

Regular immersions directed by algebraically elliptic cones

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Abstract Let M be an open Riemann surface and A be the punctured cone in $\mathbb{C}^n \setminus \{0\}$ on a smooth projective variety Y in \mathbb{P}^{n-1} . Recently, Runge approximation theorems with interpolation for holomorphic immersions $M \rightarrow \mathbb{C}^n$, directed by A , have been proved under the assumption that A is an Oka manifold. We prove analogous results in the algebraic setting, for regular immersions directed by A from a smooth affine curve M into \mathbb{C}^n . The Oka property is naturally replaced by the stronger assumption that A is algebraically elliptic, which it is if Y is uniformly rational. Under this assumption, a homotopy-theoretic necessary and sufficient condition for approximation and interpolation emerges. We show that this condition is satisfied in many cases of interest.

Keywords Riemann surface, holomorphic immersion, directed immersion, regular immersion, algebraically elliptic manifold, Oka manifold, Runge approximation, minimal surface

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1. Introduction and main results

Let A_0 be a conical closed complex subvariety of \mathbb{C}^n , $n \geq 2$. By Chow's theorem, A_0 is algebraic and is the common zero set of finitely many homogeneous holomorphic polynomials. Assume that $A = A_0 \setminus \{0\} \subset \mathbb{C}_*^n = \mathbb{C}^n \setminus \{0\}$ is smooth and connected, so A is the punctured cone on a connected submanifold Y of \mathbb{P}^{n-1} . If M is an open Riemann surface, a holomorphic immersion $M \rightarrow \mathbb{C}^n$ is said to be *directed by A* , or to be an *A -immersion*, if its complex derivative with respect to any local holomorphic coordinate on M takes its values in A . A systematic investigation of A -immersions was made by Alarcón and Forstnerič in [4]. In particular, under the assumption that A is an Oka manifold not contained in any hyperplane in \mathbb{C}^n , they proved an Oka principle for A -immersions, including the Runge and Mergelyan approximation properties. (For the theory of Oka manifolds, see the monograph [23]. We note that A is Oka if and only if Y is.) In the subsequent paper [1] by Alarcón and Castro-Infantes, interpolation was added to the picture. A parametric Oka principle for directed immersions was proved by Forstnerič and Lárusson in [24].

In this paper, we pursue the algebraic analogues of the above results for directed immersions. We prove a general uniform approximation theorem with interpolation for A -immersions of open Riemann surfaces in affine spaces in the algebraic category. To this end, throughout the paper, it is natural to assume that the punctured cone A is not only Oka but algebraically elliptic in the sense of Gromov (see [25] or [23,

Definition 5.6.13]). Likewise, in the algebraic setting, the open Riemann surface M will be a smooth affine curve, so M is the complement in a compact Riemann surface \overline{M} of finitely many points p_1, \dots, p_m , which we call the *ends* of M . We assume, without loss of generality, that M and \overline{M} are connected. A complex-valued function or a complex form on M is regular if it is holomorphic and extends meromorphically to \overline{M} . Algebraic Oka theory is more rigid than standard Oka theory. Obstructions arise that are not present in the holomorphic theory (see [29]), so it is no surprise that approximation by regular A -immersions introduces new difficulties. For instance, it is fundamental in the proofs in [4, 1, 24] that every open Riemann surface admits a holomorphic 1-form without zeros, while in general a smooth affine curve does not carry any nowhere-vanishing regular 1-forms.

The following is our first main result. Recall that a regular map $M \rightarrow \mathbb{C}^n$ is proper if and only if its meromorphic extension to \overline{M} has an effective pole (an actual pole of positive order) at each end of M . Also note that A defines a subbundle \mathcal{A} of $(T^*M)^{\oplus n}$ with fibre isomorphic to A , whose sections are n -tuples $(\alpha_1, \dots, \alpha_n)$ of $(1, 0)$ -forms with no common zeros, such that the ratio $(\alpha_1 : \dots : \alpha_n)$ takes values in Y .

Theorem 1.1. *Let $A \subset \mathbb{C}_*^n$ be the punctured cone on a connected submanifold Y of \mathbb{P}^{n-1} , $n \geq 2$, and assume that A is algebraically elliptic and not contained in a hyperplane in \mathbb{C}^n . Let M be a smooth affine curve and \mathcal{A} be the subbundle of $(T^*M)^{\oplus n}$ defined by A . Finally, let K be a holomorphically convex compact subset of M and $u : K \rightarrow \mathbb{C}^n$ be a holomorphic A -immersion. Then the following are equivalent.*

- (i) *u can be uniformly approximated on a neighbourhood of K by regular A -immersions $M \rightarrow \mathbb{C}^n$.*
- (ii) *u can be uniformly approximated on a neighbourhood of K by proper regular A -immersions $M \rightarrow \mathbb{C}^n$ agreeing with u on any given finite set in K .*
- (iii) *There is a neighbourhood U of K such that the homotopy class of continuous sections of $\mathcal{A}|_U$ that contains du also contains the restriction of a regular section of \mathcal{A} on M .*
- (iv) *There is a neighbourhood U of K such that the homotopy class of continuous sections of $\mathcal{A}|_U$ that contains du also contains the restriction of an exact regular section of \mathcal{A} on M with an effective pole at each end of M .*

Remark 1.2. (a) The only punctured cone in \mathbb{C}^* is \mathbb{C}^* itself, which is not algebraically elliptic, so Theorem 1.1 is vacuous for $n = 1$. For $n = 2$, the only punctured cone in \mathbb{C}_*^2 satisfying the assumptions of Theorem 1.1 is \mathbb{C}_*^2 itself. Our proof of Theorem 1.1 uses several results from [4, 1], which are stated for holomorphic immersions directed by punctured cones $A \subset \mathbb{C}_*^n$ with $n \geq 3$ and $A \neq \mathbb{C}_*^n$. Although an inspection of the proofs in [4, 1] shows that the results that we apply are valid for $n \geq 2$ and $A = \mathbb{C}_*^n$ (in fact, extensions of some of these results are proved for arbitrary $n \geq 1$ in [8] and [24, Section 5]), we shall give a separate proof, much simpler, of this special case of Theorem 1.1.

(b) Algebraic ellipticity may be formulated in several nontrivially equivalent ways. It was observed in [29] that the following properties of a smooth algebraic variety are equivalent: algebraic subellipticity, the algebraic homotopy Runge property, and the algebraic version of Gromov’s Ell_1 property (for the definitions of these properties, see [29]). And recently, Kaliman and Zaidenberg showed that algebraic ellipticity and algebraic subellipticity are equivalent [27].

The optimal known sufficient condition for A to be algebraically elliptic is given by a very recent theorem of Arzhantsev, Kaliman, and Zaidenberg [12, Theorem 1.3]. They showed that A is algebraically elliptic if Y is uniformly rational, meaning that Y is covered by Zariski-open sets, each isomorphic to a Zariski-open subset of affine space (see also [16], where rationally connected manifolds are called “plain”, and [17]). As a regular retract of A , Y is then algebraically elliptic as well, but we do not know whether algebraic ellipticity of Y implies that of A . The class of uniformly rational manifolds is closed under products and under blow-ups with smooth centres. Also, the total space of an algebraic fibre bundle over a uniformly rational manifold with a uniformly rational fibre is itself uniformly rational.

Most (but not all) known examples of rationally connected manifolds are in fact covered by Zariski-open sets isomorphic to affine space itself. Such manifolds are said to be A -covered or of class \mathcal{A}_0 (see [13, Section 4]). The following are examples of manifolds of class \mathcal{A}_0 : projective spaces, Grassmannians, flag manifolds, smooth projective rational surfaces, and smooth projective toric varieties.

(c) Approximating holomorphic maps by regular maps is an important theme in algebraic geometry. Such approximation is impossible in general, even for maps between very simple affine algebraic manifolds. For example, let Σ^n , $n \geq 1$, be the complex n -sphere $\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : z_0^2 + \dots + z_n^2 = 1\}$. It is a homogeneous space of the connected linear algebraic group $\text{SO}(n+1, \mathbb{C})$, which has no nontrivial characters, so Σ^n is algebraically flexible [11, Proposition 5.4] and hence algebraically elliptic. Loday showed that every regular map $\Sigma^p \times \Sigma^q \rightarrow \Sigma^{p+q}$ is nullhomotopic when p and q are odd, but there is a continuous map, and hence a holomorphic map, $\Sigma^p \times \Sigma^q \rightarrow \Sigma^{p+q}$ that is not nullhomotopic ([30]; see also [23, Example 6.15.7]) and thus not approximable by regular maps.

(d) The main task in the proof of Theorem 1.1 is to show that (iii) implies (ii). A key ingredient in our proof is Forstnerič’s approximation theorem for sections of an algebraic fibre bundle with algebraically elliptic fibre [22, Theorem 3.1] (see also [23, Theorem 6.15.3]). Applying this theorem requires a homotopy-theoretic necessary and sufficient condition, which in our setting corresponds to (iii).

The following definitions are convenient.

Definition 1.3. With notation as above, we say that a pair (A, M) is *good* if A has a regular section on M , and we say that (A, M) is *very good* if every homotopy class of continuous (or, equivalently, holomorphic) sections of \mathcal{A} on M contains a regular section. Furthermore, we say that A is *good* if (A, M) is good for every

smooth affine curve M , and we say that A is *very good* if (A, M) is very good for every smooth affine curve M .

Remark 1.4. An algebraically elliptic projective manifold is unirational and hence simply connected by Serre’s [35, Proposition 1]. Also, by the first half of [35, Lemma 2], the fundamental group of an algebraically elliptic manifold is finite. Thus, under the assumptions of Theorem 1.1, $\pi_1(Y) = 0$ and $\pi_1(A)$ is finite. Also, $\pi_1(A)$ is cyclic, being a quotient of the fundamental group of the fibre \mathbb{C}^* of the bundle $A \rightarrow Y$.

The tangent bundle TM is holomorphically trivial, so continuous sections of A may be identified with continuous maps $M \rightarrow A$. Since M has the homotopy type of a finite bouquet of circles and deformation-retracts onto an embedded bouquet, the homotopy classes of continuous maps $M \rightarrow A$ are in bijective correspondence with homomorphisms $\pi_1(M) \rightarrow \pi_1(A)$. Hence there are only finitely many such classes. If A is simply connected, then there is only one class, so (A, M) is good if and only if it is very good. In general, we are not aware of any pair (A, M) as in Theorem 1.1 that is good but not very good.

In Section 2, we establish the following sufficient conditions for a cone to be good and a pair to be good or very good.

Proposition 1.5. *Let Y , A , and M be as in Theorem 1.1.*

- (a) *If Y has a rational curve of degree 1 or 2, then A is good.*
- (b) *If M has genus 0, then (A, M) is very good.*
- (c) *If TM is algebraically trivial, for example if M has genus 1 or is planar, then (A, M) is good.*

Projective varieties of low codimension and defined by equations of low degree are covered by lines. Among the instances of this metatheorem are [28, Lemma V.4.8.1, Exercise V.4.10.5] and [32, Theorem 3.1]. Similar results exist for conics in place of lines (see [26, Proposition 2.4]). It is reasonable to ask whether every punctured cone A as in Theorem 1.1 is good.

Thus Proposition 1.5(a) applies to many cones of interest in the present context, such as those in the following examples.

Example 1.6. (a) A first important example is the “large cone” $A = \mathbb{C}_*^n$, $n \geq 2$; see Remark 1.2(a). Every immersion into \mathbb{C}^n is directed by A . Theorem 1.1 is new even for these simplest of all cones.

(b) A cone of fundamental importance in the theory of minimal surfaces in Euclidean space and satisfying the hypotheses of Theorem 1.1 is the punctured null quadric $A = \{(z_1, \dots, z_n) \in \mathbb{C}_*^n : z_1^2 + \dots + z_n^2 = 0\}$, $n \geq 3$. For this cone, Theorem 1.1 was proved in [10] by a rather involved method that heavily uses the special geometry of the cone. The initial motivation for the present work was to

generalise this result to the largest possible class of cones, while giving an easier, more conceptual proof than the proof in [10].

(c) The cone $A = \{(z_1, \dots, z_4) \in \mathbb{C}_*^4 : z_1 z_4 - z_2 z_3 = 0\}$ satisfies the requirements of Theorem 1.1 and plays a key role in the theory of surfaces of constant mean curvature 1 in hyperbolic space.

In Section 2, we prove that all the cones in Example 1.6 are in fact very good and discuss their interest in the literature. Theorem 1.1 and the following corollaries, concerning good and very good pairs, are proved in Section 4; see Remark 4.1 for a slight extension of these results.

Corollary 1.7. *Under the assumptions of Theorem 1.1, the following are equivalent.*

- (i) *There is a regular A -immersion $M \rightarrow \mathbb{C}^n$.*
- (ii) *For any finite set $\Lambda \subset M$, every map $\Lambda \rightarrow \mathbb{C}^n$ extends to a proper regular A -immersion $M \rightarrow \mathbb{C}^n$.*
- (iii) *(A, M) is good.*
- (iv) *There is an exact regular section of \mathcal{A} on M with an effective pole at each end of M .*

Corollary 1.8. *Under the assumptions of Theorem 1.1, the following are equivalent.*

- (i) *Every holomorphic A -immersion $K \rightarrow \mathbb{C}^n$, where $K \subset M$ is compact and holomorphically convex, can be uniformly approximated on K by regular A -immersions $M \rightarrow \mathbb{C}^n$.*
- (ii) *Every holomorphic A -immersion $K \rightarrow \mathbb{C}^n$, where $K \subset M$ is compact and holomorphically convex, can be uniformly approximated on K by proper regular A -immersions $M \rightarrow \mathbb{C}^n$ agreeing with the given holomorphic A -immersion $K \rightarrow \mathbb{C}^n$ on any given finite set in K .*
- (iii) *(A, M) is very good.*
- (iv) *Every homotopy class of continuous sections of \mathcal{A} on M contains an exact regular section with an effective pole at each end of M .*

For any open Riemann surface M and any punctured cone $A \subset \mathbb{C}^n$, $n \geq 3$, that is Oka and not contained in a hyperplane, Alarcón and Forstnerič proved a general existence result for proper holomorphic A -embeddings $M \rightarrow \mathbb{C}^n$ in [4]. For $n = 2$, it is an open question whether every open Riemann surface admits a proper holomorphic embedding into \mathbb{C}^2 [21, 15]. The problem seems very difficult even for smooth affine curves; see [23, Sections 9.10–9.11]. In the algebraic setting, it is well known that every smooth affine curve regularly embeds in \mathbb{C}^3 , while most of them do not regularly embed in \mathbb{C}^2 . Except for the large cones, even for the punctured null quadrics in Example 1.6(b), the following question remains open.

Question 1.9. Let A and M be as in Theorem 1.1 with $A \neq \mathbb{C}_*^n$ and assume that the pair (A, M) or the cone A is good or even very good. Is there a proper regular A -embedding $M \rightarrow \mathbb{C}^n$?

Our method of proof also gives results analogous to Theorem 1.1 for directed harmonic maps into \mathbb{R}^n , $n \geq 3$, including conformal minimal immersions. We explain this in Section 5. In fact these results follow from a more general approximation theorem for regular sections of \mathcal{A} with control on their periods, which we state and prove in Section 3 as Theorem 3.1. The following result, saying in particular that the periods of a regular section of \mathcal{A} on M can be prescribed arbitrarily whenever the pair (A, M) is good, is an example of the consequences of Theorem 3.1. As far as we know, the result is new even for the “large cones” in Example 1.6(a).

Theorem 1.10. *Let A , M , and \mathcal{A} be as in Theorem 1.1 and assume that t_0 is a regular section of \mathcal{A} on M . Then for any group homomorphism $\mathcal{F} : H_1(M, \mathbb{Z}) \rightarrow \mathbb{C}^n$ there is a regular section t of \mathcal{A} on M homotopic to t_0 such that t has an effective pole at each end of M and $\int_C t = \mathcal{F}(C)$ for every loop C in M .*

In particular, the following hold for any pair (A, M) as in Theorem 1.1.

- *(A, M) is good if and only if every class in the cohomology group $H^1(M, \mathbb{C}^n)$ contains a regular section of \mathcal{A} on M with an effective pole at each end of M .*
- *(A, M) is very good if and only if every class in the cohomology group $H^1(M, \mathbb{C}^n)$ contains a regular section of \mathcal{A} on M with an effective pole at each end of M in each homotopy class of continuous sections of \mathcal{A} on M .*

See Theorem 3.8 for a slight extension of this result. Theorem 1.10 strengthens the implications (iii) \Rightarrow (iv) in Theorem 1.1 and Corollaries 1.7 and 1.8. There is a long history of results about cohomology classes containing representatives with special properties. The holomorphic analogue of Theorem 1.10 is [8, Corollary 2(a)]. It does not require the assumption that (A, M) is good or any similar hypothesis.

2. Good and very good pairs and cones

In this section we first prove Proposition 1.5 providing sufficient conditions for a cone to be good or for a pair to be very good in the sense of Definition 1.3. Later we shall show that the cones presented in Example 1.6 are all very good, and briefly discuss their importance in the literature.

Proof of Proposition 1.5. (a) Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^{n-1}$, $[z, w] \mapsto [q_1(z, w), \dots, q_n(z, w)]$, be a rational curve of degree 1 in Y . Here, q_1, \dots, q_n are homogeneous polynomials of degree 1. Let ω and η be regular forms on M with no common zeros (such a pair exists by Riemann-Roch). Then $(q_1(\omega, \eta), \dots, q_n(\omega, \eta))$ is a regular section of \mathcal{A} . If f has degree 2, then we apply the same argument to a pair of regular sections on M with no common zeros of a square root of $T^*\overline{M}$.

(b) Since TM is algebraically trivial, regular sections of \mathcal{A} may be identified with regular maps $M \rightarrow A$. We need to show that every homomorphism $\pi_1(M) \rightarrow \pi_1(A)$ is induced by a regular map (see Remark 1.4). Consider the piece $\pi_1(\mathbb{C}^*) \rightarrow \pi_1(A) \rightarrow \pi_1(Y) = 0$ of the long exact sequence of homotopy groups

associated to the projection $A \rightarrow Y$. By the universal property of the free group $\pi_1(M)$, every homomorphism $\pi_1(M) \rightarrow \pi_1(A)$ factors through the homomorphism $\pi_1(\mathbb{C}^*) \rightarrow \pi_1(A)$. Since M has genus 0, every homomorphism $\pi_1(M) \rightarrow \pi_1(\mathbb{C}^*)$ is induced by a regular map $M \rightarrow \mathbb{C}^*$ and the proof is complete.

(c) \mathcal{A} has a regular section corresponding to a constant map $M \rightarrow A$. \square

2.1. The large cone. We begin by proving that the large cone $A = \mathbb{C}_*^n$, $n \geq 2$, is very good. In this case, an A -immersion is nothing but an immersion. We give a self-contained proof, not relying on Theorem 1.1 (see Remark 1.2(a)). We first show that condition (ii) in Theorem 1.1 always holds for the large cone, even including jet interpolation.

Proposition 2.1. *Let M be a smooth affine curve and K be a holomorphically convex compact subset K of M . Then every holomorphic immersion $K \rightarrow \mathbb{C}^n$, $n \geq 2$, may be approximated uniformly on a neighbourhood of K by proper regular immersions $M \rightarrow \mathbb{C}^n$ agreeing with the given immersion to any given finite order at any given finite subset of K . Furthermore, we can choose the approximating regular immersions such that all their component functions are proper.*

Proof. For simplicity of exposition we shall assume that $n = 2$. The proof for $n \geq 3$ follows the same lines. So assume that $K \subset M = \overline{M} \setminus \{p_1, \dots, p_m\}$ is a holomorphically convex compact subset and $u = (u_1, u_2) : K \rightarrow \mathbb{C}^2$ is a holomorphic immersion. Also, for the jet interpolation condition, let $\Lambda \subset K$ be a finite set and $d \geq 1$ be an integer. Choose a smoothly bounded connected compact domain¹ $L \supset K$ in M such that $\overline{M} \setminus \overset{\circ}{L} = D_1 \cup \dots \cup D_m$, where D_1, \dots, D_m are mutually disjoint smoothly bounded closed discs centred at p_1, \dots, p_m , respectively. By the classical Runge theorem with jet interpolation we may approximate u uniformly on a neighbourhood of K by a holomorphic map $L \rightarrow \mathbb{C}^2$ agreeing with u to order d at all points of Λ . By general position, this map can be chosen to be an immersion on a neighbourhood of L . Thus we may assume without loss of generality that u extends to L as a holomorphic immersion $u = (u_1, u_2) : L \rightarrow \mathbb{C}^2$.

Fix a finite set $F = \{q_1, \dots, q_m\}$ with $q_i \in \overset{\circ}{D}_i \setminus \{p_i\}$ for $i = 1, \dots, m$. By Royden's Runge theorem for regular functions [34, Theorem 10], we may approximate u_1 uniformly on a neighbourhood of L by a regular function $v_1 : M \rightarrow \mathbb{C}$ agreeing with u_1 to order d at all points of Λ and satisfying

$$(2.1) \quad \min\{|v_1(p)| : p \in F\} > \max\{|v_1(p)| : p \in bL\}.$$

Assuming that the approximation is close enough, the holomorphic map $(v_1, u_2) : L \rightarrow \mathbb{C}^2$ is still an immersion on a neighbourhood of L . Let B denote the set of critical points of v_1 on $M \setminus L$. Since v_1 is regular, B is finite. Also choose for $i = 1, \dots, m$ a smoothly bounded closed disc $\Delta_i \subset \overset{\circ}{D}_i \setminus \{p_i\}$ such that

¹It is customary to call a nonempty set in a topological space a compact domain if it is compact and is the closure of a connected open set. Lacking a better term, in this paper we also refer to the union of finitely many mutually disjoint compact domains as a compact domain.

$B \subset \Delta = \Delta_1 \cup \dots \cup \Delta_m$. Extend $u_2 : L \rightarrow \mathbb{C}$ to a holomorphic map $u_2 : L \cup \Delta \rightarrow \mathbb{C}$ such that u_2 is an immersion on Δ and

$$(2.2) \quad \min\{|u_2(p)| : p \in \Delta\} > \max\{|u_2(p)| : p \in bL\}.$$

(Recall that $L \cap \Delta = \emptyset$ and L and Δ are compact.) Again by [34, Theorem 10], we may approximate u_2 uniformly on a neighbourhood of $L \cup \Delta$ by a regular function v_2 on M agreeing with u_2 to order d at all points of Δ . If the approximation is close enough, then $v = (v_1, v_2) : M \rightarrow \mathbb{C}^2$ is still an immersion on a neighbourhood of L , while $v_2 : M \rightarrow \mathbb{C}$ is an immersion on a neighbourhood of Δ . Since $v_1 : M \rightarrow \mathbb{C}$ is an immersion on $M \setminus (L \cup B)$ and $B \subset \Delta$, we infer that $v : M \rightarrow \mathbb{C}^2$ is a regular immersion. Note that v is close to u uniformly on a neighbourhood of K and agrees with u to order d at all points of Δ . Finally, assuming that v_2 is sufficiently close to u_2 uniformly on $L \cup \Delta$, condition (2.2) ensures that

$$\min\{|v_2(p)| : p \in \Delta = \bigcup_{i=1}^m \Delta_i\} > \max\{|v_2(p)| : p \in bL = \bigcup_{i=1}^m bD_i\}.$$

We claim that v_2 has a pole at p_i for all $i = 1, \dots, m$, so $v_2 : M \rightarrow \mathbb{C}$ is a proper map. Indeed, otherwise, v_2 would be holomorphic on the smoothly bounded closed disc D_i , violating the maximum modulus principle since $\Delta_i \subset \overset{\circ}{D}_i$ and $\min\{|v_2(p)| : p \in \Delta_i\} > \max\{|v_2(p)| : p \in bD_i\}$. Using (2.1), the same argument shows that $v_1 : M \rightarrow \mathbb{C}$ is a proper map. In particular, $v : M \rightarrow \mathbb{C}^2$ is proper. \square

Corollary 2.2. *If $A = \mathbb{C}_*^n$, $n \geq 2$, M is a smooth affine curve, and \mathcal{A} is the subbundle of $(T^*M)^{\oplus n}$ defined by A , then conditions (i)–(iv) in Corollary 1.8 hold. In particular, A is very good in the sense of Definition 1.3.*

Proof. Proposition 2.1 implies (ii), while it is obvious that (ii) implies (i) and that (iv) implies (iii). So it remains to check (iv). For this, in view of Proposition 2.1, it suffices to show that every homotopy class of continuous or, equivalently, holomorphic sections of \mathcal{A} over M contains du for some holomorphic immersion $u : M \rightarrow \mathbb{C}^n$. By [24, Theorems 5.3 and 5.4] (see also [8, Theorem 1]), every holomorphic section of \mathcal{A} over M can be deformed to an exact holomorphic section, that is, a section of the form du . \square

2.2. The null quadric. This is the cone

$$(2.3) \quad A = \{(z_1, \dots, z_n) \in \mathbb{C}_*^n : z_1^2 + \dots + z_n^2 = 0\}, \quad n \geq 3.$$

This cone is algebraically elliptic as seen in [7, Proposition 1.15.3]. A holomorphic immersion $M \rightarrow \mathbb{C}^n$ directed by A is called a null curve and its real part $M \rightarrow \mathbb{R}^n$ is a conformal minimal immersion. Conversely, every conformal minimal immersion $M \rightarrow \mathbb{R}^n$ is the real part of a holomorphic null curve on any simply connected domain in M . The natural counterpart of a regular function in the theory of minimal surfaces is a complete minimal surface of finite total curvature. Indeed, if M is an open Riemann surface and $X : M \rightarrow \mathbb{R}^n$ is a complete conformal minimal immersion of finite total curvature, then M is a smooth affine curve and the $(1, 0)$ -part of the

exterior derivative of X is a regular section of \mathcal{A} on M with an effective pole at each end of M . (See the classical surveys [33, 14, 37] or the recent monograph [7], in particular Chapter 4.) For regular null curves and complete minimal surfaces of finite total curvature, Theorem 1.1 was recently proved by Alarcón and López in [10] (see [31, 3] for the case $n = 3$). By the results in [10], the punctured null quadric A (2.3) is very good. The proofs in [31, 3, 10], relying mainly on the theory of Riemann surfaces, are much more involved than the arguments here and heavily use the special geometry of the null quadric. It does not seem that these proofs could be adapted to more general families of regular directed immersions. For completeness, we give a simple proof that the punctured null quadric is very good. (This can also be seen as an immediate corollary of the results in [31, 10].) Thus, if M is a smooth affine curve and \mathcal{A} is the subbundle of $(T^*M)^{\oplus n}$ defined by A , then conditions (i)–(iv) in Corollary 1.8 hold.

Proposition 2.3. *The punctured cone $A = \{(z_1, \dots, z_n) \in \mathbb{C}_*^n : z_1^2 + \dots + z_n^2 = 0\}$, $n \geq 3$, is very good in the sense of Definition 1.3.*

Proof. By Proposition 1.5(a), A is good. If $n \geq 4$, then A is simply connected and hence very good by Remark 1.4. In case $n = 3$, A is biholomorphic to

$$A' = \{z = (z_1, z_2, z_3) \in \mathbb{C}_*^3 : z_1 z_2 = z_3^2\}$$

and is not simply connected (its fundamental group is \mathbb{Z}_2 : see [6, Equation (8.3)]). We will prove that the pair (A', M) is very good for every smooth affine curve M . Denote by \mathcal{A}' the subbundle of $(T^*M)^{\oplus 3}$ defined by A' . Since A' is an Oka manifold, every homotopy class of continuous sections of \mathcal{A}' on M contains a holomorphic section. Thus it suffices to see that for any holomorphically convex compact set $K \subset M$, which is a strong deformation retract of M , and for any holomorphic section η of \mathcal{A}' on a neighbourhood of K , we can approximate η on a neighbourhood of K by a regular section of \mathcal{A}' on M . For this, note that $\eta = (\eta_1, \eta_2, \eta_3)$, where each η_i is a holomorphic 1-form on a neighbourhood of K such that

$$(2.4) \quad \eta_1 \eta_2 = \eta_3^2$$

and η_1 and η_2 have no common zeros on K . By (2.4), η_1 and η_2 are spinorially equivalent on a neighbourhood of K , meaning that there is a meromorphic function f on a neighbourhood of K such that $\eta_1 = f^2 \eta_2$. Choose a regular 1-form ω on M satisfying the following conditions (see [3, Lemma 3.2]).

- (a) ω has no zeros on K .
- (b) ω is spinorially equivalent to η_1 , hence also to η_2 , on a neighbourhood of K .

By (b), there is a meromorphic function f_i on a neighbourhood of K such that

$$(2.5) \quad \eta_i = f_i^2 \omega, \quad i = 1, 2.$$

Note that (a) ensures that f_i is in fact holomorphic on a neighbourhood of K , $i = 1, 2$. By (2.4) and after replacing f_1 by $-f_1$ if necessary, we may assume that

$$(2.6) \quad \eta_3 = f_1 f_2 \omega \quad \text{on a neighbourhood of } K.$$

Denote by Z_1 the zero set of ω , a finite subset of $M \setminus K$ (see (a)). Let $\Omega_1 \subset M \setminus K$ be a compact neighbourhood of Z_1 such that each component of Ω_1 is a closed disc containing a single point in Z_1 . Choose a meromorphic function f_1 on a neighbourhood of Ω_1 such that $f_1^2 \omega$ is holomorphic and has no zeros on a neighbourhood of Ω_1 . Fix a number $\epsilon_1 > 0$. By Royden's Runge theorem for regular functions [34, Theorem 10], there is a regular function g_1 on $M \setminus Z_1$ satisfying the following conditions.

- (A1) g_1/f_1 is holomorphic and has no zeros on K .
- (B1) $|g_1 - f_1| < \epsilon_1$ on a neighbourhood of $K \cup \Omega_1$.
- (C1) $g_1^2 \omega$ is regular on M and has no zeros on Ω_1 .

Now, let Z_2 denote the zero set of g_1 on $M \setminus K$. Note that Z_2 is finite and $Z_2 \subset M \setminus (K \cup \Omega_1)$ by (C1). Let $\Omega_2 \subset M \setminus (K \cup \Omega_1)$ be a compact neighbourhood of Z_2 such that each component of Ω_2 is a closed disc containing a single point in Z_2 . Extend the function f_2 to $\Omega_1 \cup \Omega_2$ by setting $f_2 = g_1$ on Ω_1 and $f_2 = 1$ on Ω_2 ; note that the compact sets K , Ω_1 , and Ω_2 are mutually disjoint. Also note that f_2 is meromorphic on a neighbourhood of $K \cup \Omega_1 \cup \Omega_2$ and holomorphic on $(K \cup \Omega_1 \cup \Omega_2) \setminus Z_1$. Choose $\epsilon_2 > 0$. By [34, Theorem 10], there is a regular function g_2 on $M \setminus Z_1$ satisfying the following conditions.

- (A2) g_2/f_2 is holomorphic and has no zeros on K .
- (B2) $|g_2 - f_2| < \epsilon_2$ on a neighbourhood of $K \cup \Omega_1 \cup \Omega_2$.
- (C2) $g_2^2 \omega$ is regular on M and has no zeros on $\Omega_1 \cup \Omega_2$. (This is possible by (C1).)

Set $\rho_i = g_i^2 \omega$, $i = 1, 2$. By (C1), (C2), and since $Z_1 \subset \Omega_1$, these are regular forms on M . By (a), (A1), (A2), and since η_1 and η_2 have no common zeros on K , we see that ρ_1 and ρ_2 have no common zeros on K . Using also (C1), (C2), and since $Z_1 \subset \Omega_1$ and $Z_2 \subset \Omega_2$, we then infer that ρ_1 and ρ_2 have no common zeros on M . Finally, consider the regular form $\rho_3 = g_1 g_2 \omega$ on M . It is clear that $\rho_1 \rho_2 = \rho_3^2$, so $\rho = (\rho_1, \rho_2, \rho_3)$ is a regular section of \mathcal{A}' on M . Finally, (2.5), (2.6), (B1), and (B2) ensure that ρ is ϵ -close to η on a neighbourhood of K for any given $\epsilon > 0$, provided that ϵ_1 and ϵ_2 are chosen sufficiently small. \square

2.3. The Bryant cone. Consider the cone

$$(2.7) \quad A = \{(z_1, \dots, z_4) \in \mathbb{C}_*^4 : z_1 z_4 - z_2 z_3 = 0\}.$$

This is the cone on the smooth quadric in \mathbb{P}^3 given by the formula $z_1 z_4 = z_2 z_3$, which is the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in \mathbb{P}^3 . Being covered by copies of \mathbb{C}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ is uniformly rational, so A is algebraically elliptic [12, Theorem 1.3] (see Remark 1.2(b)). Bryant [18] calls a holomorphic immersion of a Riemann surface into the special linear group $\mathrm{SL}_2(\mathbb{C}) = \{(z_1, \dots, z_4) \in \mathbb{C}^4 : z_1 z_4 - z_2 z_3 = 1\}$, directed by A , a null curve. The projection of a holomorphic null curve in $\mathrm{SL}_2(\mathbb{C})$ to the hyperbolic 3-space $\mathbb{H}^3 = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$ is a conformally immersed surface of constant mean curvature 1 (also called a Bryant surface), and every simply connected surface in \mathbb{H}^3 of constant mean curvature 1 arises in this way. We refer to [36, 19] and

the introduction of the more recent paper [2] for background on this subject. It is an open question whether every open Riemann surface admits a proper holomorphic null immersion or embedding into $\mathrm{SL}_2(\mathbb{C})$ [5, Problem 1]. The same question in the algebraic category is open as well.

Let us show that the punctured cone A in (2.7), directing holomorphic null curves in $\mathrm{SL}_2(\mathbb{C}) \subset \mathbb{C}^4$, is very good. Thus, if M is a smooth affine curve and \mathcal{A} is the subbundle of $(T^*M)^{\oplus 4}$ defined by A , then conditions (i)–(iv) in Corollary 1.8 hold.

Proposition 2.4. *The punctured cone $A = \{(z_1, \dots, z_4) \in \mathbb{C}_*^4 : z_1 z_4 - z_2 z_3 = 0\}$ is very good in the sense of Definition 1.3.*

Proof. By Proposition 1.5(a), A is good. To conclude that A is very good, we show that A is simply connected. Above we described A as a \mathbb{C}^* -bundle on $\mathbb{P}^1 \times \mathbb{P}^1$. We now need an alternative description. An element z of A may be viewed as a 2×2 complex matrix of rank 1. Consider the projection $k : A \rightarrow \mathbb{P}^1$ mapping z to its kernel (a 1-dimensional linear subspace of \mathbb{C}^2 viewed as a point in \mathbb{P}^1). Let $a : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a continuous map such that $a(\zeta) \neq \zeta$ for all $\zeta \in \mathbb{P}^1$ ($a(\zeta)$ could for example be the antipodal point of ζ). The fibre $k^{-1}(\zeta)$ can then be identified with the space of nontrivial linear maps from $a(\zeta)$, viewed as a 1-dimensional linear subspace of \mathbb{C}^2 , into \mathbb{C}^2 , which in turn can be identified with \mathbb{C}_*^2 . The piece $0 = \pi_1(\mathbb{C}_*^2) \rightarrow \pi_1(A) \rightarrow \pi_1(\mathbb{P}^1) = 0$ of the long exact sequence of homotopy groups associated to the fibre bundle k now shows that A is simply connected. \square

3. Approximation by regular sections

In this section we establish the following theorem, the second main result of the paper, on approximation by regular sections with control of periods. Subsequently, we prove Theorem 1.10. Recall that the first homology group with integer coefficients $H_1(M, \mathbb{Z})$ of a smooth affine curve M is isomorphic to \mathbb{Z}^r for some integer $r \geq 0$.

Theorem 3.1. *Let $A \subset \mathbb{C}_*^n$ be the punctured cone on a connected submanifold Y of \mathbb{P}^{n-1} , $n \geq 2$, and assume that A is algebraically elliptic and not contained in a hyperplane in \mathbb{C}^n . Let M be a smooth affine curve and \mathcal{A} be the subbundle of $(T^*M)^{\oplus n}$ defined by A . Also let K be a holomorphically convex compact subset of M , s be a holomorphic section of \mathcal{A} on a neighbourhood of K , and $\mathcal{F} : H_1(M, \mathbb{Z}) \rightarrow \mathbb{C}^n$ be a group homomorphism such that*

$$(3.1) \quad \mathcal{F}(C) = \int_C s \quad \text{for every loop } C \text{ in a neighbourhood of } K.$$

Finally, let C_1, \dots, C_ℓ be smooth oriented embedded arcs in K such that $C_1 \cup \dots \cup C_\ell$ is holomorphically convex in M and each arc C_l contains a nontrivial arc disjoint from $\bigcup_{i \neq l} C_i$. Then the following are equivalent.

- (i) *s can be uniformly approximated on a neighbourhood of K by regular sections t of \mathcal{A} on M with an effective pole at each end of M , such that $\int_C t = \mathcal{F}(C)$ for every loop C in M and $\int_{C_l} t = \int_{C_l} s$ for $l = 1, \dots, \ell$.*

- (ii) *There is a neighbourhood U of K such that the homotopy class of continuous sections of $\mathcal{A}|_U$ that contains s also contains the restriction of a regular section of \mathcal{A} on M .*

Furthemore, if (ii) holds, then the regular sections $t = (t_1, \dots, t_n)$ in (i) can be chosen such that all the component 1-forms t_1, \dots, t_n have an effective pole at each end of M .

We start with some preparations. We expect the following to have been observed before, but we do not know a reference for it.

Lemma 3.2. *Let M be an open Riemann surface and Y be a connected complex manifold. Let K be a holomorphically convex compact set in M and $f : K \rightarrow Y$ be a continuous map. Then f admits a continuous extension $M \rightarrow Y$. If, in addition, Y is an Oka manifold and f is holomorphic in $\overset{\circ}{K}$, then f can be approximated uniformly on K by holomorphic maps $M \rightarrow Y$ homotopic to any given continuous extension $M \rightarrow Y$ of f .*

Proof. By the Tietze extension theorem and the existence of a smooth tubular neighbourhood of Y once smoothly embedded into some Euclidean space, f extends continuously to a neighbourhood of K . Let $K_0 \supset K$ be a smoothly bounded holomorphically convex compact domain in M , possibly disconnected, such that f admits a continuous extension $f_0 : K_0 \rightarrow Y$. Let ρ be a smooth strongly subharmonic Morse exhaustion function on M with $K_0 = \{\rho \leq 0\}$. Let $a_0 = 0 < a_1 < a_2 < \dots$ be a divergent sequence of regular values of ρ such that ρ has at most one critical point in $K_{i+1} \setminus K_i$ for all $i = 0, 1, 2, \dots$, where $K_i = \{\rho \leq i\}$. Each K_i is a smoothly bounded compact domain (possibly disconnected) with $K_i \Subset K_{i+1}$ and $M = \bigcup_{i \geq 0} K_i$. If ρ does not have critical points in $K_1 \setminus K_0$, then K_0 is a strong deformation retract of K_1 , and hence f_0 (and thus f) extends continuously to K_1 . If there is a (necessarily unique) critical point of ρ in $K_1 \setminus K_0$, then there is a smooth embedded Jordan arc E in M intersecting K_0 precisely at its two endpoints and such that the intersection of E and the boundary of K_0 is transverse and $K_0 \cup E$ is a strong deformation retract of K_1 . Since Y is path-connected, f_0 extends to a continuous map $K_0 \cup E \rightarrow Y$, and since $K_0 \cup E$ is a strong deformation retract of K_1 , there is a continuous extension $f_1 : K_1 \rightarrow Y$ of f_0 . Repeating this process we may then extend f_1 to a continuous map $f_2 : K_2 \rightarrow Y$, and continuing inductively we obtain in the limit a continuous extension $M \rightarrow Y$ of f_0 , hence of f . This proves the first assertion in the lemma. The second assertion is then guaranteed by the Mergelyan theorem for maps from Riemann surfaces to Oka manifolds; see e.g. [20, Corollary 8]. \square

A set S in a smooth surface M is said to be *admissible* if it is of the form $S = K \cup E$, where K is a (possibly empty) finite union of mutually disjoint compact connected domains with piecewise \mathcal{C}^1 boundaries in M and $E = S \setminus \overset{\circ}{K}$ is a finite union of mutually disjoint smooth Jordan arcs and closed Jordan curves meeting K only at

their endpoints (if at all) and such that their intersections with the boundary bK of K are transverse; see [7, Definition 1.12.9]. Let us record the following result.

Lemma 3.3. *Let $A \subset \mathbb{C}_*^n$ be the punctured cone on a connected submanifold Y of \mathbb{P}^{n-1} , $n \geq 2$, and assume that A is an Oka manifold and is not contained in a hyperplane in \mathbb{C}^n . Let M be an open Riemann surface, $S = K \cup E \subset M$ be a holomorphically convex admissible set, and C_1, \dots, C_ℓ be smooth oriented embedded Jordan arcs and closed Jordan curves in S such that $C_1 \cup \dots \cup C_\ell$ is holomorphically convex in M and each C_l contains a nontrivial arc disjoint from $\bigcup_{i \neq l} C_i$. Finally, let θ be a holomorphic 1-form vanishing nowhere on M . Then every map $f : S \rightarrow A$ which is of class \mathcal{C}^r ($r \geq 0$) on a neighbourhood of S and holomorphic on $\mathring{S} = \mathring{K}$ can be approximated in $\mathcal{C}^r(S)$ by holomorphic maps $F : M \rightarrow A$ such that $F(M)$ is not contained in a hyperplane in \mathbb{C}^n , F agrees with f on any given finite set in S , F agrees with f to any given finite order on any given finite set in \mathring{S} , and $\int_{C_l} F\theta = \int_{C_l} f\theta$ for all $l = 1, \dots, \ell$.*

Note that the given map f in the lemma extends to a continuous map $M \rightarrow A$ by Lemma 3.2, and hence there is no topological obstruction to holomorphic approximation. When A is the punctured null quadric (2.3), Lemma 3.3 coincides with [7, Lemma 3.3.1]. The proof for an arbitrary cone A as in the statement follows the proof in [7] word for word, but using the tools from [4, 1] that deal with maps into an arbitrary cone. See also [1, Lemma 4.2] for an analogous result in a slightly different topological setting. We leave the details of the proof to the reader.

Lemma 3.4. *Let $A \subset \mathbb{C}_*^n$ be the punctured cone on a connected submanifold Y of \mathbb{P}^{n-1} , $n \geq 2$, and assume that A is an Oka manifold, not contained in a hyperplane in \mathbb{C}^n . Also let $\sigma : [0, 1] \rightarrow A$ be a path. Then for any $\mu \in \mathbb{C}^n$, there is a homotopy of paths $\sigma^t : [0, 1] \rightarrow A$, $t \in [0, 1]$, such that $\sigma^0 = \sigma$, $\sigma^t(0) = \sigma(0)$ and $\sigma^t(1) = \sigma(1)$ for all $t \in [0, 1]$, and $\int_0^1 \sigma^t(x) dx = \mu$.*

Proof. Applying a small deformation to σ keeping $\sigma(0)$ and $\sigma(1)$ fixed, we can assume that $\sigma([0, 1])$ is not contained in any hyperplane in \mathbb{C}^n . This enables us to use the analogue of [24, Lemma 3.1] with the punctured null quadric replaced by the cone A (see [8, p. 10] or [24, proof of Theorem 5.3]) to obtain a homotopy σ^t satisfying the conclusion of the lemma. Note that the approximate condition in [24, Lemma 3.1] can be made exact since $\sigma([0, 1])$ is not contained in any hyperplane in \mathbb{C}^n , as is seen in [8, proof of Theorem 1] and [24, proof of Theorem 5.3]. We also refer to [1, Lemma 3.3] for a closely related result. \square

With Lemmas 3.3 and 3.4 in hand, we are now ready to prove the following first main technical step in the proof of Theorem 3.1.

Proposition 3.5. *Let A and M be as in Theorem 3.1. Also let $\mathcal{F} : H_1(M, \mathbb{Z}) \rightarrow \mathbb{C}^n$ be a group homomorphism and θ be a nowhere-vanishing holomorphic 1-form on M . Then every continuous map $f : M \rightarrow A$ is homotopic to a holomorphic map*

$\tilde{f} : M \rightarrow A$ such that $\tilde{f}(M)$ is not contained in a hyperplane in \mathbb{C}^n and

$$\int_C \tilde{f}\theta = \mathcal{F}(C) \quad \text{for every loop } C \text{ in } M.$$

Furthermore, if f is holomorphic on a neighbourhood of a holomorphically convex compact set K in M ,

$$(3.2) \quad \int_C f\theta = \mathcal{F}(C) \quad \text{for every loop } C \text{ in a neighbourhood of } K,$$

and C_1, \dots, C_ℓ are smooth oriented embedded arcs in K such that $C_1 \cup \dots \cup C_\ell$ is holomorphically convex in M and each arc C_l contains a nontrivial arc disjoint from $\bigcup_{i \neq l} C_i$, then a holomorphic map \tilde{f} as above can be chosen to approximate f uniformly on a neighbourhood of K , to agree with f to any given finite order on any given finite set in K , and to satisfy

$$\int_{C_l} \tilde{f}\theta = \int_{C_l} f\theta \quad \text{for } l = 1, \dots, \ell.$$

We point out that the interpolation condition in this proposition (or in Lemma 3.3) will not be required for further developments in this paper. In particular, it is not used to prove Theorem 1.1.

Proof. We may assume without loss of generality that $K \neq \emptyset$ is a smoothly bounded compact domain, possibly disconnected.

We claim that K can be assumed to be connected. Indeed, otherwise denote by K_0, K_1, \dots, K_N , $N \geq 1$, the connected components of K . Attach to K a union E of N mutually disjoint smooth embedded Jordan arcs $\gamma_1, \dots, \gamma_N$ in M meeting K precisely at their two endpoints and such that their intersections with the boundary bK of K are transverse. Then the compact set $S = K \cup E \subset M$ is admissible [7, Definition 1.12.9]. Further, choose the arc γ_i to have an endpoint in bK_0 and the other endpoint in bK_i , $i = 1, \dots, N$. Then S is connected and holomorphically convex in M . Note that the first homology group $H_1(S, \mathbb{Z})$ of S is isomorphic to \mathbb{Z}^r for some integer $r \geq 0$ (see [7, Lemma 1.12.10]) and the inclusion $K \hookrightarrow S$ induces an isomorphism $H_1(K, \mathbb{Z}) \rightarrow H_1(S, \mathbb{Z})$. Let $\ell' = \ell + r$. By [7, proof of Proposition 3.3.2], there are smooth oriented embedded Jordan curves $C_{\ell+1}, \dots, C_{\ell'}$ in S forming a homology basis of S such that $\bigcup_{l=1}^{\ell'} C_l$ is holomorphically convex in M and each C_l , $l = 1, \dots, \ell'$, contains a nontrivial arc disjoint from $\bigcup_{i \neq l} C_i$. By Lemma 3.3, we may approximate f uniformly on S by a holomorphic map $f' : M \rightarrow A$ agreeing with f to any given order on any given finite set in K and satisfying $\int_{C_l} f'\theta = \int_{C_l} f\theta$ for all $l = 1, \dots, \ell'$. In particular, $\int_C f'\theta = \mathcal{F}(C)$ for every loop C in a neighbourhood of S by (3.2) and the fact that $C_{\ell+1}, \dots, C_{\ell'}$ generate $H_1(S, \mathbb{Z}) \cong H_1(K, \mathbb{Z})$. Moreover, if the approximation of f by f' on S is sufficiently close, then f' and f are homotopic on a neighbourhood of S and there is a continuous map $f'' : M \rightarrow A$ homotopic to f such that $f'' = f'$ on a neighbourhood of S (and also outside a bigger neighbourhood of S if desired). Therefore, replacing f by f'' and K by S , we may assume that K is connected, as claimed.

We now fix Jordan curves $C_{\ell+1}, \dots, C_{\ell'}$ in the connected compact domain K forming a homology basis of K , where $\ell' = \ell + \dim(H_1(K, \mathbb{Z}))$, such that $C_1 \cup \dots \cup C_{\ell'}$ is holomorphically convex in M and each C_l , $l = 1, \dots, \ell'$, contains a nontrivial arc disjoint from $\bigcup_{i \neq l} C_i$. We proceed by induction on the Euler characteristic $\chi(M \setminus K)$ of $M \setminus K$, which is a nonpositive integer. Assume for the basis of the induction that $\chi(M \setminus K) = 0$, so K is a strong deformation retract of M and the inclusion $K \hookrightarrow M$ induces an isomorphism $H_1(K, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z})$. Applying Lemma 3.3 again, we may approximate f uniformly on a neighbourhood of K by a holomorphic map $\tilde{f} : M \rightarrow A$ agreeing with f to any given finite order on any given finite set in K , such that $\tilde{f}(M)$ is not contained in a hyperplane in \mathbb{C}^n and $\int_{C_l} \tilde{f}\theta = \int_{C_l} f\theta$ for $l = 1, \dots, \ell'$. Since $C_{\ell+1}, \dots, C_{\ell'}$ generate the homology of M , this and (3.2) ensure that $\int_C \tilde{f}\theta = \mathcal{F}(C)$ for every loop C in M . Moreover, if the approximation is close enough, then \tilde{f} is homotopic to f over K , and since K is a strong deformation retract of M , \tilde{f} is homotopic to f over all of M as well.

For the inductive step, assume now that the theorem holds whenever K is a connected smoothly bounded compact domain with $-\chi(M \setminus K) = k$ for some integer $k \geq 0$, and suppose that $-\chi(M \setminus K) = k+1$. Attach to K a smooth embedded Jordan arc E in M meeting K precisely at its two endpoints and such that its intersection with the boundary bK of K is transverse. Then the compact set $S = K \cup E$ is connected and admissible. Choose E so that S is holomorphically convex in M and $-\chi(M \setminus S) = k$, that is, there is a smooth embedded closed Jordan curve in S which contains E whose homology class belongs to $H_1(S, \mathbb{Z})$ but not to $H_1(K, \mathbb{Z})$. In view of (3.2), we can use Lemma 3.4 in order to deform $f : S \rightarrow A$ to a continuous map $f' : S \rightarrow A$ such that $f' = f$ on a neighbourhood of K and $\int_C f'\theta = \mathcal{F}(C)$ for every loop C in S . Since f' is homotopic to f on S , it admits a continuous extension to M which is homotopic to f on all of M (we can in fact choose f' to equal f outside a neighbourhood of E if desired). Reasoning as above, we can assume that f' is holomorphic on a neighbourhood of S , thereby reducing the proof to the case when $-\chi(M \setminus K) = k$. The induction hypothesis then completes the proof. \square

Remark 3.6. Proposition 3.5 remains true with M replaced by an arbitrary open Riemann surface and with the cone A being Oka but not necessarily algebraically elliptic. This follows by a standard recursive application of the arguments in the proof we have given and we leave further details to the reader.

Proof of Theorem 3.1. (i) \Rightarrow (ii): If s is sufficiently close on a neighbourhood U of K to a regular section t of \mathcal{A} on M , then s and t are homotopic through continuous sections of $\mathcal{A}|_U$.

(ii) \Rightarrow (i): We proceed in two steps. The first one is given by the following claim.

Claim 3.7. *If (ii) holds, then s can be uniformly approximated on a neighbourhood of K by regular sections t of \mathcal{A} on M such that $\int_C t = \mathcal{F}(C)$ for every loop C in M and $\int_{C_l} t = \int_{C_l} s$ for $l = 1, \dots, \ell$.*

Note that the section t given by this claim satisfies (i) except it may not have a pole at each end of M .

Proof. Let s_0 be a regular section of \mathcal{A} on M that is homotopic to s over a neighbourhood of K ; it exists by (ii). Then s_0 is homotopic over all of M to a continuous section s' of \mathcal{A} on M that equals s on a smaller neighbourhood of K (see Lemma 3.2). So by replacing s by s' , we can assume that s is continuous on M and s and s_0 are homotopic on all of M . Fix a holomorphic 1-form θ vanishing nowhere on M and write

$$(3.3) \quad f = s/\theta : M \rightarrow A \quad \text{and} \quad f_0 = s_0/\theta : M \rightarrow A.$$

Note that f and f_0 are homotopic, f_0 is holomorphic, and f is continuous on M and holomorphic on a neighbourhood of K . By Proposition 3.5, f can be deformed to a holomorphic map $\tilde{f} : M \rightarrow A$ that approximates f uniformly on a neighbourhood of K , such that $\tilde{f}(M)$ is not contained in a hyperplane in \mathbb{C}^n ,

$$(3.4) \quad \int_C \tilde{f}\theta = \mathcal{F}(C) \quad \text{for every loop } C \text{ in } M,$$

and

$$(3.5) \quad \int_{C_l} \tilde{f}\theta = \int_{C_l} f\theta \quad \text{for } l = 1, \dots, \ell$$

(see (3.1)). In particular, $\tilde{f} : M \rightarrow A$ is homotopic to f_0 on M and is nonflat in the sense of [1, Definition 3.1] and nondegenerate in the sense of [4, Definition 2.2]. Note that $\tilde{f}\theta$ is a holomorphic section of \mathcal{A} on M satisfying the conclusion of the claim, except it may not be regular.

Let L be a connected smoothly bounded compact domain in $M = \overline{M} \setminus \{p_1, \dots, p_m\}$ with $K \subset \overset{\circ}{L}$ and $\overline{M} \setminus \overset{\circ}{L} = D_1 \cup \dots \cup D_m$, where D_1, \dots, D_m are mutually disjoint smoothly bounded closed discs centred at p_1, \dots, p_m , respectively. Recall that the first homology group $H_1(M, \mathbb{Z})$ is isomorphic to \mathbb{Z}^r for some $r \geq 0$ and that the inclusion $L \hookrightarrow M$ induces an isomorphism $H_1(L, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z})$. Write $\ell' = \ell + r$. By [7, proof of Proposition 3.3.2], there are smooth oriented embedded Jordan curves $C_{\ell+1}, \dots, C_{\ell'}$ in L forming a homology basis of L such that

$$\Gamma = C_1 \cup \dots \cup C_{\ell'}$$

is holomorphically convex in M and each C_l , $l = 1, \dots, \ell'$, contains a nontrivial arc disjoint from $\bigcup_{i \neq l} C_i$. Consider the *period-interpolation map*

$$\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{\ell'}) : \mathcal{C}(\Gamma, \mathbb{C}^n) \rightarrow (\mathbb{C}^n)^{\ell'} = \mathbb{C}^{\ell'n}$$

defined by

$$(3.6) \quad \mathcal{P}_l(h) = \int_{C_l} h\theta \in \mathbb{C}^n, \quad h \in \mathcal{C}(\Gamma, \mathbb{C}^n), \quad l = 1, \dots, \ell'.$$

Arguing as in [7, proof of Lemma 3.3.1, Step 2] but with the punctured null quadric replaced by the cone A (see also [1, Lemma 4.2] and [4, Lemma 5.1]), we can find an open ball V centred at the origin 0 in some \mathbb{C}^k and a holomorphic \mathcal{P} -dominating

spray $\phi : V \times M \rightarrow A$ with core \tilde{f} , that is, a holomorphic map such that $\phi(0, \cdot) = \tilde{f}$ and

$$\frac{\partial}{\partial \zeta} \Big|_{\zeta=0} \mathcal{P}(\phi(\zeta, \cdot)) : (\mathbb{C}^n)^{\ell'} \rightarrow (\mathbb{C}^n)^{\ell'} \quad \text{is an isomorphism.}$$

(Note that [1, Lemma 4.2] and [4, Lemma 5.1] deal with the case when M is a compact bordered Riemann surface, but the same proof applies when M is an arbitrary open Riemann surface since we only need to control the periods on the finitely many arcs and curves $C_1, \dots, C_{\ell'}$.) We may assume that the spray ϕ extends continuously to a map $\phi : \mathbb{C}^k \times M \rightarrow A$. Since A satisfies the basic Oka property with approximation and interpolation (see [23, Corollary 5.4.5]), ϕ may be deformed to a holomorphic spray $\sigma : \mathbb{C}^k \times M \rightarrow A$ such that $\sigma(0, \cdot) = \tilde{f}$, with sufficiently close approximation on a neighbourhood of $\{0\} \times L \supset \{0\} \times \Gamma$ to ensure that σ is still \mathcal{P} -dominating:

$$(3.7) \quad \frac{\partial}{\partial \zeta} \Big|_{\zeta=0} \mathcal{P}(\sigma(\zeta, \cdot)) : (\mathbb{C}^n)^{\ell'} \rightarrow (\mathbb{C}^n)^{\ell'} \quad \text{is an isomorphism.}$$

Using θ , we turn σ into a holomorphic spray $\sigma\theta$ of sections of \mathcal{A} on M parametrised by \mathbb{C}^k such that $\sigma(0, \cdot)\theta = \tilde{f}\theta$. We may view the spray $\sigma\theta$ as a holomorphic section of the pullback of \mathcal{A} by the projection $\mathbb{C}^k \times M \rightarrow M$. By Forstnerič's algebraic approximation theorem [22, Theorem 3.1] (see also [23, Theorem 6.15.3]), since A is algebraically elliptic and $\tilde{f}\theta$ is homotopic to the regular section $f_0\theta = s_0$ of \mathcal{A} on M (see (3.3)), we may uniformly approximate the spray $\sigma\theta$ over $W \times L$ by a regular spray τ of sections of \mathcal{A} on M parametrised by \mathbb{C}^k , where W is a closed ball centred at the origin in \mathbb{C}^k . By (3.7), if the approximation is close enough then near the origin is a parameter $\zeta_0 \in \mathbb{C}^k$ such that

$$(3.8) \quad \mathcal{P}(\tau(\zeta_0)/\theta) = \mathcal{P}(\sigma(0, \cdot)) = \mathcal{P}(\tilde{f}).$$

Then $t = \tau(\zeta_0)$ is a regular section of \mathcal{A} on M which is close to $\tilde{f}\theta = \sigma(0, \cdot)\theta$ on L and hence to $s = f\theta$ on K ; see (3.3). It follows from (3.6) and (3.8) that

$$(3.9) \quad \int_{C_l} t = \int_{C_l} \tilde{f}\theta \quad \text{for } l = 1, \dots, \ell'.$$

This, (3.3), and (3.5) show that $\int_{C_l} t = \int_{C_l} f\theta = \int_{C_l} s$ for $l = 1, \dots, \ell'$. Finally, since the curves $C_{\ell'+1}, \dots, C_{\ell'}$ form a basis of $H_1(M, \mathbb{Z})$, (3.4) and (3.9) imply that $\int_C t = \int_C \tilde{f}\theta = \mathcal{F}(C)$ for every loop C in M . \square

To complete the proof that (ii) \Rightarrow (i), it remains to see that we can find a regular section t as in Claim 3.7 with an effective pole at each end of M . For this, we adapt the argument in the proof of Proposition 2.1 to our current more general framework. By Claim 3.7, we may assume that s extends to a regular section of \mathcal{A} on M . Let $L \ni K$ be a smoothly bounded compact domain in $M = \overline{M} \setminus \{p_1, \dots, p_m\}$ such that $\overline{M} \setminus \overset{\circ}{L} = D_1 \cup \dots \cup D_m$, where D_1, \dots, D_m are mutually disjoint smoothly bounded closed discs centred at p_1, \dots, p_m , respectively. In particular, $bL = bD_1 \cup \dots \cup bD_m$. Fix a holomorphic 1-form ϑ vanishing nowhere on a neighbourhood of $D = D_1 \cup \dots \cup D_m \supset bL$. Also choose for each $i = 1, \dots, m$ a

smoothly bounded closed disc $\Delta_i \subset \mathring{D}_i \setminus \{p_i\}$, set $\Delta = \Delta_1 \cup \dots \cup \Delta_m$, and consider a holomorphic section \tilde{s} of \mathcal{A} on a neighbourhood of $L \cup \Delta$ such that $\tilde{s} = s$ on a neighbourhood of L and the map

$$(3.10) \quad g = (g_1, \dots, g_n) = \frac{\tilde{s}}{\vartheta} : \Delta \cup bL \rightarrow A$$

satisfies

$$(3.11) \quad \min\{|g_k(p)| : p \in \Delta\} > \max\{|g_k(p)| : p \in bL\} \quad \text{for } k = 1, \dots, n.$$

(Simply define \tilde{s} on Δ as $\tilde{s} = g\vartheta$ where $g = (g_1, \dots, g_n) : \Delta \rightarrow A$ is a holomorphic map with $\min\{|g_k(p)| : p \in \Delta\} > \max\{|(s/\vartheta)(p)| : p \in bL\}$ for $k = 1, \dots, n$; note that Δ is compact, $L \cap \Delta = \emptyset$, and $g = s/\vartheta$ on bL . We can even choose g to be constant on Δ .) Since Δ is simply connected and $\tilde{s} = s$ on L , \tilde{s} and the regular section s of \mathcal{A} on M are homotopic over a neighbourhood of $L \cup \Delta$, so condition (ii) is satisfied by the holomorphic section \tilde{s} of \mathcal{A} on $L \cup \Delta$. In view of Claim 3.7, \tilde{s} can be uniformly approximated on a neighbourhood of the holomorphically convex compact set $L \cup \Delta \subset M$ by a regular section t of \mathcal{A} on M such that $\int_C t = \mathcal{F}(C)$ for every loop C in M and $\int_{C_i} t = \int_{C_i} s$ for $i = 1, \dots, \ell$. Take such a section t so close to \tilde{s} on $L \cup \Delta$ that the holomorphic map $h = (h_1, \dots, h_n) = t/\vartheta : D \setminus \{p_1, \dots, p_m\} \rightarrow A$ satisfies

$$(3.12) \quad \min\left\{|h_k(p)| : p \in \Delta = \bigcup_{i=1}^m \Delta_i\right\} > \max\left\{|h_k(p)| : p \in bL = \bigcup_{i=1}^m bD_i\right\};$$

for $k = 1, \dots, n$; see (3.10) and (3.11). For each $i = 1, \dots, m$ and $k = 1, \dots, n$, since $t = (t_1, \dots, t_n)$ is regular on M and ϑ is holomorphic at p_i , either $h_k = t_k/\vartheta$ has an effective pole at p_i or h_k is holomorphic at p_i . To complete the proof it now suffices to check that h_k has a pole at p_i for all $i = 1, \dots, m$ and $k = 1, \dots, n$. Indeed, since ϑ is holomorphic and nowhere vanishing on $D \supset \{p_1, \dots, p_m\}$, this would imply that $t_k|_{D \setminus \{p_1, \dots, p_m\}} = h_k\vartheta$ has a pole at p_i , $i = 1, \dots, m$, and so t_k would have a pole at each end p_i of M as well. So fix $i \in \{1, \dots, m\}$ and $k = 1, \dots, n$, and suppose that h_k is holomorphic at p_i . Then h_k is holomorphic on the smoothly bounded closed disc D_i , violating the maximum modulus principle, since $\Delta_i \subset \mathring{D}_i$ and $\min\{|h_k(p)| : p \in \Delta_i\} > \max\{|h_k(p)| : p \in bD_i\}$ by (3.12). Therefore, each h_k has an effective pole at each end p_i of M . \square

Here is a more precise version of Theorem 1.10.

Theorem 3.8. *Let A , M , and \mathcal{A} be as in Theorem 1.1 and assume that t_0 is a regular section of \mathcal{A} on M . Then for any group homomorphism $\mathcal{F} : H_1(M, \mathbb{Z}) \rightarrow \mathbb{C}^n$ there is a regular section $t = (t_1, \dots, t_n)$ of \mathcal{A} on M , homotopic to t_0 , such that all the components 1-forms t_1, \dots, t_n have an effective pole at each end of M and $\int_C t = \mathcal{F}(C)$ for every loop C in M .*

Proof. Let θ be a holomorphic 1-form vanishing nowhere on M and consider the holomorphic map $f_0 = t_0/\theta : M \rightarrow A$. Recall that $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^r$ for some integer $r \geq 0$. Fix a point $p_0 \in M$ and let C_1, \dots, C_r be a family of smooth oriented

embedded Jordan curves forming a basis of $H_1(M, \mathbb{Z})$ such that $C_i \cap C_j = \{p_0\}$ for all $i \neq j$ and the compact set $\Gamma = C_1 \cup \dots \cup C_r$ is holomorphically convex in M , and hence a strong deformation retract of M . By Lemma 3.4, we can deform $f_0|_\Gamma$ outside a neighbourhood of p_0 to a continuous map $f : \Gamma \rightarrow A$ with $\int_{C_i} f\theta = \mathcal{F}(C_i)$ for $i = 1, \dots, r$. Then Lemma 3.3 enables us to approximate f uniformly on Γ by a holomorphic map $f' : M \rightarrow A$ with $\int_C f'\theta = \mathcal{F}(C)$ for every loop C in M . If the approximation is close enough, then f' is homotopic to f on Γ , so f' and f_0 are homotopic on M . Set $s = f'\theta$, which is a holomorphic section of \mathcal{A} on M , homotopic to the given regular section $t_0 = f_0\theta$. Let $K \subset M$ be a smoothly bounded compact domain that is a strong deformation retract of M . By Theorem 3.1, s can be approximated uniformly on K by a regular section $t = (t_1, \dots, t_n)$ of \mathcal{A} on M , such that t_k has an effective pole at each end of M for all $k = 1, \dots, n$, and $\int_C t = \mathcal{F}(C)$ for every loop C in M . If the approximation on K is close enough, then t is homotopic to s , hence to t_0 . \square

4. Regular directed immersions

In this section we first derive Theorem 1.1 on regular immersions directed by algebraically elliptic cones from Theorem 3.1. Then we prove Corollaries 1.7 and 1.8.

Proof of Theorem 1.1. By Corollary 2.2, we may assume that $n \geq 3$ and $A \neq \mathbb{C}_*^n$. We proceed in three steps.

(ii) \Rightarrow (i) and (iv) \Rightarrow (iii) are obvious.

(i) \Rightarrow (iii) and (ii) \Rightarrow (iv): If u is near a regular A -immersion $v : M \rightarrow \mathbb{C}^n$ on a neighbourhood of K , then du is near dv on a smaller neighbourhood U of K . If du and dv are sufficiently close on U , then they are homotopic through continuous sections of $\mathcal{A}|_U$. Clearly, dv is exact, and since v is a regular A -immersion, dv is a regular section of \mathcal{A} . If in addition v is proper, then v , and hence dv , has an effective pole at each end of M .

(iii) \Rightarrow (ii): Assume without loss of generality that K is a smoothly bounded compact domain, possibly disconnected, and let $\Lambda = \{q_1, \dots, q_\ell\} \subset K$ be a finite set. We may assume that $\Lambda \subset \mathring{K}$.

Set $s = du$, so s is an exact holomorphic section of \mathcal{A} on a neighbourhood of K :

$$(4.1) \quad \int_C s = 0 \quad \text{for every loop } C \text{ in a neighbourhood of } K.$$

Assume first that K is connected. Fix a point $p_0 \in \mathring{K} \setminus \Lambda$. It is clear that

$$(4.2) \quad u(p) = u(p_0) + \int_{p_0}^p du \quad \text{for any point } p \in K,$$

where the integral is computed over any oriented arc in K connecting p_0 and p . Let C_1, \dots, C_ℓ be smooth oriented embedded arcs in \mathring{K} such that $C_i \cap C_j = \{p_0\}$ for all $i, j \in \{1, \dots, \ell\}$, $i \neq j$, and the endpoints of C_l are p_0 and q_l , $l = 1, \dots, \ell$. By virtue

of (iii), we may use Theorem 3.1 to approximate s uniformly on a neighbourhood of K by a regular section $t = (t_1, \dots, t_n)$ of \mathcal{A} on M , such that t_k has an effective pole at each end of M for all $k = 1, \dots, n$,

$$(4.3) \quad \int_C t = 0 \quad \text{for every loop } C \text{ in } M$$

(see (4.1) and take $\mathcal{F} = 0$ everywhere on $H_1(M, \mathbb{Z})$) and

$$(4.4) \quad \int_{C_l} t = \int_{C_l} s \quad \text{for } l = 1, \dots, \ell.$$

By (4.3), t is exact. It is also regular, and hence the map $v = (v_1, \dots, v_n) : M \rightarrow \mathbb{C}^n$ given by

$$v(p) = u(p_0) + \int_{p_0}^p t, \quad p \in M,$$

is a regular A -immersion. It is clear that v is close to u uniformly on K , since K is connected and t is close to $s = du$ uniformly on K ; see (4.2). Moreover, since t_k has an effective pole at each end of M , so does v_k , so $v_k : M \rightarrow \mathbb{C}$ is proper for all $k = 1, \dots, n$. In particular, $v : M \rightarrow \mathbb{C}^n$ is proper as well. Finally, by (4.4), $v = u$ on Λ . This proves (ii) in the case when K is connected.

Assume now that the compact domain K is disconnected. Denote by K_0, \dots, K_N , $N \geq 1$, the connected components of K . As in the first part of the proof of Proposition 3.5, attach to K the union $E = \gamma_1 \cup \dots \cup \gamma_N$ of N mutually disjoint smooth embedded Jordan arcs in M , such that the arc γ_i has an endpoint in bK_0 and the other endpoint in bK_i , and is otherwise disjoint from K , $i = 1, \dots, N$, in order to construct a connected holomorphically convex compact admissible set $S = K \cup E \subset M$. Since this procedure does not create new loops, the inclusion $K \hookrightarrow S$ induces an isomorphism $H_1(K, \mathbb{Z}) \rightarrow H_1(S, \mathbb{Z})$. In this case, condition (4.1) implies that (4.2) holds for any pair of points p and p_0 in the same connected component of K . In view of (iii), there is a regular section t of \mathcal{A} on M which is homotopic to s over a neighbourhood of K . Fix a holomorphic 1-form θ vanishing nowhere on M and consider the holomorphic maps

$$(4.5) \quad f = s/\theta : K \rightarrow A \quad \text{and} \quad g = t/\theta : M \rightarrow A$$

which are homotopic over a neighbourhood of K . Extend f to a continuous map $f : S \rightarrow A$ which is holomorphic on a neighbourhood of K such that f and g are homotopic on S . Fix orientations for the arcs $\gamma_1, \dots, \gamma_N$ and choose points $\mu_1, \dots, \mu_N \in \mathbb{C}^n$ which will be specified later. Reasoning as in the final part of the proof of Proposition 3.5, we can now use Lemma 3.4 in order to deform $f : S \rightarrow A$ to a continuous map $f' : S \rightarrow A$ such that $f' = f$ on a neighbourhood of K and

$$(4.6) \quad \int_{\gamma_i} f' \theta = \mu_i \quad \text{for } i = 1, \dots, N.$$

Then f' and g are homotopic over S . Moreover, the continuous 1-form $f' \theta$ on S is exact, since so is $f \theta = s = du$ on K (see (4.1)), $f' = f$ on a neighbourhood of K , and there is no closed curve in S whose homology class belongs to $H_1(S, \mathbb{Z})$ but not

to $H_1(K, \mathbb{Z})$. Thus, fixing a point $p_0 \in \overset{\circ}{K}_0 \setminus \Lambda$, $f'\theta$ integrates to a continuous map $u' : S \rightarrow \mathbb{C}^n$ given by

$$(4.7) \quad u'(p) = u(p_0) + \int_{p_0}^p f'\theta, \quad p \in S.$$

By (4.2), (4.7), and since $f' = f$ on K , we have that $u' = u$ in a neighbourhood of K_0 , while $u' - u$ is constant on a neighbourhood of K_i for $i = 1, \dots, N$. In fact, suitably choosing the constants μ_i in (4.6), we can ensure that

$$(4.8) \quad u' = u \text{ on a neighbourhood of } K = K_0 \cup K_1 \cup \dots \cup K_N.$$

Choose smooth oriented embedded Jordan arcs C_1, \dots, C_ℓ in S such that $C_1 \cup \dots \cup C_\ell$ is holomorphically convex in M and each arc C_l has initial point p_0 and final point $q_l \in \Lambda$, $l = 1, \dots, \ell$. It follows from (4.7) and (4.8) that

$$(4.9) \quad \int_{C_l} f'\theta = u(q_l) - u(p_0) \quad \text{for } l = 1, \dots, \ell.$$

Set $\ell' = \ell + \dim(H_1(S, \mathbb{Z}))$. By [7, proof of Proposition 3.3.2], there are smooth oriented embedded Jordan arcs $C_{\ell+1}, \dots, C_{\ell'}$ in S forming a homology basis of S such that $\bigcup_{l=1}^{\ell'} C_l$ is holomorphically convex in M and each arc C_l , $l = 1, \dots, \ell'$, contains a nontrivial arc disjoint from $\bigcup_{i \neq l} C_i$. By Lemma 3.3, we can approximate f' uniformly on S by a holomorphic map $F : M \rightarrow A$ such that

$$(4.10) \quad \int_{C_l} F\theta = \int_{C_l} f'\theta \quad \text{for } l = 1, \dots, \ell'.$$

Since $f'\theta$ is exact on S and $C_{\ell+1}, \dots, C_{\ell'}$ generate $H_1(S, \mathbb{Z})$, it follows that $F\theta$ is exact on a connected neighbourhood U of S , and hence it integrates to an A -immersion $v : U \rightarrow \mathbb{C}^n$ given by

$$(4.11) \quad v(p) = u(p_0) + \int_{p_0}^p F\theta, \quad p \in U.$$

Since F is close to f' on the connected compact set S , (4.7) and (4.11) ensure that v is close to u' on a neighbourhood of S , and hence to u on a neighbourhood of K ; see (4.8). Moreover, by (4.9) and (4.10), $v = u$ on Λ . Finally, if the approximation of f' by F is close enough, then F is homotopic to f' , and hence to g , over S . Thus $F\theta$ is homotopic to the regular section $t = g\theta$ of \mathcal{A} on M over a neighbourhood of S ; see (4.5). This reduces the proof to the already settled case when K is connected. \square

Proof of Corollary 1.7. The implications (ii) \Rightarrow (iv) \Rightarrow (i) \Rightarrow (iii) are obvious. To show that (iii) implies (ii), we simply apply Theorem 1.1 to any holomorphic A -immersion $u : K \rightarrow \mathbb{C}^n$, where $K \subset M$ is a simply connected, smoothly bounded compact neighbourhood of Λ , such that u extends the given map $\Lambda \rightarrow \mathbb{C}^n$. Such an A -immersion trivially exists; note that a translate of an A -immersion is still an A -immersion. Since K is simply connected, there is only one homotopy class of continuous sections of $\mathcal{A}|_U$ on a suitable neighbourhood of K , hence du is homotopic to the restriction of the regular section of \mathcal{A} on M given by (iii) and the theorem applies. \square

Proof of Corollary 1.8. (ii) \Rightarrow (i) and (iv) \Rightarrow (iii) are obvious, while (i) \Rightarrow (iii) and (ii) \Rightarrow (iv) are seen as in the proof of Corollary 2.2.

(iii) \Rightarrow (ii): Let $u : K \rightarrow \mathbb{C}^n$ be a holomorphic A -immersion as in (i). By [1, Theorem 1.3], u can be uniformly approximated on K by holomorphic A -immersions $v : M \rightarrow \mathbb{C}^n$ agreeing with u on any given finite subset of K . Assuming (iii), apply Theorem 1.1 to approximate and interpolate v and hence u on K to obtain (ii). \square

Remark 4.1. An inspection of the proof of Theorem 1.1 shows, assuming the equivalent conditions hold, that the approximation in (ii) can be done by A -immersions $M \rightarrow \mathbb{C}^n$ all of whose component functions are proper regular functions on M . Likewise, the exact regular section of \mathcal{A} on M in (iv) can be chosen so that all its component 1-forms have an effective pole at each end of M . The same applies to conditions (ii) and (iv) in Corollaries 1.7 and 1.8.

5. Directed harmonic maps

Our method of proof yields a result analogous to Theorem 1.1 for directed harmonic maps into \mathbb{R}^n . If M is an open Riemann surface and A and \mathcal{A} are as in Theorem 1.1, a harmonic map $\phi : M \rightarrow \mathbb{R}^n$ is said to be *directed by A* , or to be an *A -map*, if the $(1,0)$ -part $\partial\phi$ of its exterior derivative $d\phi$ is a holomorphic section of \mathcal{A} on M . If M is a smooth affine curve, we say that ϕ is regular if $\partial\phi$ is. The *flux map* (or just the *flux*) of ϕ is the group homomorphism $\text{Flux}_\phi : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}^n$ defined by

$$\text{Flux}_\phi(C) = \int_C d^c\phi \quad \text{for every loop } C \text{ in } M,$$

where $d^c\phi = i(\bar{\partial}\phi - \partial\phi)$ is the conjugate differential of ϕ . A harmonic A -map $\phi : M \rightarrow \mathbb{R}^n$ is the real part of a holomorphic A -immersion $M \rightarrow \mathbb{C}^n$ if and only if $\text{Flux}_\phi(C) = 0$ for every loop C in M or, equivalently, $d^c\phi$ is exact.

We record the following third main result of this paper. The proof is very similar to that of Theorem 1.1 and we leave the details to the reader.

Theorem 5.1. *Let A , M , and \mathcal{A} be as in Theorem 1.1. Let K be a holomorphically convex compact subset of M , $\phi : K \rightarrow \mathbb{R}^n$ be a harmonic A -map on a neighbourhood of K , and $\mathcal{F} : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}^n$ be a group homomorphism with $\mathcal{F}(C) = \text{Flux}_\phi(C)$ for every loop C in a neighbourhood of K . Then the following are equivalent.*

- (i) ϕ can be uniformly approximated on a neighbourhood of K by regular harmonic A -maps $M \rightarrow \mathbb{R}^n$.
- (ii) ϕ can be uniformly approximated on a neighbourhood of K by regular harmonic A -maps $\psi : M \rightarrow \mathbb{R}^n$ agreeing with ϕ on any given finite set in K and with flux map \mathcal{F} , such that $\partial\psi$ has an effective pole at each end of M .
- (iii) There is a neighbourhood U of K such that the homotopy class of continuous sections of $\mathcal{A}|_U$ that contains $\partial\phi$ also contains the restriction of a regular section of \mathcal{A} on M .

- (iv) *There is a neighbourhood U of K such that the homotopy class of continuous sections of $\mathcal{A}|_U$ that contains $\partial\phi$ also contains the restriction of a regular section t of \mathcal{A} on M with an effective pole at each end of M and representing any given class in the cohomology group $H^1(M, \mathbb{C}^n)$.*

Note that (iii) \Leftrightarrow (iv) is guaranteed by Theorem 1.10. As in Remark 4.1, the sections $\partial\psi$ in (ii) and t in (iv) can be chosen with all their component 1-forms having an effective pole at each end of M .

In general, harmonic A -maps need not be immersions; by a harmonic A -immersion we mean a harmonic A -map which is an immersion. Given a harmonic map $\phi = (\phi_1, \dots, \phi_n) : M \rightarrow \mathbb{R}^n$ on an open Riemann surface M , the holomorphic 2-form $\mathfrak{H}_\phi = \sum_{i=1}^n (\partial\phi_i)^2$ on M is called the *Hopf differential* of ϕ . It turns out that ϕ is an immersion if and only if $|\mathfrak{H}_\phi| < |\partial\phi|^2 = \sum_{i=1}^n |\partial\phi_i|^2$ everywhere on M ; see [9, Lemma 2.4]. This happens if and only if $\partial\phi/\theta$ takes its values in $\mathbb{C}^n \setminus (\mathbb{C} \cdot \mathbb{R}^n)$ for any nowhere-vanishing holomorphic 1-form θ on M . In particular, if A and \mathcal{A} are as in Theorem 5.1 with $A \cap \mathbb{R}^n = \emptyset$ (hence $n \geq 3$) and $\partial\phi$ is a section of \mathcal{A} , then ϕ is a harmonic A -immersion. So, the following holds.

Remark 5.2. Assuming in Theorem 5.1 that $A \cap \mathbb{R}^n = \emptyset$, we have that the given map ϕ is a harmonic A -immersion. If, in addition, the equivalent conditions (i)–(iv) hold, then the approximating maps ψ in (ii) are harmonic A -immersions as well.

The punctured null quadric $A \subset \mathbb{C}_*^n$ in (2.3), directing holomorphic null curves in \mathbb{C}^n and minimal surfaces in \mathbb{R}^n , is very good by Proposition 2.3. It also satisfies $A \cap \mathbb{R}^n = \emptyset$. Thus, by a direct application of Theorem 5.1, we recover the following result from [10] (see [31, 3] for the case $n = 3$); see Section 2.2 for background.

Corollary 5.3. *Let M be a smooth affine curve, $K \subset M$ be a holomorphically convex compact set, $\phi : K \rightarrow \mathbb{R}^n$, $n \geq 3$, be a conformal minimal immersion on a neighbourhood of K , and $\mathcal{F} : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}^n$ be a group homomorphism such that $\mathcal{F}(C) = \text{Flux}_\phi(C)$ for every loop C in a neighbourhood of K . Then ϕ can be uniformly approximated on a neighbourhood of K by complete conformal minimal immersions $\psi : M \rightarrow \mathbb{R}^n$ of finite total curvature agreeing with ϕ on any given finite set in K and satisfying $\text{Flux}_\psi = \mathcal{F}$.*

As above, the approximating conformal minimal immersions $\psi = (\psi_1, \dots, \psi_n) : M \rightarrow \mathbb{R}^n$ in the corollary can be chosen such that $\partial\psi_k$ has an effective pole at each end of M for all $k = 1, \dots, n$.

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