

The proof shows that (1) implies (2) for any manifold X such that every plurisubharmonic function on X which is bounded above is constant.

6.2. Remark. Let us note that when constructing envelopes of disc functionals on a manifold X, we cannot restrict ourselves to the family of analytic discs $f : \overline{\mathbb{D}} \to X$ that extend holomorphically to a single larger disc D_r, $r > 1$. More precisely, there is a manifold X in the class $\mathcal{P}$, and an upper semi-continuous function φ on X, such that the function v defined by the formula

$$v(x) = \inf\{\int_{\mathbb{T}} \varphi \circ f \, d\lambda \,;\, f \in \mathcal{O}(D_r, X), f(0) = x\}, \qquad x \in X,$$

is not plurisubharmonic on X for any $r > 1$.

To see this, let X be a compact manifold whose universal covering space is the unit ball in $\mathbb{C}^n$. There are many examples of such manifolds. Then X is in $\mathcal{P}$. Let $p \in X$, and let (U_n) be a decreasing neighbourhood basis of a point $q \neq p$ in X. Let φ_n equal 0 on U_n and 1 on $X \setminus U_n$. Fix $r > 1$. If v defined as above using φ_n is plurisubharmonic, then $v = 0$, so as in the proof of Proposition 6.1, we get holomorphic maps $f_n : D_r \to X$ such that $f_n(0) = p$ and $\lambda(\mathbb{T} \setminus f_n^{-1}(U_n)) < 1/n$. Now X is taut, so there is a subsequence of (f_n) that converges uniformly on compact sets to a holomorphic map $f : D_r \to X$. Then $f(0) = p$ but $f(\mathbb{T}) = \{q\}$, which is absurd.

Now we turn to the Lelong functional.

6.3. Proposition. *The following are equivalent for a complex manifold X such that every plurisubharmonic function on X which is bounded above is constant.*

(1) *The Lelong functional has plurisubharmonic envelopes on X.*
(2) *For every $p, q \in X$, $p \neq q$, and $M > 0$, there is $f \in \mathcal{A}_X$ such that $f(0) = p$ and*

$$\sum_{z \in f^{-1}(q)} m_z(f) \log|z| < -M.$$

Proof. (1) $\Rightarrow$ (2): Let α equal 1 at q and 0 on $X \setminus \{q\}$. By assumption, $u = EH_3^\alpha$ is plurisubharmonic. Also, $u(q) = -\infty$, so $u = -\infty$, and (2) is immediate.

(2) $\Rightarrow$ (1): Let $\alpha : X \to [0, \infty)$. If $\alpha = 0$, then $u = EH_3^\alpha = 0$. Say $\alpha(q) > 0$. Then $u(q) = -\infty$. Applying (2) to any $p \neq q$, we get $u(p) = -\infty$. Hence, $u = -\infty$. □

The proof shows that (2) implies (1) for all manifolds X.

The following is an immediate consequence of Propositions 6.1 and 6.3.

6.4. Proposition. *Suppose X and Y are complex manifolds with no non-constant negative plurisubharmonic functions, and $h : X \to Y$ is a surjective holomorphic map.*

If the Lelong functional has plurisubharmonic envelopes on X, then the Lelong functional has plurisubharmonic envelopes on Y.

If Y is compact and the Poisson functional has plurisubharmonic envelopes on X, then the Poisson functional has plurisubharmonic envelopes on Y.

Recall that a compact complex manifold X is called Moishezon if the transcendence degree of its field of meromorphic functions equals its dimension. Then X can be made into a projective manifold by blowing up finitely many submanifolds. In particular, X is the image of a holomorphic map from a projective manifold.

6.5. Corollary. *The Poisson functional and the Lelong functional have plurisubharmonic envelopes on Moishezon manifolds.*

Proof. By Proposition 6.4 and the remarks above, it suffices to show that the Poisson functional and the Lelong functional have plurisubharmonic envelopes on a projective manifold X.

Let $\alpha : X \to [0, \infty)$. If $\alpha = 0$, then $EH_3^\alpha = 0$. Otherwise, let $\beta(p) = \alpha(p) > 0$ for some $p \in X$, and $\beta = 0$ on $X \setminus \{p\}$. By Theorem 5.13, EH_3^β is plurisubharmonic, so $EH_3^\beta = -\infty$, and $EH_3^\alpha = -\infty$.

To show that the Poisson functional has plurisubharmonic envelopes on X we can invoke Theorem 3.4 and Corollary 3.10, but a more elementary proof which avoids the latter can be given. Namely, let $p \in X$ and $U \neq \varnothing$ be open in X. By embedding X in some projective space and intersecting it transversely with a linear subspace of the appropriate dimension, we obtain a smooth 1-dimensional subvariety Y in X containing p and intersecting U. By Theorem 3.4 and Proposition 3.7, the Poisson functional has plurisubharmonic envelopes on the compact Riemann surface Y. By Proposition 6.1, there are $f \in \mathcal{A}_Y$ such that $f(0) = p$ and $\lambda(\mathbb{T} \setminus f^{-1}(U \cap Y))$ is arbitrarily small. This shows that the Poisson functional has plurisubharmonic envelopes on X, again by Proposition 6.1. $\square$

We do not know if all Moishezon manifolds belong to the class $\mathcal{P}$.

7. Final remarks

Let us recall the classical Kontinuitätssatz; see for instance Krantz [1992].

7.1. Kontinuitätssatz. *Let X be a domain in $\mathbb{C}^n$. The following are equivalent.*

(1) *X is pseudoconvex.*
(2) *If $f_n \in \mathcal{A}_X$, $n \in \mathbb{N}$, and $\bigcup f_n(\mathbb{T}) \subset\subset X$, then $\bigcup f_n(\overline{\mathbb{D}}) \subset\subset X$.*

The statement (2) is called the Kontinuitätsprinzip.

Since we can reparametrize analytic discs at will, we see that (2) is equivalent to the following statement:

(2') If $f_n \in \mathcal{A}_X$, $n \in \mathbb{N}$, and $\bigcup f_n(\mathbb{T}) \subset\subset X$, then $f_n(0) \not\to \infty$.

Here, ∞ denotes the point at infinity in the one-point compactification of X.

We do not know if the Kontinuitätssatz generalizes to arbitrary manifolds. However, the theory of the Poisson functional provides a new Kontinuitätssatz with a stronger Kontinuitätsprinzip, which holds for a great many manifolds.

7.2. Theorem. *Let X be a manifold on which the Poisson functional has plurisubharmonic envelopes, such as a manifold in the class $\mathcal{P}$. The following are equivalent.*

(1) *X is pseudoconvex, meaning that X has a plurisubharmonic exhaustion function.*

(2) *There is an upper semi-continuous function* φ *on* X *such that if* $f_n \in \mathcal{A}_X$*,* $n \in \mathbb{N}$*, and*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{T}} \varphi \circ f_n \, d\lambda < \infty,$$

then $f_n(0) \not\to \infty$*.*

Proof. If X is pseudoconvex, take φ to be a plurisubharmonic exhaustion of X. For the converse, if φ is as in (2), then EH_1^{φ} is a plurisubharmonic exhaustion on X. □

The contrapositive of this theorem is also of interest.

7.3. Theorem. *Let* X *be a manifold on which the Poisson functional has plurisubharmonic envelopes. The following are equivalent.*

(1) X *is not pseudoconvex.*
(2) *For every upper semi-continuous function* $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ *there are* $f_n \in \mathcal{A}_X$*,* $n \in \mathbb{N}$*, such that* $f_n(0) \to \infty$ *and*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{T}} \varphi \circ f_n \, d\lambda < \infty.$$

Taking φ to be an exhaustion, we see that (2) implies that there are $f_n \in \mathcal{A}_X$ with $f_n(0) \to \infty$, such that for every $\varepsilon > 0$ there is a compact set K in X with

$$\lambda(\mathbb{T} \setminus f_n^{-1}(K)) < \varepsilon \qquad \text{for all } n.$$

Roughly speaking, X fails to be pseudoconvex because it contains arbitrarily large analytic discs whose boundaries stay mostly within a compact set.

This is a very interesting feature of the theory of envelopes of disc functionals, viewed as a method for constructing plurisubharmonic functions. The method tries to construct a plurisubharmonic function with specified properties (here, an exhaustion), and if it fails, it tells us why. It gives an obstruction in terms of the existence of analytic discs with certain properties. Here, it is easy to see that (2) implies (1) for all manifolds, but what the method shows is that if X is not pseudoconvex, the reason is the existence of analytic discs as in (2). Thus the method gives a general answer to the question, *why is this manifold not pseudoconvex?*, in terms of the existence, rather than the *non*-existence of something (at least for a large class of manifolds). Before, this question did not even seem to make sense in general.

Here is a sample problem about plurisubharmonic functions. *When does a plurisubharmonic function on a submanifold extend to a plurisubharmonic function on the ambient manifold?* It is known that the answer is affirmative when the ambient manifold is Stein, but otherwise this is essentially an open question. A good answer would have many interesting applications, for instance to pseudoconvexity of covering spaces. There are various results in this vein for holomorphic functions, but one would like to tackle the question without using holomorphic functions, because while the two classes of functions are closely related locally, globally this relation is very subtle, as evidenced by the existence of non-compact pseudoconvex manifolds with no non-constant holomorphic functions. But here we run into the problem that for plurisubharmonic functions we have nothing comparable to the powerful methods for constructing holomorphic functions, such as the celebrated $\overline{\partial}$-method.

The theory of envelopes of disc functionals is a new candidate for a general method for constructing plurisubharmonic functions. Although existing work has primarily focussed on developing the basic theory, there is already at least one important application: Poletsky's characterization of the polynomial hull of a pluriregular compact set in $\mathbb{C}^n$ [1993]. Let us conclude by giving a simple proof of a variant of this result.

7.4. Theorem. *For a compact set K and a point p in $\mathbb{C}^n$, the following are equivalent.*

(1) *p is in the polynomial hull of K.*

(2) *There is an open ball B containing K and p such that for every neighbourhood U of K and every $\varepsilon > 0$, there is $f \in \mathcal{A}_B$ with $f(0) = p$ and*

$$\lambda(\mathbb{T} \setminus f^{-1}(U)) < \varepsilon.$$

Proof. (1) $\Rightarrow$ (2): Let B be an open ball containing K and p. Suppose p is in the polynomial hull of K. Then p is in the plurisubharmonic hull of K in B. Let U be a neighbourhood of K in B, and set $\varphi = 0$ on U and $\varphi = 1$ on $B \setminus U$. Then φ is upper semi-continuous on B, and $u = EH_1^\varphi$ is plurisubharmonic on B. Since $u = 0$ on $U \supset K$, we have $u(p) = 0$. By the definition of the Poisson functional, this means that for every $\varepsilon > 0$ there is $f \in \mathcal{A}_B$ such that $f(0) = p$ and $\varepsilon > 2\pi H_1^\varphi(f) = \lambda(\mathbb{T} \setminus f^{-1}(U))$.

(2) $\Rightarrow$ (1): Let P be a polynomial. Then

$$|P(p)| \leq \frac{1}{2\pi} \int_{\mathbb{T}} |P| \circ f \leq \sup_U |P| + \frac{1}{2\pi} \lambda(\mathbb{T} \setminus f^{-1}(U)) \sup_B |P| \to \sup_K |P|$$

as $U \to K$ and $\varepsilon \to 0$. $\square$

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