AN EXTENSION THEOREM FOR HOLOMORPHIC FUNCTIONS OF SLOW GROWTH ON COVERING SPACES OF PROJECTIVE MANIFOLDS

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Abstract. Let X be a projective manifold of dimension $n \geq 2$ and $Y \to X$ be an infinite covering space. Embed X into projective space by sections of a sufficiently ample line bundle. We prove that any holomorphic function of sufficiently slow growth on the preimage of a transverse intersection of X by a linear subspace of codimension $< n$ extends to Y. The proof uses a Hausdorff duality theorem for L_2 cohomology. We also show that every projective manifold has a finite branched covering whose universal covering space is Stein.

1. Introduction.

Infinite covering spaces of projective algebraic manifolds form an interesting and natural class of non-compact complex manifolds, whose function theory is still not well understood. The central problem in this area is the conjecture of Shafarevich that the universal covering space of any projective manifold is holomorphically convex. There are no known counterexamples to this conjecture, and it has been verified only in a number of fairly special cases. For a survey of results up to 1985, see [Gur]. For more recent results, see [ABR], [Nap1], [Nap2] and [Ram].

It is well known that any holomorphic function on a subvariety of a Stein space extends to the whole space. In this paper, we prove such an extension theorem

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for covering spaces of projective manifolds, of necessity restricted to certain well-behaved functions and subvarieties.

Let X be a projective manifold of dimension $n \geq 2$, and $\pi : Y \to X$ be an infinite covering space. Suppose X is embedded in some projective space by sections of a very ample line bundle L. The generic linear subspace of codimension $k < n$ intersects X transversely in a submanifold C of codimension k, whose preimage D in Y is connected. The main result of the paper states that if L is sufficiently positive, then a holomorphic function f on D extends to all of Y if it does not grow too fast. More precisely, we must have $|f| \leq ce^{\epsilon r}$ for $\epsilon > 0$ small enough, where r is the distance from a fixed point in D in a metric pulled back from C. In particular, if f is bounded or grows polynomially with respect to r, then f extends to Y.

For the proof we need a vanishing theorem in L_2 cohomology for negative vector bundles. We deduce it from the L_2 Kodaira-Nakano vanishing theorem for positive vector bundles by means of a Hausdorff duality theorem, which seems not to have been previously recorded in the literature.

The extension theorem reduces many questions about covering spaces of projective manifolds to questions about holomorphic functions of slow growth on covering spaces of compact Riemann surfaces. For example, it is not known if Y can contain a non-compact hypersurface Z, whose irreducible components Z_i are all compact. The non-existence of such a hypersurface is an obvious necessary condition for holomorphic convexity. A curve C as above intersects each $\pi(Z_i)$, so $D = \pi^{-1}(C)$ intersects Z in an infinite discrete subset E. If there are two points in E that can be separated by a holomorphic function of slow growth on D, then we have a contradiction.

To take another example, suppose D is convex with respect to functions of slow growth for every curve C as above. If K is a compact subset of Y with holomorphically convex hull $\hat{K}$, then $\hat{K} \cap D$ is compact. In particular, $\hat{K} \cap \pi^{-1}(x)$ is finite for every $x \in X$. It is not clear how close this is to implying that Y is holomorphically convex.

At present, very little is known about the existence of bounded or slowly growing functions on covering spaces of compact Riemann surfaces. Applications of the extension theorem must await further study of this interesting subject.

At the end of the paper, we point out that projective manifolds whose universal covering space is not only holomorphically convex, but actually Stein, are plentiful in the sense that every projective manifold has such a manifold as a finite branched covering space.

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2. Duality for L_2 cohomology.

We start by reviewing the definition and some basic properties of L_2 cohomology. For more details, see [Che] and [SaZu].

Let X be a complex manifold of dimension n with a hermitian metric and E be a holomorphic vector bundle over X with a hermitian metric. Let $L_2^{p,q}(X,E)$ be the space of L_2 E-valued (p,q)-forms on X with the L_2 norm, and $W_2^{p,q}(X,E)$ be the subspace of forms η such that $\bar\partial\eta$ is L_2. The forms η may be taken to be either smooth or, as we shall do, just measurable, in which case $\bar\partial\eta$ is understood in the distributional sense. The cohomology of the resulting L_2 Dolbeault complex $(W_2^{\cdot,\cdot},\bar\partial)$ is the L_2 cohomology

$$H_{(2)}^{p,q}(X,E) = Z_2^{p,q}(X,E)/B_2^{p,q}(X,E),$$

where $Z_2^{p,q}(X,E)$ and $B_2^{p,q}(X,E)$ are the spaces of $\bar\partial$-closed and $\bar\partial$-exaxt forms in $L_2^{p,q}(X,E)$ respectively. The space $Z_2^{p,q}(X,E)$ is closed in $L_2^{p,q}(X,E)$.

The L_2 cohomology group $H_{(2)}^{p,q}(X,E)$ is Hausdorff in the quotient topology if and only if $B_2^{p,q}(X,E)$ is closed in $Z_2^{p,q}(X,E)$. The separation space of $H_{(2)}^{p,q}(X,E)$ is the reduced L_2 cohomology group

$$\bar H_{(2)}^{p,q}(X,E) = Z_2^{p,q}(X,E)/\overline{B_2^{p,q}(X,E)}.$$

Assume now that the hermitian metric on X is complete. Let $\mathcal{H}_{(2)}^{p,q}(X,E) \subset L_2^{p,q}(X,E)$ be the subspace of harmonic forms, say in the distributional sense. The general Hodge theorem states that there is an isomorphism

$$\mathcal{H}_{(2)}^{p,q}(X,E) \cong \bar H_{(2)}^{p,q}(X,E),$$

so the star operator on harmonic forms induces a conjugate-linear isomorphism

$$\bar H_{(2)}^{p,q}(X,E) \longrightarrow \bar H_{(2)}^{n-p,n-q}(X,E^\vee).$$

Here, $E^\vee$ denotes the dual bundle of E with the dual metric.

To obtain duality for the L_2 cohomology itself, this must be complemented by a Hausdorff result such as theorem 2.3 below. Our proof of this theorem stems from Henkin and Leiterer's exposition in [HeLe] of the duality between ordinary cohomology and cohomology with compact supports. Serre's classical paper [Ser] on that topic reduces the problem to a general lemma on Fréchet complexes. In the L_2 case, the approach taken here seems easier.

2.1. Lemma. *Let E be a real vector bundle of rank m on a smooth manifold X. Then E has measurable sections $s_1, \dots, s_m$ that generate E almost everywhere, i.e., outside a nullset.*

Proof. There is a family $(U_i)_{i \in I}$ of mutually disjoint open subsets of X such that $\bigcup U_i$ is conull and E has a frame $s_1^i, \dots, s_m^i$ on each U_i. These sets may e.g. be taken to be the top-dimensional open simplices of a sufficiently fine triangulation of X. The sections s_ν^i, $i \in I$, piece together to give a measurable section s_ν of E, defined as zero, say, outside $\bigcup U_i$, and $s_1, \dots, s_m$ generate E on $\bigcup U_i$. $\square$

2.2. Lemma. *Let E be a hermitian vector bundle on a hermitian manifold X of dimension n. Then the dual space of $L_2^{p,q}(X, E)$ is $L_2^{n-p,n-q}(X, E^\vee)$. More precisely, if λ is a continuous linear functional on $L_2^{p,q}(X, E)$, then $\lambda = \int \cdot \wedge \eta$ for a unique form η in $L_2^{n-p,n-q}(X, E^\vee)$.*

Proof. By lemma 2.1, the bundle E has measurable sections $s_1, \dots, s_m$, $m = \operatorname{rank} E$, that generate it a.e. By Gram-Schmidt, we may assume that they form an orthonormal frame a.e. The dual sections $s_\nu^\vee$ of $E^\vee$, defined as zero where $s_\nu = 0$, are measurable and form an orthonormal frame for $E^\vee$ a.e. Also, the holomorphic cotangent bundle $T^\vee X$ of X has measurable sections $\phi_1, \dots, \phi_n$ that form an orthonormal frame a.e., so $\phi_1 \wedge \overline{\phi}_1 \wedge \cdots \wedge \phi_n \wedge \overline{\phi}_n$ is, up to a constant, the volume form Ω of X a.e.

The frame $s_\nu \phi_{i_1} \wedge \cdots \wedge \phi_{i_p} \wedge \overline{\phi}_{j_1} \wedge \cdots \wedge \overline{\phi}_{j_q}$ for $E \otimes \Lambda^{p,q} T^\vee X$ induces an isomorphism

$$\Phi : L_2(X)^N \to L_2^{p,q}(X, E),$$

preserving the inner products up to a constant. Similarly, we get an isomorphism

$$\Psi : L_2(X)^N \to L_2^{n-p,n-q}(X, E^\vee),$$

such that for any $f, g \in L_2(X)^N$,

$$f \cdot g \, \Omega = c \, \Phi(f) \wedge \Psi(g),$$

where c is a suitable constant and $f \cdot g = f_1 g_1 + \cdots + f_N g_N$.

Now if λ is a continuous linear functional on $L_2^{p,q}(X, E)$, then $\lambda \circ \Phi$ is a continuous linear functional on $L_2(X)^N$, so there is $g \in L_2(X)^N$ such that

$$(\lambda \circ \Phi)(f) = \int f \cdot g \, \Omega$$

for all $f \in L_2(X)^N$. If $\alpha \in L_2^{p,q}(X, E)$, then

$$\lambda(\alpha) = (\lambda \circ \Phi)(\Phi^{-1}(\alpha)) = \int \Phi^{-1}(\alpha) \cdot g \, \Omega = c \int \alpha \wedge \Psi(g),$$

so $\lambda = \int \cdot \wedge \eta$ with $\eta = \Psi(cg)$. Clearly, η is unique. $\square$

2.3. Theorem. *Let E be a hermitian vector bundle on a complete hermitian manifold X of dimension n. If $H_{(2)}^{n-p,n-q+1}(X,E)$ is Hausdorff for some p,q, then $H_{(2)}^{p,q}(X,E^\vee)$ is also Hausdorff.*

Proof. Let $\eta \in \overline{B_2^{p,q}(E^\vee)}$; say η is the L_2 limit of a sequence $\bar\partial\theta_\nu$ with $\theta_\nu \in L_2^{p,q-1}(E^\vee)$. We need to show that $\eta = \bar\partial\xi$ for some $\xi \in L_2^{p,q-1}(E^\vee)$, so $\eta \in B_2^{p,q}(E^\vee)$. If $\alpha \in Z_2^{n-p,n-q}(E)$, then $\int \alpha\wedge\eta$ is the limit as $\nu\to\infty$ of

$$\int \alpha\wedge\bar\partial\theta_\nu = \int \bar\partial(\alpha\wedge\theta_\nu) = \int d(\alpha\wedge\theta_\nu) = 0,$$

by Stokes' theorem for L_1 forms on a complete manifold [Gaf], so $\int\alpha\wedge\eta = 0$.

Hence we can define a linear functional λ on $B_2^{n-p,n-q+1}(E)$ by the formula

$$\lambda(\beta) = \int \alpha\wedge\eta \qquad \text{if } \beta = \bar\partial\alpha, \quad \alpha\in L_2^{n-p,n-q}(E).$$

We claim that λ is continuous. Let W' be $W_2^{n-p,n-q}(E)$ with the complete Sobolev norm $\|\cdot\|_{L_2} + \|\bar\partial\cdot\|_{L_2}$. Then $\bar\partial : W' \to B_2^{n-p,n-q+1}(E)$ is a continuous epimorphism. By assumption, $B_2^{n-p,n-q+1}(E)$ is closed in $Z_2^{n-p,n-q+1}(E)$, which is complete, so $B_2^{n-p,n-q+1}(E)$ is complete. Therefore $\bar\partial$ is open by the open mapping theorem. Now $\lambda\circ\bar\partial$ is the functional $\int\cdot\wedge\eta$ on W', which is clearly continuous. Hence, λ is continuous.

By the Hahn-Banach theorem, λ extends to a continuous linear functional on $L_2^{n-p,n-q+1}(E)$. By lemma 2.2 there is $\xi\in L_2^{p,q-1}(E^\vee)$ such that

$$\lambda = (-1)^{p+q+1}\int\cdot\wedge\xi.$$

Then $\bar\partial\xi = \eta$, because

$$(-1)^{p+q+1}\int\bar\partial\alpha\wedge\xi = \lambda(\bar\partial\alpha) = \int\alpha\wedge\eta$$

for all smooth E-valued $(n-p,n-q)$-forms α with compact support. $\square$

2.4. Corollary. *Let E be a hermitian vector bundle with curvature Θ on a complex manifold X of dimension n with a complete Kähler form ω. If $\Theta \geq \epsilon\omega$ for some $\epsilon > 0$ in the sense of Nakano, then*

$$H_{(2)}^{q}(X,E^\vee) = 0 \qquad \textit{for } q < n.$$

Proof. By the L_2 Kodaira-Nakano vanishing theorem [Dem], [Ohs], $H^{n,r}_{(2)}(X, E) = 0$ for $r > 0$. Hence, $\bar{H}^q_{(2)}(X, E^\vee) = 0$ for $q < n$ by duality. Also, $H^q_{(2)}(X, E^\vee)$ is Hausdorff for $1 \leq q \leq n$ by theorem 2.3, and obviously for $q = 0$. □

Remark added in July 1994. Using the Bochner-Kodaira-Nakano identity along the lines of the usual proof of the Kodaira-Nakano vanishing theorem, one can prove that $H^q_{(2)}(X, E^\vee) = 0$ for $q < n$ if the curvature of $E^\vee$ is at most $-\epsilon\,\omega$ for some $\epsilon > 0$ in the sense of Nakano. This was pointed out to me by Professor M. Ramachandran. This is sufficient to prove theorem 3.1 and does not require theorem 2.3.

3. Extension theorems.

Now let $\pi : Y \to X$ be a covering space of a compact n-dimensional Kähler manifold X with a Kähler form ω. Let Y have the pullback metric. We will always assume that $n \geq 2$.

3.1. Theorem. *Let*

(1) *ϕ be a smooth function on Y such that $d\phi$ is bounded,*
(2) *L be a line bundle on X with canonical connection ∇ and curvature Θ in a hermitian metric h, and*
(3) *C be the zero locus of a section s of L over X with $\nabla s \neq 0$ at each point of C.*

If

$$\Theta \geq \mathcal{L}(\phi) + \epsilon\,\omega$$

for some $\epsilon > 0$, then every holomorphic function f on $D = \pi^{-1}(C)$ such that $f^2 e^{-\phi}$ is integrable on D can be extended to a holomorphic function F on Y such that $F^2 e^{-\phi}$ is integrable on Y.

The condition in (3) means that for every $p \in C$ there is a coordinate neighbourhood (U, z) centred at p and a holomorphic frame e for L on U such that $s = z_1 e$ on U. Hence C is a smooth (possibly disconnected) hypersurface in X, unless C is empty, in which case the theorem is trivial.

By induction, the theorem generalizes to the case where C is the common zero locus of sections $s_1, \ldots, s_k$, $k < n$, of L over X, which, in a trivialization, can be completed to a set of local coordinates at each point of C.

If L is very ample, and therefore the pullback of the hyperplane bundle by an embedding of X in some projective space, then this condition simply means that the linear space $\{s_1, \ldots, s_k = 0\}$ intersects X transversely in a smooth subvariety of codimension k.

Proof. Assume $C \neq \varnothing$. Let U_0 be the pullback of the complement of a closed neighbourhood of C, and $U_1, \ldots, U_N$ be the pullbacks of shrunk coordinate polydiscs covering a larger neighbourhood of C, in which $C = \{z_1 = 0\}$. Also pull back a smooth partition of unity (χ_i) subordinate to (U_i).

Let f be a holomorphic function on D such that $f^2 e^{-\phi}$ is integrable on D. For $i \geq 1$, extend f to a holomorphic function f_i on U_i which is constant on each line $\{z_2, \ldots, z_n \text{ constant}\}$. Let $f_0 = 0$ on U_0. By (3) and since $f_i = f = f_j$ on D, we can define a holomorphic section of the dual bundle $L^\vee$ on $U_{ij} = U_i \cap U_j$ by the formula

$$u_{ij} = (f_i - f_j)s^\vee.$$

Then

$$v_i = \sum_j u_{ij}\chi_j$$

is a smooth section of $L^\vee$ on U_i and $v_i - v_j = u_{ij}$. Hence, $\bar\partial v_i = \bar\partial v_j$ on U_{ij}, so we get a $\bar\partial$-closed $L^\vee$-valued $(0,1)$-form η on Y defined as $\bar\partial v_i$ on U_i.

Let us show that $|\eta|^2 e^{-\phi}$ is integrable on Y. On U_0, s is bounded away from 0 and

$$\eta = \bar\partial v_0 = -\sum_j f_j s^\vee \bar\partial \chi_j,$$

so

$$|\eta|^2 \leq c \sum_j |f_j|^2.$$

Here and in the following, we denote by the same letter c any constant not depending on the particular function f. Also,

$$\int_{U_j} |f_j|^2 e^{-\phi}\omega^n \leq c \int_{D\cap U_j} |f|^2 e^{-\phi}\omega^{n-1}$$

because $d\phi$ is bounded. Since $f^2 e^{-\phi}$ is integrable on D, so is $|\eta|^2 e^{-\phi}$ on U_0. For $i \geq 1$,

$$\eta = \bar\partial v_i = \sum_j (f_i - f_j)s^\vee \bar\partial \chi_j$$

on U_i, so it remains to show that

$$(*) \qquad \sum_{i,j\geq 1} \int_{U_{ij}} |f_i - f_j|^2 |s|^{-2} e^{-\phi}\omega^n < \infty.$$

For $x \in U_{ij}$, $i, j \geq 1$, there are $x_i \in D \cap U_i$ and $x_j \in D \cap U_j$ such that $f_i(x) = f(x_i)$ and $f_j(x) = f(x_j)$ and $\mathrm{dist}(x_i, x_j) \leq c|s(x)|$, so

$$|f_i(x) - f_j(x)||s(x)|^{-1} \leq c \sup |df|,$$

where the supremum is taken over $D \cap (U_i \cup U_j)$. By the Cauchy inequalities and since $d\phi$ is bounded,

$$\int_{U_{ij}} |f_i - f_j|^2 |s|^{-2} e^{-\phi} \omega^n \leq c \int_{U_{ij}} \sup |df|^2 e^{-\phi} \omega^n \leq c \int_{D \cap (V_i \cup V_j)} |f|^2 e^{-\phi} \omega^{n-1},$$

where $V_i \supset U_i$, $V_j \supset U_j$ are pullbacks of larger polydiscs. Since $f^2 e^{-\phi}$ is integrable on D, (*) follows.

The weighted metric $e^{\phi} h$ in L has curvature $-\mathcal{L}(\phi) + \Theta \geq \epsilon\, \omega$, so corollary 2.4 implies that $H^1_{(2)}(Y, L^\vee) = 0$ with respect to the weighted dual metric $e^{-\phi} h^\vee$. Since η is L_2 in this metric, there is a smooth section w of $L^\vee$ such that $|w|^2 e^{-\phi}$ is integrable and $\bar{\partial} w = \eta$. Let $u_i = v_i - w$. Then u_i is a holomorphic section of $L^\vee$ on U_i and $u_i - u_j = u_{ij}$, so

$$f_i - u_i \otimes s = f_j - u_j \otimes s \qquad \text{on } U_{ij}.$$

Hence we obtain a holomorphic extension F of f to Y by setting

$$F = f_i - u_i \otimes s = f_i + w \otimes s - \sum_j (f_i - f_j) \chi_j$$

on U_i. The term $w \otimes s$ is L_2 with respect to $e^{-\phi}$ by construction of w and since s is bounded. The other two terms on the right hand side can be shown to be L_2 with respect to $e^{-\phi}$ by arguments similar to those used for η above. Hence, $F^2 e^{-\phi}$ is integrable. $\square$

Let $\delta(x)$ be the distance from a fixed point in Y to $x \in Y$ in the pullback metric. By a result of Napier [Nap1], there is a smooth function τ on Y, obtained by smoothing δ, such that

(1) $c_1 \delta \leq \tau \leq c_2 \delta + c_3$ for some constants $c_1, c_2, c_3 > 0$,
(2) $d\tau$ is bounded, and
(3) $\mathcal{L}(\tau)$ is bounded.

By (1) and since the curvature of Y is bounded below, there is $c > 0$ such that $e^{-c\tau}$ is integrable on Y. Let $a \geq 0$ be the infimum of such numbers c.

Assume now that X is projective algebraic with a very ample line bundle L. We may think of X as embedded in some projective space and of L as the restriction to X of the hyperplane bundle with the standard positively curved metric. Then zero loci of sections of L are hyperplane sections of X. By Bertini's theorem, the generic linear subspace of codimension $k < n$ intersects X transversely in a smooth subvariety C of codimension k. By the Lefschetz hyperplane theorem, C is connected and the map $\pi_1(C) \to \pi_1(X)$ is surjective, which implies that the preimage $D = \pi^{-1}(C)$ in Y is also connected.

Suppose

$$\Theta \geq a\mathcal{L}(\tau) + \epsilon\,\omega$$

for some $\epsilon > 0$. Since $\mathcal{L}(\tau)$ is bounded, this can be achieved by replacing L by a sufficiently high tensor power of itself. Then we also have

$$\Theta \geq (a + b + \epsilon_1)\mathcal{L}(\tau) + \epsilon\,\omega$$

for some $b, \epsilon, \epsilon_1 > 0$.

Let f be a holomorphic function on D such that $f^2 e^{-b\tau}$ is bounded. Since $e^{-(a+\epsilon_1)\tau}$ is integrable on Y and $d\tau$ is bounded, $e^{-(a+\epsilon_1)\tau}$ is integrable on D. Hence, $f^2 e^{-(a+b+\epsilon_1)\tau}$ is integrable on D. By theorem 3.1 applied with $\phi = (a+b+\epsilon_1)\tau$, f extends to a holomorphic function on Y.

Let $r(x)$ be the distance from a fixed point in D to $x \in D$ in some metric pulled back from C. Then $\epsilon_0 r \leq \tau + c$ for some $\epsilon_0, c > 0$. By the above, if f is a holomorphic function on D such that $f^2 e^{-\epsilon_0 b r}$ is bounded, then f extends to Y. In particular, if f is bounded or grows polynomially with respect to r, then f extends from D to Y.

The above discussion may be summarized as follows.

3.2. Theorem. *Let $Y \to X$ be a covering space of a projective manifold X of dimension $n \geq 2$. Embed X into projective space by sections of a sufficiently ample line bundle. Then any holomorphic function of sufficiently slow growth on the preimage of a transverse intersection of X by a linear subspace of codimension $< n$ extends to all of Y.*

The distance r seems a more natural measure of growth than τ for functions on D. When C is a curve, we have in mind the distance in the standard metric of constant curvature. It would be interesting to have some estimates on the size of $\epsilon_0 b$, and especially to know how this number changes when L is replaced by tensor powers of itself.

4. Projective manifolds with Stein universal coverings.

In [Gro], Gromov points out that every projective manifold can be dominated by a Kähler hyperbolic manifold. This is also true for manifolds whose universal covering space is Stein.

4.1. Proposition. *Every projective algebraic manifold X has a finite branched covering $Y \to X$, where Y is a projective manifold whose universal covering space is Stein.*

In general, suppose P is a property of projective manifolds such that

(1) if $Y \to X$ is a finite branched covering and X is P, then Y is P, and
(2) manifolds Z_n which are P exist in every dimension n.

Let X be an arbitrary n-dimensional projective manifold. By intersecting n sufficiently ample smooth hypersurfaces in $X \times Z_n$, we obtain an n-dimensional submanifold Y in $X \times Z_n$ such that the projections $Y \to X$ and $Y \to Z_n$ are finite. Since Z_n is P, so is Y by (1). Therefore, X is dominated by a projective manifold which is P.

4.2. Lemma. *Let X be a projective manifold with a positive line bundle L and $\pi : Y \to X$ be a covering space. If π^*L is holomorphically trivial and has a bounded trivialization, then Y is Stein.*

Proof. Let s be a bounded trivialization of π^*L. Then $u = -\log|s|^2$ is a strictly plurisubharmonic function which is bounded below and its Levi form is bounded away from zero. Let τ be the smoothed distance from a point in Y as described in section 3. It is an exhaustion of Y with bounded Levi form. Then $cu + \tau$ is a strictly plurisubharmonic exhaustion of Y for $c > 0$ large enough, so Y is Stein. □

By the lemma, to prove the proposition, it suffices to show that conditions (1) and (2) above are satisfied by the property of having a positive line bundle which has a bounded trivialization on the universal covering.

So suppose X is a projective manifold with a positive line bundle L which has a bounded trivialization s on the universal covering $\tilde{X}$. Let $f : Y \to X$ be a finite branched covering and $p : \tilde{Y} \to Y$ be the universal covering of Y. Then f lifts to a possibly infinite branched covering $\tilde{f} : \tilde{Y} \to \tilde{X}$. By Grauert's ampleness criterion [Gra], f^*L is an ample line bundle on Y, so f^*L has a positively curved metric. The trivialization $\tilde{f}^*s$ of p^*f^*L is bounded with respect to the pullback of this metric.

As for (2), let C be a compact Riemann surface of genus at least 2. The canonical bundle of C is ample, and its pullback to the universal covering of C, which is the unit disc Δ, is trivial. In the Poincaré metric

$$\frac{idz \wedge d\bar{z}}{(1-|z|^2)^2}$$

the trivialization dz has length $1-|z|^2$, which is bounded. In dimension $n > 1$, the product C^n also has an ample canonical bundle, whose pullback to the universal covering Δ^n is trivialized by the form $dz_1 \wedge \cdots \wedge dz_n$, which has length $(1-|z_1|^2)\dots(1-|z_n|^2)$ in the product Poincaré metric. Therefore, we can take $Z_n = C^n$.

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