SUFFICIENT CONDITIONS FOR HOLOMORPHIC LINEARISATION

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Abstract. Let G be a reductive complex Lie group acting holomorphically on $X = \mathbb{C}^n$. The (holomorphic) Linearisation Problem asks if there is a holomorphic change of coordinates on $\mathbb{C}^n$ such that the G-action becomes linear. Equivalently, is there a G-equivariant biholomorphism $\Phi\colon X \to V$ where V is a G-module? There is an intrinsic stratification of the categorical quotient Q_X, called the Luna stratification, where the strata are labeled by isomorphism classes of representations of reductive subgroups of G. Suppose that there is a Φ as above. Then Φ induces a biholomorphism $\varphi\colon Q_X \to Q_V$ which is stratified, i.e., the stratum of Q_X with a given label is sent isomorphically to the stratum of Q_V with the same label.

The counterexamples to the Linearisation Problem construct an action of G such that Q_X is not stratified biholomorphic to any Q_V. Our main theorem shows that, for most X, a stratified biholomorphism of Q_X to some Q_V is *sufficient* for linearisation. In fact, we do not have to assume that X is biholomorphic to $\mathbb{C}^n$, only that X is a Stein manifold.

Contents

1. Introduction

The problem of linearising the action of a reductive group G on $\mathbb{C}^n$ has attracted much attention both in the algebraic and holomorphic settings ([Huc90],[KR04],[Kra96]).

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First there was work in the algebraic category. If X is an affine G-variety, then the quotient is the affine variety Q_X with coordinate ring $\mathscr{O}_{\mathrm{alg}}(X)^G$. An early high point is a consequence of Luna's slice theorem [Lun73]. Suppose that Q_X is a point and that X is contractible. Then X is algebraically G-isomorphic to a G-module. The structure theorem for the group of algebraic automorphisms of $\mathbb{C}^2$ shows that any action on $\mathbb{C}^2$ is linearisable [Kra96, Section 5]. As a consequence of a long series of results by many people, it was finally shown in [KR14] that an effective action of a positive dimensional G on $\mathbb{C}^3$ is linearisable. The case of finite groups acting on $\mathbb{C}^3$ remains open.

The first counterexamples to the algebraic linearisation problem were constructed by Schwarz [Sch89] for $n \geq 4$. His examples came from negative solutions to the equivariant Serre problem, i.e., there are algebraic G-vector bundles with base a G-module which are not isomorphic to the trivial ones (those of the form $\mathrm{pr}\colon W \times W' \to W$ where G acts diagonally on the G-modules W and W'). It is interesting to note that in these counterexamples, the nonlinearisable actions may have the same stratified quotient as a G-module.

By the equivariant Oka principle of Heinzner and Kutzschebauch [HK95], all holomorphic G-vector bundles over a G-module are trivial. Thus the algebraic counterexamples to linearisation are not counterexamples in the holomorphic category. But Derksen and Kutzschebauch [DK98] showed that for G nontrivial, there is an $N_G \in \mathbb{N}$ such that there are nonlinearisable holomorphic actions of G on $\mathbb{C}^n$, for every $n \geq N_G$.

Consider Stein G-manifolds X and Y. There are the categorical quotients Q_X and Q_Y. We have the Luna stratifications of Q_X and Q_Y labeled by isomorphism classes of representations of reductive subgroups of G (see Section 2). We say that a biholomorphism $\varphi\colon Q_X \to Q_Y$ is *stratified* if it sends the Luna stratum of Q_X with a given label to the Luna stratum of Q_Y with the same label. If $\Phi\colon X \to Y$ is a G-biholomorphism, then the induced mapping $\varphi\colon Q_X \to Q_Y$ is stratified. The counterexamples of Derksen and Kutzschebauch are actions on $\mathbb{C}^n$ whose quotients are not isomorphic, via a stratified biholomorphism, to the quotient of a linear action. We will show that, under a mild assumption, this is the only way to get a counterexample to linearisation.

Suppose that we have a stratified biholomorphism $\varphi\colon Q_X \to Q_V$ where V is a G-module. Then we may identify Q_X and Q_V and call the common quotient Q. We have quotient mappings $p\colon X \to Q$ and $r\colon V \to Q$. Assume there is an open cover $\{U_i\}_{i\in I}$ of Q and G-equivariant biholomorphisms $\Phi_i : p^{-1}(U_i) \to r^{-1}(U_i)$ over U_i (meaning that Φ_i descends to the identity map of U_i). We express the assumption by saying that X and V are *locally G-biholomorphic over a common quotient*. Equivalently, our original $\varphi\colon Q_X \to Q_V$ locally lifts to G-biholomorphisms of X to V.

Our first main result is the following.

Theorem 1.1. *Suppose that X is a Stein G-manifold, V is a G-module and X and V are locally G-biholomorphic over a common quotient. Then X and V are G-biholomorphic.*

Remark 1.2. Assume that X and V are locally G-biholomorphic over a common quotient Q and let $\{U_i\}$ and Φ_i be as above. The maps $\Phi_j \circ {\Phi_i}^{-1}$ give us an element of $H^1(Q, \mathcal{F})$, where for U open in Q, $\mathcal{F}(U)$ is the group of G-biholomorphisms of $r^{-1}(U)$ which induce the identity on U. Theorem 1.1 says that the cohomology class associated to X is trivial. On the other hand, given any element of $H^1(Q, \mathcal{F})$ one constructs a corresponding Stein G-manifold X [KLS, Theorem 5.11]. Hence the theorem is equivalent to the statement that $H^1(Q, \mathcal{F})$ is trivial.

We now consider two conditions on X. Assume that the set of closed orbits with trivial isotropy group is open in X and that the complement, a closed subvariety of X, has complex codimension at least two. We say that X is *generic*. Let $X_{(n)}$ denote the subset of X whose isotropy groups have dimension n. Following the terminology of [Sch95] we say that X is 2-*large* (or just *large*) if X is generic and $\operatorname{codim} X_{(n)} \geq n+2$ for $n \geq 1$. For other conditions equivalent to 2-largeness see [Sch95, Section 9]. The conditions "generic" and "large" hold for "most" X (see Remark 2.1 below).

Remark 1.3. In [KLS15, Corollary 14] we established Theorem 1.1 with the extra hypothesis that X (equivalently V) is generic. Removing the hypothesis is technically very difficult and requires the techniques of [KLS].

Our second main result is the following.

Theorem 1.4. *Suppose that X is a Stein G-manifold and V is a G-module satisfying the following conditions.*

(1) *There is a stratified biholomorphism φ from Q_X to Q_V.*
(2) *V (equivalently, X) is large.*

Then, by perhaps changing φ, one can arrange that X and V are locally G-biholomorphic over $Q_X \simeq Q_V$, hence X and V are G-biholomorphic.

The proofs of our main theorems use the flows of the Euler vector field on V and an analogous vector field on X. They give us smooth deformation retractions of $Q_X \simeq Q_V$ to a point which are covered by G-equivariant retractions of X and V to fixed points. The difficult part of the proof of Theorem 1.4 is to construct local G-biholomorphic lifts of φ. Then we can reduce to Theorem 1.1.

One can ask if assumption (2) of Theorem 1.4 can be removed. In Section 5 we show it can if $\dim Q \leq 1$ or $G = \mathrm{SL}_2(\mathbb{C})$. We would be surprised if there is a counterexample to Theorem 1.4 with (2) omitted.

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2. Background

We start with some background. For more information, see [Lun73] and [Sno82, Section 6]. Let X be a normal Stein space with a holomorphic action of a reductive complex Lie group G. The categorical quotient $Q_X = X/\!\!/G$ of X by the action of G is the set of closed orbits in X with a reduced Stein structure that makes the quotient map $p\colon X \to Q_X$ the universal G-invariant holomorphic map from X to a Stein space. When X is understood, we drop the subscript X in Q_X. Since X is normal, $Q = Q_X$ is normal. If U is an open subset of Q, then $\mathscr{O}_X(p^{-1}(U))^G \simeq \mathscr{O}_Q(U)$. We say that a subset of X is *G-saturated* if it is a union of fibres of p. If X is a G-module, then Q is just the complex space corresponding to the affine algebraic variety with coordinate ring $\mathscr{O}_{\mathrm{alg}}(X)^G$.

If Gx is a closed orbit, then the stabiliser (or isotropy group) G_x is reductive. We say that closed orbits Gx and Gy have the same *isotropy type* if G_x is G-conjugate to G_y. Thus we get the *isotropy type stratification* of Q with strata whose labels are conjugacy classes of reductive subgroups of G.

Assume that X is smooth and let Gx be a closed orbit. Then we can consider the *slice representation* which is the action of G_x on $T_xX/T_x(Gx)$. We say that closed orbits Gx and Gy have the same *slice type* if they have the same isotropy type and, after arranging that $G_x = G_y$, the slice representations are isomorphic representations of G_x. The slice type (Luna) strata are locally closed smooth subvarieties of Q. The Luna stratification is finer than the isotropy type stratification, but the Luna strata are unions of connected components of the isotropy type strata [Sch80, Proposition 1.2]. Hence if the isotropy strata are connected, the Luna strata and isotropy type strata are the same. This occurs for the case of a G-module [Sch80, Lemma 5.5]. Alternatively, one can show directly that in a G-module, the isotropy group of a closed orbit determines the slice representation (see [Sch80, proof of Proposition 1.2]).

There is a unique open stratum $Q_{\mathrm{pr}} \subset Q$, corresponding to the closed orbits with minimal stabiliser. We call this the *principal stratum* and the closed orbits above Q_{pr} are called *principal orbits*. The isotropy groups of principal orbits are called *principal isotropy groups*. By definition, X is generic when the principal isotropy groups are trivial and the closed subvariety $p^{-1}(Q \setminus Q_{\mathrm{pr}})$ has codimension at least 2 in X. Recall that X is large if it is generic and $\operatorname{codim} X_{(n)} \geq n+2$ for $n \geq 1$.

Remark 2.1. If G is simple, then, up to isomorphism, all but finitely many G-modules V with $V^G = 0$ are large [Sch95, Corollary 11.6 (1)]. The same result holds for semisimple groups but one needs to assume that every irreducible component of V is a faithful module for the Lie algebra of G [Sch95, Corollary 11.6 (2)]. A "random" $\mathbb{C}^*$-module is large, although infinite families of counterexamples exist. More precisely, a faithful n-dimensional $\mathbb{C}^*$-module without zero weights is large if and only if it has at least two positive weights and at least two negative weights and any $n-1$ weights have no common prime factor. Finally, X is large if and only if every slice representation is large and the property of being large only depends upon the Luna stratification of Q.

3. Proof of Theorem 1.1

We will need to show that some vector fields on X are *complete*, i.e., can be integrated for all real time t.

Lemma 3.1. *Let A be a G-invariant holomorphic vector field on X and let F be a fibre of $p\colon X \to Q$. Then there is a G-saturated neighbourhood U of F and an $\epsilon > 0$ such that the local 1-parameter group ψ_t of A exists on $(-\epsilon, \epsilon) \times U$.*

Proof. Let K be a maximal compact subgroup of G. By [HK95, Section 5 Corollary 1] there is a K-orbit O in F and a neighbourhood basis $\{U_i\}_{i\in\mathbb{N}}$ of O consisting of K-stable open sets with the following properties.

(1) For all i, $G \cdot U_i$ is G-saturated.
(2) For all i, any K-equivariant holomorphic map from U_i to X has a unique extension to a G-equivariant holomorphic map of $G \cdot U_i$ to X.

Since O is compact, there is an $\epsilon > 0$ and a neighbourhood U of O such that ψ_t is defined on $(-\epsilon, \epsilon) \times U$. Since A is K-invariant and holomorphic, so is each ψ_t. We may assume that U is one of the U_i. Then each ψ_t extends uniquely to a G-equivariant holomorphic map $\widetilde{\psi}_t$ of $G \cdot U$ to X and the $\widetilde{\psi}_t$ are easily seen to be a local 1-parameter group corresponding to A. □

Definition 3.2. Let B be a holomorphic vector field on Q (derivation of $\mathscr{O}_Q$). We say that a G-equivariant holomorphic vector field A on X is a *lift of* B if $A(p^*f) = p^*B(f)$ for every $f \in \mathscr{O}(Q)$.

We leave the proof of the following to the reader.

Corollary 3.3. *Let A be a G-invariant vector field on X which is a lift of the holomorphic vector field B on Q. Then A is complete if and only if B is complete.*

Recall that we are assuming that the Stein G-manifold X and the G-module V are locally G-biholomorphic over their common quotient Q.

The scalar action of $\mathbb{C}^*$ on V descends to a $\mathbb{C}^*$-action on Q (see below), in particular, we have an action of $\mathbb{R}^{>0} = \{u \in \mathbb{R} \mid u > 0\}$ on Q. We will now show how to lift the $\mathbb{R}^{>0}$-action to X. Let $\mathscr{A}_Q$ denote the sheaf of holomorphic vector fields on Q and let $\mathscr{A}(Q)$ denote the global sections. Similarly we have the sheaf of holomorphic vector fields $\mathscr{A}_X$ on X. Let U be open in Q and let $\mathscr{A}_X^G(U)$ denote $\mathscr{A}_X(p^{-1}(U))^G$. Then $\mathscr{A}_X^G$ is a coherent sheaf of $\mathscr{O}_Q$-modules [Rob86] as is $\mathscr{A}_Q$. We have $p_*\colon \mathscr{A}_X(p^{-1}(U))^G \to \mathscr{A}_Q(U)$ where $p_*(A)(f) = A(p^*(f))$ for $f \in \mathscr{O}_Q(U) \simeq \mathscr{O}_X(p^{-1}(U))^G$. Then $p_*\colon \mathscr{A}_X^G \to \mathscr{A}_Q$ is a morphism of coherent sheaves of $\mathscr{O}_Q$-modules. Hence the kernel $\mathcal{M}$ of p_* is coherent.

Let $r_1, \dots, r_m$ be homogeneous generators of $\mathscr{O}_{\mathrm{alg}}(V)^G$ where r_i has degree d_i. Then $(r_1, \dots, r_m)\colon V \to \mathbb{C}^m$ induces a map $f\colon Q \to \mathbb{C}^m$ which is an algebraic isomorphism of Q onto the image of f. Hence we can think of the quotient map $r\colon V \to Q$ as the polynomial map with entries r_i. Let $t \in \mathbb{C}^*$ and $q = (q_1, \dots, q_m) \in Q$. Define $t \cdot q = (t^{d_1} q_1, \dots, t^{d_m} q_m)$. This is a $\mathbb{C}^*$-action on Q and $r(tv) = t \cdot r(v)$, $t \in \mathbb{C}^*$, $v \in V$. We have the Euler vector field $E = \sum_i x_i \partial/\partial x_i$ on V, where the x_i are the coordinate functions on V. Let $y_1, \dots, y_m$ be the usual coordinate functions on $\mathbb{C}^m$. Then $r_*(E) = \sum d_i y_i \partial/\partial y_i \in \mathscr{A}(Q)$.

Lemma 3.4. *Let $B = r_*E$. Then B lifts to a G-invariant holomorphic vector field A on X.*

Proof. Let U be open and G-saturated in V and let $\Phi\colon U \to \Phi(U) \subset X$ be a G-biholomorphism inducing the identity on $r(U)$. Let A_U denote the image of $E|_U$ in $\mathscr{A}_X(\Phi(U))^G$ under the action of Φ. Then A_U is a lift of $B|_{r(U)}$. The various A_U differ by elements in the kernel $\mathcal{M}$ of p_*, hence a global lift of B is obstructed by an element of $H^1(Q, \mathcal{M})$, which vanishes by Cartan's Theorem B. Hence A exists. □

Choose a lift A of B and let ψ_t denote the flow of A on X. From Corollary 3.3 we have:

Lemma 3.5. *The flow ψ_t exists for all $t \in \mathbb{R}$.*

We have an $\mathbb{R}^{>0}$-action on X where $t \cdot x = \psi_{\log t}(x)$, $t \in \mathbb{R}^{>0}$, $x \in X$, and it is the promised lift of the $\mathbb{R}^{>0}$-action on Q. We now need to find retractions of Q, X and V. Choose positive integers e_i such that $d_i e_i = d$ is independent of $i = 1, \ldots, m$. For $q = (q_1, \ldots, q_m) \in Q$ let $\rho(q) = \sum_i |q_i|^{2e_i}$. Choose $u \in \mathbb{R}^{>0}$ and set $Q_u = \{q \in Q \mid \rho(q) < u\}$. Let $h\colon [0, \infty) \to [0, u)$ be a diffeomorphism which is the identity in a neighbourhood of 0. Set

$$a(q) = \left(\frac{h(\rho(q))}{\rho(q)}\right)^{1/2d} \text{ and } \alpha(q) = a(q) \cdot q,\ q \in Q.$$

Here $a(q) \cdot q$ denotes the $\mathbb{C}^*$-action. Now α is a diffeomorphism of Q with Q_u with inverse

$$\beta(q) = b(q) \cdot q \text{ where } b(q) = \left(\frac{h^{-1}(\rho(q))}{\rho(q)}\right)^{1/2d},\ q \in Q_u.$$

Proof of Theorem 1.1. For $u > 0$, let X_u denote $p^{-1}(Q_u)$ and let V_u denote $r^{-1}(Q_u)$. Choose $u > 0$ so that we have a local G-biholomorphism $\Phi\colon X_u \to V_u$ inducing the identity on Q_u. Let η_t be the flow of the Euler vector field on V. Then we have a G-diffeomorphism σ of V with V_u which sends $v \in V$ to $\eta_{\log a(r(v))}(v)$. Using the flow ψ_t of the vector field A that we constructed on X, we have a G-diffeomorphism τ of X with X_u which sends $x \in X$ to $\psi_{\log a(p(x))}(x)$. By construction, σ and τ map fibres G-biholomorphically to fibres and $\sigma^{-1} \circ \Phi \circ \tau$ is a G-diffeomorphism of X and V inducing the identity map on Q. By [KLS, Theorem 1.1], X and V are G-biholomorphic. □

Remark 3.6. We used the fact that X and V are locally G-biholomorphic over Q to construct our special G-invariant vector field A. But given any lift A of $B = r_* E$, we can construct our G-diffeomorphism which is biholomorphic on fibres, as long as we have a G-biholomorphism of neighbourhoods of $p^{-1}(r(0))$ and $r^{-1}(r(0))$ inducing the identity on Q.

4. Proof of Theorem 1.4

Assume for now that we have a biholomorphism $\varphi\colon Q_X \to Q_V$ which preserves the Luna stratifications. Note that X^G is smooth and closed in X. We may identify X^G with its image in Q_X and similarly for V^G. Then φ induces a biholomorphism (which we also call φ) from X^G to V^G. We have $V = V^G \oplus V'$ where V' is a G-module. Since $X^G \simeq V^G$ is contractible, the normal bundle $\mathscr{N}(X^G) = (TX|_{X^G})/T(X^G)$ is a trivial G-vector bundle [HK95] and we have an isomorphism $\Phi\colon \mathscr{N}(X^G) \to V^G \times V'$ (viewing the target as the G-vector bundle $V^G \times V' \to V^G$). Since $TX|_{X^G}$ is also G-trivial, we may think of $\mathscr{N}(X^G)$ as a G-subbundle of $TX|_{X^G}$. Note that Φ restricts to φ on the zero section.

The following proposition does not need the hypothesis that X or V is large.

Proposition 4.1. *Let $\varphi\colon X^G \to V^G$ and Φ be as above. Then there is a G-saturated neighbourhood U of X^G in X and a G-saturated neighbourhood U' of V^G in V and a G-biholomorphism $\Psi\colon U \to U'$ whose differential induces Φ on $\mathscr{N}(X^G)$.*

Proof. Let $v_1, \ldots, v_k$ be a basis of V' and let $A_1, \ldots, A_k$ denote the corresponding constant vector fields on V. Let $X_1, \ldots, X_k$ denote their inverse images under Φ. Then the

X_i are holomorphic vector fields defined on X^G and they extend to global vector fields on X, which we also denote as X_i. Let $\rho_t^{(i)}$ denote the complex flow of X_i, $i = 1, \dots, k$. For $(v, v') \in V^G \oplus V'$, $v' = \sum a_i v_i$, let

$$F(v, v') = \rho_{a_1}^{(1)} \rho_{a_2}^{(2)} \cdots \rho_{a_k}^{(k)} (\varphi^{-1}(v)).$$

Then F is defined and biholomorphic on a neighbourhood of $V^G \times \{0\}$ and the derivative of F along $V^G \times \{0\}$ is Φ^{-1}. The inverse of F gives us a biholomorphism $\Psi \colon U \to U'$ with the following properties:

(1) U is a neighbourhood of X^G in X and U' is a neighbourhood of V^G in V.
(2) Ψ restricts to φ on X^G.
(3) $d\Psi$ restricted to $\mathscr{N}(X^G)$ gives Φ.

Let K be a maximal compact subgroup of G. Averaging Ψ over K gives us a new holomorphic map (also called Ψ) which still satisfies the conditions above, perhaps with respect to smaller neighbourhoods U_0 and U_0'. Shrinking we may assume that U_0 and U_0' are K-stable. It follows from [HK95, Section 5, Lemma 1, Proposition 1 and Corollary 1] that shrinking further we may achieve the following:

(4) The restriction of Ψ to U_0 extends to a G-equivariant holomorphic map on $U = G \cdot U_0$ (which we also call Ψ).
(5) The restriction of Ψ^{-1} to U_0' extends to a G-equivariant holomorphic map Θ on $U' = G \cdot U_0'$.

We can reduce to the case that U and U' are connected. Then it is clear that $\Psi \circ \Theta$ and $\Theta \circ \Psi$ are identity maps. Removing $p^{-1}(p(X \setminus U))$ from U we can arrange that it is G-saturated, and similarly for U'. □

Note that our Ψ only induces φ on X^G. Let ψ denote the biholomorphism of $U/\!\!/G$ and $U'/\!\!/G$ induced by Ψ and let τ denote $\varphi \circ \psi^{-1}$. Then τ is a strata preserving biholomorphism defined on a neighbourhood of V^G in Q_V and τ is the identity on V^G. Our goal is to show that τ has a local G-biholomorphic lift to V if we modify φ without changing its restriction to X^G. Let $x_0 = \varphi^{-1}(0) \in X^G$.

We will need to use some results from [Sch14]. We say that X *is admissible* if X is generic and every holomorphic differential operator on Q lifts to a G-invariant holomorphic differential operator on X. In particular, holomorphic vector fields on Q lift to G-invariant holomorphic vector fields on X. Admissibility is a local condition over Q, hence X is admissible if and only if each slice representation of X is admissible. Since X is large, [Sch95, Theorem 0.4] and [Sch13, Remark 2.4] show that X is admissible. The only role of largeness in this paper is that it implies admissibility.

Let $s_1, \dots, s_n$ denote homogeneous invariant polynomials generating $\mathscr{O}_{\mathrm{alg}}(V')^G$, where δ_i is the degree of s_i, $i = 1, \dots, n$. Let $s = (s_1, \dots, s_n) \colon V' \to \mathbb{C}^n$. We can identify $Q' = V'/\!\!/G$ with the image of s and we can identify Q_V with $V^G \times Q'$. We have an action of $\mathbb{C}^*$ on Q' where $t \in \mathbb{C}^*$ sends $(q_1, \dots, q_n) \in Q'$ to $(t^{\delta_1} q_1, \dots, t^{\delta_n} q_n)$.

Let $\mathrm{Aut}_{\mathrm{q}\ell}(Q')$ denote the *quasilinear* automorphisms of Q', that is, the automorphisms which commute with the $\mathbb{C}^*$-action. An element of $\mathrm{Aut}_{\mathrm{q}\ell}(Q')$ is determined by its (linear) action on the invariant polynomials of degrees δ_i, $i = 1, \dots, n$. Hence

$\mathrm{Aut}_{q\ell}(Q')$ is a linear algebraic group. Let σ be a germ of a strata preserving automorphism of Q' at $0 = s(0)$. Then $\sigma(0) = 0$. Let σ_t denote the germ of an automorphism of Q' which sends q' to $t^{-1} \cdot \sigma(t \cdot q')$, $t \in \mathbb{C}^*$, $q' \in Q'$. It is not automatic that the limit of σ_t exists as $t \to 0$. One needs to have the vanishing of certain terms of the Taylor series of σ (see [Sch14, Section 2]). But this occurs in the case that V is admissible [Sch14, Theorem 2.2]. Moreover, the limit σ_0 lies in $\mathrm{Aut}_{q\ell}(Q')$ and $\sigma_t(q')$ is holomorphic in all $t \in \mathbb{C}$ and $q' \in Q'$ such that $\sigma_t(q')$ is defined. We consider $\mathrm{Aut}_{q\ell}(Q')$ as the subgroup of $\mathrm{Aut}(V^G \times Q')$ whose elements send (v, q') to $(v, \rho(q'))$, $v \in V^G$, $q' \in Q'$, $\rho \in \mathrm{Aut}_{q\ell}(Q')$.

Consider our automorphism τ defined on a neighbourhood of V^G in $Q_V = V^G \times Q'$. Write $\tau = (\tau_1, \tau_2)$, where τ_1 takes values in V^G and τ_2 in Q'. Then $\tau_1(v, 0) = v$ and $\tau_2(v, 0) = 0$ for all $v \in V^G$. It follows from the inverse function theorem that $\tau_2(v, \cdot)$ is a germ of a strata preserving automorphism of Q'. Thus we have a holomorphic family of automorphisms $\tau_2(v, \cdot)_t$ with $\tau_{2,v} := \tau_2(v, \cdot)_0 \in \mathrm{Aut}_{q\ell}(Q')$. Set $\tau_1(v, q)_t = \tau_1(v, t \cdot q)$. Then

$$\tau_t(v, q) = (\tau_1(v, q)_t, \tau_2(v, q)_t)$$

is a homotopy connecting τ with τ_0 where $\tau_0(v, q) = (v, \tau_{2,v}(q))$. The homotopy is holomorphic in all v, q and t such that $(v, t \cdot q)$ lies in the domain of τ. The connected component of $\mathrm{Aut}_{q\ell}(Q')$ containing $\tau_{2,v}$ is independent of v. Let $\rho \in \mathrm{Aut}_{q\ell}(Q')$ denote any of the $\tau_{2,v}$, say $\tau_{2,0}$, which we can consider as an automorphism of Q_V. Change our original φ to $\rho^{-1} \circ \varphi$. This does not change φ restricted to X^G or the biholomorphism Ψ that we constructed. We then find ourselves in the situation where $\tau_{2,v}$ lies in the identity component of $\mathrm{Aut}_{q\ell}(Q')$ for every $v \in V^G$. Since V is admissible, the identity component of $\mathrm{Aut}_{q\ell}(Q')$ is the image of $\mathrm{GL}(V')^G$ [Sch14, Proposition 2.8]. Then there is a neighbourhood of $0 \in V^G$ on which we have a holomorphic lift of the $\tau_{2,v}$ to elements of $\mathrm{GL}(V')^G$. Hence we can reduce to the case that τ_0 is the identity. Now let $B = \{t \in \mathbb{C} : |t| \leq 2\}$. Shrinking our neighbourhood U' (which we now just consider as a neighbourhood of $0 \in V$), we can arrange that τ_t is defined on $\Omega := U' /\!\!/ G$ for $t \in B$. Thus τ_t is a homotopy, $t \in B$, fixing $(V^G \times \{0\}) \cap \Omega$ and starting at the identity. Let

$$\Delta = \{(t, q) \in B \times \Omega \mid q \in \tau_t(\Omega)\}.$$

Then Δ is a neighbourhood of $B \times \{(0, 0)\}$ where $(0, 0)$ is the origin in $V^G \times Q'$. Hence there is a neighbourhood Ω_1 of $(0, 0)$ such that $B \times \Omega_1 \subset \Delta$. We have a (complex) time dependent vector field C_t defined by

$$C_t(\tau_t(q)) = \frac{d\,\tau_s(q)}{ds}\Big|_{t=s}, \ t \in B, \ q \in \Omega,$$

and by definition of Ω_1, $C_t(q)$ is defined for all $t \in B$ and $q \in \Omega_1$. Let

$$\Delta' = \{(t, q) \in [0, 1] \times \Omega_1 \mid \tau_t(q) \in \Omega_1\}.$$

Then, as before, Δ' contains an open set of the form $[0, 1] \times \Omega_2$. On Ω_2, τ_t is obtained by integrating the time dependent vector field C_t where now we only consider $t \in [0, 1]$. Let B^0 denote the interior of B. Since V is admissible, we can lift (time-dependent) holomorphic vector fields on $B^0 \times \Omega_1$ to G-invariant holomorphic vector fields on $B^0 \times p^{-1}(\Omega_1)$. Hence C_t lifts to a G-invariant holomorphic vector field A_t on $B^0 \times p^{-1}(\Omega_1)$. Now we know that integrating C_t on Ω_2 lands us in Ω_1 for $t \in [0, 1]$. As in Lemma 3.5 we can integrate A_t for $t \in [0, 1]$ and $x \in p^{-1}(\Omega_2)$ and we end up in $p^{-1}(\Omega_1)$. We thus have a homotopy whose value Θ at time 1 is a G-biholomorphism of $p^{-1}(\Omega_2)$ which covers $\tau = \varphi \circ \psi^{-1}$. Then $\Theta \circ \Psi$ is a G-biholomorphism inducing φ sending a G-saturated neighbourhood U_X of x_0 onto a G-saturated neighbourhood U_V of $0 \in V$.

Proof of Theorem 1.4. Let E denote the Euler vector field on V. Since X is admissible we can lift the vector field r_*E to a G-invariant vector field A on X. Recall the G-equivariant flows η_t of E on V and ψ_t of A on X. Let X_u and V_u be as before, $u > 0$. Perhaps modifying φ by composition with an element of $\mathrm{Aut}_{q\ell}(Q') \subset \mathrm{Aut}(Q_V)$, we can find a G-biholomorphism $\Phi \colon U_X \to U_V$ inducing φ as above. Then there is a $t \in \mathbb{R}$ such that $\psi_t(X_u) \subset U_X$ and $\eta_t(V_u) \subset U_V$. The composition $\eta_{-t} \circ \Phi \circ \psi_t$ is a G-biholomorphism of X_u with V_u which induces φ. Hence X and V are locally G-biholomorphic over a common quotient and we can apply Theorem 1.1 or [KLS15, Corollary 14]. □

5. Small representations

Suppose that we have a strata preserving biholomorphism $\tau \colon Q_X \to Q_V$ as in Theorem 1.4. We know that X and V are G-equivariantly biholomorphic if V is large. In this section we investigate "small" G-modules V which are not large and see if we can still prove that X and V are G-equivariantly biholomorphic. The proof of Theorem 1.4 goes through if we can establish the following two statements where $V = V^G \oplus V'$ and $Q' = V'/\!\!/G$.

- $(*)$ Let φ be a germ of a strata preserving automorphism near the origin of Q' and let $\varphi_t = t^{-1} \circ \varphi \circ t$. Then $\lim_{t \to 0} \varphi_t$ exists.
- $(**)$ Let B be a holomorphic vector field on Q' which preserves the strata, that is, $B(s) \in T_s(S)$ for every $s \in S$, where S is any stratum of Q'. Then B lifts to a G-invariant holomorphic vector field on V'.

Remark 5.1. Suppose that the minimal homogeneous generators of $\mathscr{O}_{\mathrm{alg}}(V')^G$ have the same degree. Then $\varphi_0 = \varphi'(0)$ exists.

The following theorem is one of the results in [Jia92].

Theorem 5.2. *Suppose that* $\dim Q \leq 1$. *Then* X *and* V *are* G*-biholomorphic.*

Proof. The case $\dim Q = 0$ is an immediate consequence of Luna's slice theorem, so let us assume that $\dim Q = 1$. Then $\mathscr{O}_{\mathrm{alg}}(V)^G$ is normal of dimension one, hence regular, and it is graded. Thus $\mathscr{O}_{\mathrm{alg}}(V)^G = \mathbb{C}[f]$, where f is homogeneous and $Q \simeq \mathbb{C}$. First suppose that Q has one stratum. Then the closed orbits in V are the fixed points and Proposition 4.1 gives the required biholomorphism. The remaining case is where the strata of Q are $\mathbb{C} \setminus \{0\}$ and $\{0\}$. Then $(*)$ follows from Remark 5.1. As for $(**)$, our vector field is of the form $h(z)z\partial/\partial z$ where $h(z)$ is holomorphic. The vector field lifts to an invariant holomorphic function times the Euler vector field on V. □

Theorem 5.3. *Suppose that* $G = \mathrm{SL}_2(\mathbb{C})$. *Then* X *and* V *are* G*-biholomorphic.*

Proof. Let R_d denote the representation of G on $S^d\mathbb{C}^2$. Then the G-modules V' where $(V')^G = 0$ and V' is not large are [Sch95, Theorem 11.9]

(1) kR_1, $1 \leq k \leq 3$.

(2) R_2, $2R_2$, $R_2 \oplus R_1$.
(3) R_3, R_4.

In all cases the quotient is $\mathbb{C}^k$ for some $k \leq 3$. The cases R_1, $2R_1$, R_2, R_3 have quotient of dimension at most 1, hence they present no problem. Suppose that $V' = 3R_1$. Then the generating invariants are determinants of degree 2, so we have $(*)$. Let z_{ij} be the variable on $Q' = \mathbb{C}^3$ corresponding to the ith and jth copy of $\mathbb{C}^2$. Then the strata preserving vector fields are generated by the $z_{ij}\partial/\partial z_{k\ell}$. Thus we have 9 generators. But we have a canonical action of $\mathrm{GL}_3(\mathbb{C})$ on V' commuting with the action of G and the image of $\mathfrak{gl}_3(\mathbb{C})$ is the span of the 9 generators. Hence we have $(**)$. For the case of $2R_2$ we have $(*)$ because the generators are polynomials of degree 2 and we have $(**)$ because $2R_2$ is an orthogonal representation [Sch80, Theorems 3.7 and 6.7].

Suppose that $V' = R_4$. Then V' is orthogonal, so $(**)$ holds. The quotient Q' is isomorphic to the quotient of $\mathbb{C}^2$ by S_3, and it is known that strata preserving automorphisms have local lifts [Lya83], [KLM03, Theorem 5.4], hence we certainly have $(*)$. Finally, there is the case $V' = R_2 \oplus R_1$. Then there are generating invariants homogeneous of degrees 2 and 3 and the zeroes of the degree 3 invariant define the closure of the codimension one stratum. Thus we may think of Q' as $\mathbb{C}^2$ with coordinate functions z_2 and z_3 where z_i has weight i for the action of $\mathbb{C}^*$. A strata preserving φ has to send z_3 to a multiple of z_3 (and fix the origin), so that $\varphi = (\varphi_2, \varphi_3)$ where $\varphi_3(z_2, z_3) = \alpha(z_2, z_3)z_3$. It follows easily that φ_0 exists and we have $(*)$. The strata preserving vector fields must all vanish at the origin and preserve the ideal of z_3, so they are generated by $z_3\partial/\partial z_3$, $z_2\partial/\partial z_2$ and $z_3\partial/\partial z_2$. Since V' is self dual, we can change the differentials of the generators f_2 and f_3 into invariant vector fields A_2 and A_3, and one can see that our three strata preserving vector fields below are in the span of the images of A_2, A_3 and the Euler vector field. Hence we have $(**)$. □

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