

The space of Gauss maps of complete minimal surfaces

Antonio Alarcón and Finnur Lárusson

Abstract The Gauss map of a conformal minimal immersion of an open Riemann surface M into \mathbb{R}^3 is a meromorphic function on M . In this paper, we prove that the Gauss map assignment, taking a full conformal minimal immersion $M \rightarrow \mathbb{R}^3$ to its Gauss map, is a Serre fibration. We then determine the homotopy type of the space of meromorphic functions on M that are the Gauss map of a complete full conformal minimal immersion, and show that it is the same as the homotopy type of the space of all continuous maps from M to the 2-sphere. We obtain analogous results for the generalised Gauss map of conformal minimal immersions $M \rightarrow \mathbb{R}^n$ for arbitrary $n \geq 3$.

Keywords Riemann surface, minimal surface, complete minimal surface, Gauss map, h-principle, Oka manifold, Serre fibration, weak homotopy equivalence

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1. Introduction and main results

The Gauss map of a minimal surface in \mathbb{R}^3 , parametrised as a conformal minimal immersion from an open Riemann surface M into \mathbb{R}^3 , may be viewed as a meromorphic function on M . Bonnet first observed this fact in 1860 [7] and Christoffel proved in 1867 [8] that it characterises minimal surfaces in \mathbb{R}^3 . Via the Gauss map, complex-analytic methods have ever since played a major role in the classical theory of minimal surfaces. The literature is vast. We refer to [18, Chapter 12] and [4, Chapter 5] for historical background and further references.

It is a long-standing unsolved problem in the global theory of minimal surfaces to usefully characterise those meromorphic functions that are the Gauss map of a *complete* minimal surface. Several decades of research on Picard-type theorems for Gauss maps of complete minimal surfaces culminated in the 1988 theorem of Fujimoto that such a map can omit at most four values in the Riemann sphere unless the surface is a plane [13]. This result is sharp. Some further restrictions were given by Osserman [17] and by Weitsman and Xavier [22]. As an example in the other direction, Su and Li recently produced a sufficient Nevanlinna-theoretic condition for a meromorphic function on the plane or the disc to be the Gauss map of a complete minimal surface [20, 21].

In this paper, we take a new approach to the problem. We investigate the space of meromorphic functions on M that are the Gauss map of a complete minimal surface from a homotopy-theoretic viewpoint. We determine the homotopy type of

this space. One of our main results is that the inclusion of this space in the space of all meromorphic functions on M is a weak homotopy equivalence and, when M has finite topological type, even a genuine homotopy equivalence.

It was discovered only recently that every meromorphic function on M is the Gauss map of a conformal minimal immersion $M \rightarrow \mathbb{R}^3$ [3]. We show that the Gauss map assignment is not only surjective: it is in fact a Serre fibration. This is a key ingredient in the proof of the main result described above, along with the strong parametric h-principle for complete minimal surfaces in our previous paper [5].

Our results extend to all higher dimensions. To more precisely present them, we need to introduce some notation. Let M be an open Riemann surface, throughout assumed connected, and let $n \geq 3$. If $u = (u_1, \dots, u_n) : M \rightarrow \mathbb{R}^n$ is a conformal minimal immersion, then the $(1, 0)$ -differential ∂u of u determines the Kodaira-type holomorphic map $\mathcal{G}(u)$ from M into the hyperquadric

$$\mathbf{Q}^{n-2} = \{[z_1 : \dots : z_n] \in \mathbb{C}\mathbb{P}^{n-1} : z_1^2 + \dots + z_n^2 = 0\}$$

in $\mathbb{C}\mathbb{P}^{n-1}$ given by

$$\mathcal{G}(u)(p) = [\partial u_1(p) : \dots : \partial u_n(p)], \quad p \in M.$$

The map $\mathcal{G}(u)$ is called the *generalised Gauss map* of u , or, in this paper, simply the *Gauss map* of u . A conformal immersion $M \rightarrow \mathbb{R}^n$ is minimal if and only if its Gauss map is holomorphic [15, Theorem 1.1].¹

By [3, Theorem 1.1], the Gauss map assignment $\mathcal{G} : \text{CMI}(M, \mathbb{R}^n) \rightarrow \mathcal{O}(M, \mathbf{Q}^{n-2})$ is surjective. Here, $\text{CMI}(M, \mathbb{R}^n)$ and $\mathcal{O}(M, \mathbf{Q}^{n-2})$ denote the spaces of all conformal minimal immersions $M \rightarrow \mathbb{R}^n$ and of all holomorphic maps $M \rightarrow \mathbf{Q}^{n-2}$, respectively. The key to this and other recent applications of Oka theory in the theory of minimal surfaces is the fact that \mathbf{Q}^{n-2} is an Oka manifold (see [1] or [9, Example 5.6.2]).

A holomorphic map $M \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ is said to be *full* if its image is not contained in any hyperplane; we denote by $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ the open subspace of $\mathcal{O}(M, \mathbf{Q}^{n-2})$ consisting of full maps. A conformal minimal immersion $u \in \text{CMI}(M, \mathbb{R}^n)$ is called *full* if its Gauss map $\mathcal{G}(u)$ is full, and we denote by $\text{CMI}_{\text{full}}(M, \mathbb{R}^n)$ the open subspace of $\text{CMI}(M, \mathbb{R}^n)$ consisting of all such immersions. We endow these spaces with the compact-open topology.

The *flux* $\text{Flux}(u)$ of an immersion $u \in \text{CMI}(M, \mathbb{R}^n)$ is the cohomology class of its conjugate differential $d^c u = i(\bar{\partial} u - \partial u)$ in $H^1(M, \mathbb{R}^n)$. The flux is naturally identified with the group homomorphism $\text{Flux}(u) : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}^n$ given by

$$\text{Flux}(u)([C]) = \int_C d^c u = -2i \int_C \partial u, \quad [C] \in H_1(M, \mathbb{Z}).$$

We view the cohomology group $H^1(M, \mathbb{C}^n)$ as the de Rham group of n -tuples of holomorphic 1-forms on M modulo exact forms, endowed with the quotient topology

¹The Gauss map defined in [15] is the conjugate of the Gauss map defined here.

induced from the compact-open topology. The subgroup $H^1(M, \mathbb{R}^n)$ carries the subspace topology.

The first main result of this paper states that the Gauss map assignment for full conformal minimal immersions $\mathcal{G} : \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbb{Q}^{n-2})$ is a Serre fibration, that is, satisfies the homotopy lifting property with respect to all CW-complexes. In fact, we prove the following stronger result.

Theorem 1.1. *If M is an open Riemann surface and $n \geq 3$, then the map*

$$(\mathcal{G}, \text{Flux}) : \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbb{Q}^{n-2}) \times H^1(M, \mathbb{R}^n)$$

is a Serre fibration.

The fact that the flux map $\text{CMI}_{\text{full}}(M, \mathbb{R}^n) \rightarrow H^1(M, \mathbb{R}^n)$ is a Serre fibration was already known ([5, Theorem 6.1(a)] is stated for complete immersions, but in its proof completeness may be ignored). We prove Theorem 1.1 in Section 3 as a consequence of the main technical result of the paper, Theorem 2.1, which is stated and proved in Section 2. Our proofs rely on the maps we are working with being full. The key applications of fullness, in the proofs of Lemma 2.2 and Corollary 2.6, have been highlighted for the reader's convenience.

Our second main result is a contribution to the open problem of determining which holomorphic maps $M \rightarrow \mathbb{Q}^{n-2}$ are Gauss maps of complete conformal minimal immersions. As already mentioned for $n = 3$, the study of the value distribution properties of the Gauss map of complete minimal surfaces in \mathbb{R}^n for $n \geq 3$ has been one of the main foci of interest in this theory. Some restrictions are known. Ru proved that the Gaussian image of a complete nonflat minimal surface in \mathbb{R}^n cannot omit more than $n(n+1)/2$ hyperplanes in $\mathbb{C}\mathbb{P}^{n-1}$ in general position [19]. This is sharp whenever n is odd or at most 17 [12]. The same result for full minimal surfaces was previously obtained by Fujimoto [11, 14]. Let $\text{CMI}_{\text{full}}^c(M, \mathbb{R}^n)$ denote the subspace of $\text{CMI}_{\text{full}}(M, \mathbb{R}^n)$ of complete conformal minimal immersions. It follows from the parametric h-principle that is the main result of our paper [5] that the inclusion $\text{CMI}_{\text{full}}^c(M, \mathbb{R}^n) \hookrightarrow \text{CMI}_{\text{full}}(M, \mathbb{R}^n)$ is a weak homotopy equivalence with dense image. Let $\mathcal{O}_{\text{full}}^c(M, \mathbb{Q}^{n-2}) = \mathcal{G}(\text{CMI}_{\text{full}}^c(M, \mathbb{R}^n))$.

Theorem 1.2. *Let M be an open Riemann surface and $n \geq 3$.*

(a) *The inclusion $\mathcal{O}_{\text{full}}^c(M, \mathbb{Q}^{n-2}) \hookrightarrow \mathcal{O}_{\text{full}}(M, \mathbb{Q}^{n-2})$ is a weak homotopy equivalence.*

(b) *If M has finite topological type,² then the inclusion is a homotopy equivalence.*

(c) *The inclusion $\mathcal{O}_{\text{full}}^c(M, \mathbb{Q}^{n-2}) \hookrightarrow \mathcal{C}(M, \mathbb{Q}^{n-2})$ is a weak homotopy equivalence, and, if M has finite topological type, a homotopy equivalence.*

²We recall the following equivalent definitions of finite topological type: the fundamental group of M is finitely generated; M has the homotopy type of a finite bouquet of circles; M can be obtained from a compact Riemann surface by removing a finite number of mutually disjoint points and closed discs; M has a strictly subharmonic Morse exhaustion with finitely many critical points.

Part (a) means that the inclusion induces a bijection of path components $\pi_0(\mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})) \rightarrow \pi_0(\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}))$ and an isomorphism of homotopy groups

$$\pi_k(\mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2}), g) \rightarrow \pi_k(\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}), g)$$

for every integer $k \geq 1$ and every base point $g = \mathcal{G}(u) \in \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$, with $u \in \text{CMI}_{\text{full}}^c(M, \mathbb{R}^n)$. By (b), when M is of finite topological type, there is a homotopy inverse $\xi : \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}) \rightarrow \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ to the inclusion. This means that there is a way to associate to every map $g \in \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ a map $\xi(g) \in \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ that is homotopic to g . Moreover, if $g \in \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ to begin with, then there is such a homotopy through maps in $\mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$. The main point is that $\xi(g)$ and the homotopies depend continuously on g . Finally, part (c) reduces the determination of the homotopy type of $\mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ to a purely topological problem.

The main ingredients in the proof of the theorem, which is given in Section 3, are the parametric h-principle in [5], the result that the Gauss map assignment is a fibration (Theorem 1.1), and, for part (b), the theory of absolute neighbourhood retracts (ANRs) in the category of metric spaces and [16, Theorem 9], which uses Oka theory to show that certain spaces of holomorphic maps are ANRs.

We observe that since the space $\text{CMI}_{\text{full}}^c(M, \mathbb{R}^n)$ is dense in $\text{CMI}_{\text{full}}(M, \mathbb{R}^n)$ ([2, Theorem 7.1]; the case of $n = 3$ follows from [6, Theorem 5.6]) and the map $\mathcal{G} : \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ is surjective [3, Theorem 1.1], $\mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ is dense in $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$. As a consequence of Theorem 1.1, $\mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ is dense in $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ in the following stronger sense. See [5, Corollary 1.3] for an analogous result for the subspace $\text{CMI}_{\text{full}}^c(M, \mathbb{R}^n)$ of $\text{CMI}_{\text{full}}(M, \mathbb{R}^n)$.

Corollary 1.3. *If M is an open Riemann surface, P is a contractible finite CW-complex, and $Q \subset P$ is a retract of P , then every continuous map $Q \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$, $n \geq 3$, extends to a continuous map $P \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ that takes $P \setminus Q$ into $\mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$.*

Remark 1.4. Consider the commuting square

$$\begin{array}{ccc} \text{CMI}_{\text{full}}^c(M, \mathbb{R}^n) & \xrightarrow{i} & \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \\ \downarrow \mathcal{G}^c & & \downarrow \mathcal{G} \\ \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2}) & \xrightarrow{j} & \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}) \end{array}$$

where \mathcal{G}^c is the restriction of \mathcal{G} to $\text{CMI}_{\text{full}}^c(M, \mathbb{R}^n)$, that is, the Gauss map assignment for full complete conformal minimal immersions. We know that the inclusions i and j are weak homotopy equivalences by [5, Theorem 6.1(c)] and Theorem 1.2(a), respectively, while \mathcal{G} is a fibration by Theorem 1.1. It remains an open question whether \mathcal{G}^c is a fibration as well.

In the classical case of $n = 3$, a conformal minimal immersion u is full if and only if it is nonflat, that is, its image does not lie in an affine 2-plane in \mathbb{R}^3 .

Equivalently, the Gauss map of u is not constant. Also, \mathbf{Q}^1 may be identified with the Riemann sphere $\mathbb{P} = \mathbb{C}\mathbb{P}^1$ and the Gauss map of a conformal minimal immersion $u = (u_1, u_2, u_3) : M \rightarrow \mathbb{R}^3$ viewed, via the stereographic projection, as the holomorphic function $M \rightarrow \mathbb{P}$ given by

$$\mathcal{G}(u) = \frac{\partial u_3}{\partial u_1 - i\partial u_2},$$

often called the *complex Gauss map* of u (see [4, Section 2.5] for more details). In the following corollary, the subscript nf stands for nonflat and nc for nonconstant.

Corollary 1.5. *Let M be an open Riemann surface.*

(a) $(\mathcal{G}, \text{Flux}) : \text{CMI}_{\text{nf}}(M, \mathbb{R}^3) \rightarrow \mathcal{O}_{\text{nc}}(M, \mathbb{P}) \times H^1(M, \mathbb{R}^3)$ is a Serre fibration.

(b) The inclusion $\mathcal{O}_{\text{nc}}^c(M, \mathbb{P}) \hookrightarrow \mathcal{O}_{\text{nc}}(M, \mathbb{P})$ is a weak homotopy equivalence, whose image is dense in the strong sense of Corollary 1.3. The inclusion $\mathcal{O}_{\text{nc}}^c(M, \mathbb{P}) \hookrightarrow \mathcal{C}(M, \mathbb{P})$ is a weak homotopy equivalence. If M has finite topological type, then the inclusions are homotopy equivalences.

Remark 1.6. Let us briefly indicate how our results can be adapted to null curves. A conformal minimal immersion $u : M \rightarrow \mathbb{R}^n$ has a harmonic conjugate v if and only if its flux vanishes. Then the holomorphic map $u + iv : M \rightarrow \mathbb{C}^n$ is a so-called null curve. The subspace of $\text{CMI}(M, \mathbb{R}^n)$ of immersions with vanishing flux, that is, real parts of holomorphic null curves, is denoted $\mathfrak{RNC}(M, \mathbb{C}^n)$. By [3, Theorem 1.1], the Gauss map assignment $\mathcal{G} : \mathfrak{RNC}_{\text{full}}(M, \mathbb{C}^n) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ is surjective, and Theorem 1.1 implies that it is a fibration (where the subscripts have the usual meaning).

The proof of Theorem 1.2(a) is then easily adapted, using the control on the flux provided by [5, Theorem 6.1(a)], to show that the inclusion into $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ of the space $\mathcal{G}(\mathfrak{RNC}_{\text{full}}^c(M, \mathbb{R}^n))$ of full holomorphic maps $M \rightarrow \mathbf{Q}^{n-2}$ that are the Gauss map of the real part of a complete holomorphic null curve is a weak homotopy equivalence. Moreover, the inclusion has dense image (using the analogue for full immersions of [5, Corollary 1.3], which is an immediate consequence of [5, Theorem 6.1(a)]). We also see that the inclusion $\mathcal{G}(\mathfrak{RNC}_{\text{full}}^c(M, \mathbb{R}^n)) \hookrightarrow \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ is a weak homotopy equivalence. It is an open question whether the two spaces are in fact the same.

More generally, if we fix $\alpha \in H^1(M, \mathbb{R}^n)$, the corresponding results hold for conformal minimal immersions with flux α .

Theorem 1.1 implies that the space $\mathcal{G}^{-1}(g)$ of full conformal minimal immersions $M \rightarrow \mathbb{R}^n$ with fixed Gauss map g has the same weak homotopy type for all $g \in \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$. In Section 4 we determine this homotopy type.

Theorem 1.7. *Let M be an open Riemann surface and $n \geq 3$. The fibre of the Gauss map assignment $\mathcal{G} : \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ has the weak homotopy type of a countably infinite disjoint union of circles, unless M is the plane or the disc, in which case the fibre has the weak homotopy type of a circle.*

Whether $\mathcal{G} : \text{CMI}(M, \mathbb{R}^n) \rightarrow \mathcal{O}(M, \mathbf{Q}^{n-2})$ is a fibration or whether there are singularities of some sort over the non-full maps that prevent \mathcal{G} from being a fibration is an open question that is beyond our techniques at present. For $n = 3$, the fibre of \mathcal{G} over a constant map essentially consists of the holomorphic immersions $M \rightarrow \mathbb{C}$. As determined in [10], the space of such maps has the weak homotopy type of $\mathcal{C}(M, \mathbb{C}^*)$. As shown in Section 4, the fibre of \mathcal{G} over a full immersion has that same homotopy type, suggesting that the question might have an affirmative answer.

2. The main technical theorem

According to [4, Definition 1.12.9], a compact subset $S \neq \emptyset$ of an open Riemann surface M is *admissible* if it is $\mathcal{O}(M)$ -convex and of the form $S = K \cup \Gamma$, where K is the union of finitely many pairwise disjoint smoothly bounded compact domains in M and $\Gamma = S \setminus K$ is a finite union of pairwise disjoint smooth Jordan arcs meeting K only at their endpoints (or not at all) and such that their intersections with the boundary of K are transverse. Given such a set $S = K \cup \Gamma$ and a complex submanifold $Z \subset \mathbb{C}^n$, we denote by $\mathcal{A}(S, Z)$ the space of all continuous maps $S \rightarrow Z$ that are holomorphic on $\mathring{S} = \mathring{K}$. For simplicity, we write $\mathcal{A}(S) = \mathcal{A}(S, \mathbb{C})$.

In this section we prove the following theorem, which is the technical heart of the paper.

Theorem 2.1. *Let M be an open Riemann surface, θ be a holomorphic 1-form vanishing nowhere on M , $S = K \cup \Gamma \subset M$ be an admissible subset, k and n be positive integers, and $f_p : M \rightarrow \mathbb{C}^n$ and $F_p \in H^1(M, \mathbb{C}^n)$ ($p \in [0, 1]^k$) be continuous families of full holomorphic maps and cohomology classes. Then, every continuous family of functions $\varphi_p : S \rightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ($p \in [0, 1]^k$) of class $\mathcal{A}(S)$ satisfying*

$$(2.1) \quad \int_C \varphi_p f_p \theta = F_p([C]) \quad \text{for all closed curves } C \subset S \text{ and all } p \in [0, 1]^k$$

can be approximated uniformly on $[0, 1]^k \times S$ by continuous families of holomorphic functions $\tilde{\varphi}_p : M \rightarrow \mathbb{C}^$ ($p \in [0, 1]^k$) such that*

$$\int_C \tilde{\varphi}_p f_p \theta = F_p([C]) \quad \text{for all closed curves } C \subset M \text{ and all } p \in [0, 1]^k.$$

Furthermore, if φ_p is holomorphic on M , vanishes nowhere on M , and satisfies

$$\int_C \varphi_p f_p \theta = F_p([C]) \quad \text{for all closed curves } C \subset M, \text{ for all } p \in [0, 1]^{k-1} \times \{0\},$$

then we can choose $\tilde{\varphi}_p = \varphi_p$ for all $p \in [0, 1]^{k-1} \times \{0\}$.

The case $k = 1$ of Theorem 2.1 was proved in [3, Theorem 4.1]; the proof relies in an essential way on the parameter space $[0, 1]$ being 1-dimensional (see [3, proof of Lemma 2.3]). The proof of Theorem 2.1 follows the scheme of the proof of [3, Theorem 4.1], but with an additional idea that enables us to work with a parameter space of arbitrary dimension. In particular, we shall make use of [3, Lemmas 3.2

and 4.2]. The main new technical ingredient in our proof is the following extension of [3, Lemma 4.3] to the parameter space $[0, 1]^k$ for arbitrary $k \geq 1$.

Lemma 2.2 (The critical case). *Let M , θ , k , n , and f_p ($p \in [0, 1]^k$) be as in Theorem 2.1. Also let $\rho : M \rightarrow [0, +\infty)$ be a smooth strongly subharmonic Morse exhaustion function and let $0 < a < b$ be a pair of regular values of ρ such that ρ has a single critical point in $L \setminus \mathring{K}$, where $K = \{\rho \leq a\}$ and $L = \{\rho \leq b\}$. Assume that we have continuous families of functions $\varphi_p : K \rightarrow \mathbb{C}^*$ of class $\mathcal{A}(K)$ and cohomology classes $F_p \in H^1(M, \mathbb{C})$ ($p \in [0, 1]^k$) such that*

$$\int_C \varphi_p f_p \theta = F_p([C]) \quad \text{for all closed curves } C \subset K \text{ and all } p \in [0, 1]^k.$$

Then, the family φ_p can be approximated uniformly on $[0, 1]^k \times K$ by continuous families of functions $\tilde{\varphi}_p : L \rightarrow \mathbb{C}^$ ($p \in [0, 1]^k$) of class $\mathcal{A}(L)$ such that*

$$\int_C \tilde{\varphi}_p f_p \theta = F_p([C]) \quad \text{for all closed curves } C \subset L \text{ and all } p \in [0, 1]^k.$$

Furthermore, if φ_p is of class $\mathcal{A}(L)$, vanishes nowhere on L , and satisfies

$$\int_C \varphi_p f_p \theta = F_p([C]) \quad \text{for all closed curves } C \subset L, \text{ for all } p \in [0, 1]^{k-1} \times \{0\},$$

then we can choose $\tilde{\varphi}_p = \varphi_p$ for all $p \in [0, 1]^{k-1} \times \{0\}$.

We note that [3, Lemma 4.2] holds in our more general framework with the same proof but replacing the parameter space $[0, 1]$ by $[0, 1]^k$ for the given integer $k \geq 1$. We record this result for later reference.

Lemma 2.3 (The noncritical case). *Let M , $S = K \cup \Gamma$, θ , k , n , and f_p ($p \in [0, 1]^k$) be as in Theorem 2.1. If $L \subset M$ is a smoothly bounded $\mathcal{O}(M)$ -convex compact domain such that $S \subset \mathring{L}$ and S is a deformation retract of L , then every continuous family of functions $\varphi_p : S \rightarrow \mathbb{C}^*$ ($p \in [0, 1]^k$) of class $\mathcal{A}(S)$ can be uniformly approximated on $[0, 1]^k \times S$ by continuous families of functions $\tilde{\varphi}_p : L \rightarrow \mathbb{C}^*$ ($p \in [0, 1]^k$) of class $\mathcal{A}(L)$ such that $(\tilde{\varphi}_p - \varphi_p) f_p \theta$ is exact on S for all $p \in [0, 1]^k$.*

Furthermore, if φ_p is of class $\mathcal{A}(L)$ and vanishes nowhere on L for all $p \in [0, 1]^{k-1} \times \{0\}$, then we can choose $\tilde{\varphi}_p = \varphi_p$ for all $p \in [0, 1]^{k-1} \times \{0\}$.

Proof of Theorem 2.1 assuming Lemma 2.2. Choose a smooth strongly subharmonic Morse exhaustion function $\rho : M \rightarrow \mathbb{R}$ and a divergent sequence of regular values $0 < a_1 < a_2 < \dots$ of ρ such that, setting $K_0 = S$ and $K_j = \{\rho \leq a_j\}$ for all $j \geq 1$, the following conditions are satisfied.

- $K_0 \subset \mathring{K}_1$ and K_0 is a strong deformation retract of K_1 .
- ρ has at most a single critical point in $K_{j+1} \setminus \mathring{K}_j$ (which lies in $\mathring{K}_{j+1} \setminus K_j$) for all $j \geq 1$.

It turns out that K_j is a smoothly bounded $\mathcal{O}(M)$ -convex compact domain for all $j \geq 1$, and

$$S = K_0 \Subset K_1 \Subset K_2 \cdots \Subset \bigcup_{j \geq 0} K_j = M$$

is an exhaustion of M . Let $\varphi_p : S \rightarrow \mathbb{C}^*$ ($p \in [0, 1]^k$) be a continuous family of functions of class $\mathcal{A}(S)$ satisfying condition (2.1), and fix $\epsilon > 0$. Call $\varphi_p^0 = \varphi_p$ for all $p \in [0, 1]^k$. A standard recursive application of Lemmas 2.2 and 2.3 provides a sequence of continuous families of functions $\varphi_p^j : K_j \rightarrow \mathbb{C}^*$ ($p \in [0, 1]^k$), $j \geq 1$, of class $\mathcal{A}(K_j)$ satisfying the following conditions for all $j \geq 1$.

- φ_p^j is as close as desired to φ_p^{j-1} uniformly on $[0, 1]^k \times K_{j-1}$.
- $\int_C \varphi_p^j f_p \theta = F_p([C])$ for all closed curves $C \subset K_j$ and all $p \in [0, 1]^k$.
- If φ_p is holomorphic on M , vanishes nowhere on M , and satisfies

$$\int_C \varphi_p f_p \theta = F_p([C]) \quad \text{for all closed curves } C \subset M, \text{ for all } p \in [0, 1]^{k-1} \times \{0\},$$

then we can choose $\varphi_p^j = \varphi_p^0 = \varphi_p$ for all $p \in [0, 1]^{k-1} \times \{0\}$.

If we take φ_p^j sufficiently close to φ_p^{j-1} on $[0, 1]^k \times K_{j-1}$ at each step, we obtain in the limit a continuous family of holomorphic functions

$$\tilde{\varphi}_p := \lim_{j \rightarrow \infty} \varphi_p^j : M \rightarrow \mathbb{C}^* \quad (p \in [0, 1]^k)$$

which is ϵ -close to the family φ_p on $[0, 1]^k \times S$ and satisfies the requirements in the theorem. \square

To complete the proof of Theorem 2.1 it remains to prove Lemma 2.2. We begin with some preparations. Given an integer $n \geq 1$, we shall say that a continuous map $f : [0, 1] \rightarrow \mathbb{R}^n$ is \mathbb{R} -full if its image is contained in no real linear hyperplane, that is, the real span of $f([0, 1])$ equals \mathbb{R}^n . Likewise, a continuous map $f : [0, 1] \rightarrow \mathbb{C}^n$ is said to be \mathbb{C} -full if the complex span of $f([0, 1])$ equals \mathbb{C}^n . It is clear that every \mathbb{R} -full map $[0, 1] \rightarrow \mathbb{C}^n = \mathbb{R}^{2n}$ is \mathbb{C} -full, but the converse does not hold true in general.

Lemma 2.4. *Let k and n be positive integers, $P = [0, 1]^k$, and $f : P \times [0, 1] \rightarrow \mathbb{R}^n$ and $\alpha : P \rightarrow \mathbb{R}^n$ be continuous maps. If the path $f_p := f(p, \cdot) : [0, 1] \rightarrow \mathbb{R}^n$ is \mathbb{R} -full for every $p \in P$, then for any $\epsilon > 0$ there exists a continuous function $x : P \times [0, 1] \rightarrow \mathbb{R}$ such that $x(p, s) = 0$ for $p \in P$ and $s \in \{0, 1\}$ and*

$$\left| \int_0^1 x(p, s) f(p, s) ds - \alpha(p) \right| < \epsilon \quad \text{for all } p \in P.$$

If in addition $\alpha(p) = 0$ for all $p \in Q = [0, 1]^{k-1} \times \{0\} \subset P$, then we can choose x with $x(p, s) = 0$ for all $p \in Q$ and $s \in [0, 1]$.

In case $k = 1$, we identify Q with $\{0\} \subset P = [0, 1]$. In the proof we shall use the following observation, which corresponds to the case $k = 0$ in the lemma.

Claim 2.5. *If $f : [0, 1] \rightarrow \mathbb{R}^n$ ($n \geq 1$) is continuous and \mathbb{R} -full, then for any $\alpha \in \mathbb{R}^n$ and any $\epsilon > 0$ there exists a continuous function $x : [0, 1] \rightarrow \mathbb{R}$ such that $x(0) = x(1) = 0$ and*

$$\left| \int_0^1 x(s)f(s) ds - \alpha \right| < \epsilon.$$

Proof. If $\alpha = 0 \in \mathbb{R}^n$, then we simply choose $x = 0$. Assume that $\alpha \neq 0$. By \mathbb{R} -fullness of f there are points $0 < s_1 < s_2 < \dots < s_n < 1$ in $[0, 1]$ such that

$$(2.2) \quad \text{span}_{\mathbb{R}}\{f(s_1), \dots, f(s_n)\} = \mathbb{R}^n.$$

Thus, there is a number $\sigma > 0$ so small that the intervals $[s_j - \sigma, s_j + \sigma]$, $j \in \{1, \dots, n\}$, lie in $(0, 1)$ and are pairwise disjoint. Set

$$v_j = \int_{s_j - \sigma}^{s_j + \sigma} f(s) ds \in \mathbb{R}^n, \quad j = 1, \dots, n,$$

and note that v_j is close to $2\sigma f(s_j)$ provided that σ is small. We assume as we may by (2.2) that $\sigma > 0$ is so small that $\text{span}_{\mathbb{R}}\{v_1, \dots, v_n\} = \mathbb{R}^n$, and write

$$\alpha = \sum_{j=1}^n \lambda_j v_j$$

for (unique) $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. The step function $x_0 : [0, 1] \rightarrow \mathbb{R}$ given by

$$x_0(s) = \begin{cases} \lambda_j & \text{if } s \in [s_j - \sigma, s_j + \sigma], j = 1, \dots, n, \\ 0 & \text{if } s \in [0, 1] \setminus \bigcup_{j=1}^n [s_j - \sigma, s_j + \sigma], \end{cases}$$

satisfies $x_0(0) = x_0(1) = 0$ and $\int_0^1 x_0(s)f(s) ds = \alpha$. Suitably deforming x_0 in a small neighbourhood of the points $s_j \pm \sigma$, $j = 1, \dots, n$, to make it continuous, we obtain a function $x : [0, 1] \rightarrow \mathbb{R}$ satisfying the required conditions. \square

Proof of Lemma 2.4. We proceed by induction on the positive integer k . For the base case when $k = 1$, we have $Q = \{0\} \subset P = [0, 1]$ and a pair of continuous maps $f : P \times [0, 1] \rightarrow \mathbb{R}^n$ and $\alpha : P \rightarrow \mathbb{R}^n$. Since the map $f_p : [0, 1] \rightarrow \mathbb{R}^n$ is \mathbb{R} -full for every $p \in P$, Claim 2.5 gives a continuous function $y_p : [0, 1] \rightarrow \mathbb{R}$ such that

$$(2.3) \quad y_p(0) = y_p(1) = 0$$

and

$$\left| \int_0^1 y_p(s)f_p(s) ds - \alpha(p) \right| < \epsilon \quad \text{for all } p \in P.$$

If $\alpha(0) = 0$ then we choose $y_0 = 0$. The problem now is that y_p does not depend continuously on $p \in P$, so we have to do some more work. By continuity of f and α and compactness of P , there is a partition $0 = p_0 < p_1 < \dots < p_k = 1$ of $P = [0, 1]$ such that

$$(2.4) \quad \left| \int_0^1 y_{p_j}(s)f_p(s) ds - \alpha(p) \right| < \epsilon \quad \text{for all } p \in [p_{j-1}, p_j], j = 1, \dots, k.$$

The function $x : P \times [0, 1] \rightarrow \mathbb{R}$ given by

$$x(p, \cdot) = \frac{p_j - p}{p_j - p_{j-1}} y_{p_{j-1}} + \frac{p - p_{j-1}}{p_j - p_{j-1}} y_{p_j} \quad \text{for all } p \in [p_{j-1}, p_j], j = 1, \dots, k,$$

is continuous and, in view of (2.3), satisfies $x(p, 0) = x(p, 1) = 0$ for all $p \in P$. Moreover, (2.4) ensures that

$$\left| \int_0^1 x(p, s) f(p, s) ds - \alpha(p) \right| < \epsilon \quad \text{for all } p \in P.$$

Finally, note that $x(0, \cdot) = x(p_0, \cdot) = y_0$; hence $x(0, s) = 0$ for all $s \in [0, 1]$ provided that $\alpha(0) = 0$. This proves the base case.

For the inductive step, fix an integer $k \geq 2$ and assume that the lemma holds for $P' = [0, 1]^{k-1}$ and $Q' = [0, 1]^{k-2} \times \{0\}$. If $k = 2$, we identify Q' with $\{0\} \subset P' = [0, 1]$. We write $P = [0, 1]^k = P' \times [0, 1]$ and $Q = [0, 1]^{k-1} \times \{0\} = P' \times \{0\}$. We use the same argument as above. Fix $t \in [0, 1]$. The map $f_q^t := f_{(q,t)} : [0, 1] \rightarrow \mathbb{R}^n$ is \mathbb{R} -full for every $q \in P'$, and hence the induction hypothesis provides a continuous function $y_t : P' \times [0, 1] \rightarrow \mathbb{R}$ such that $y_t(q, 0) = y_t(q, 1) = 0$ for all $q \in P'$, and

$$\left| \int_0^1 y_t(q, s) f_q^t(s) ds - \alpha(q, t) \right| < \epsilon \quad \text{for all } q \in P'.$$

If $\alpha(q, 0) = 0$ for all $q \in P'$ then we choose $y_0 = 0$. Again, the problem is that the map y_t does not depend continuously on $t \in [0, 1]$. To arrange this, take a partition $0 = t_0 < t_1 < \dots < t_k = 1$ of $[0, 1]$ such that

$$\left| \int_0^1 y_{t_j}(q, s) f_q^{t_j}(s) ds - \alpha(q, t) \right| < \epsilon \quad \text{for all } q \in P' \text{ and } t \in [t_{j-1}, t_j], j = 1, \dots, k.$$

The function $x : P \times [0, 1] = P' \times [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ given by

$$x(\cdot, t, \cdot) = \frac{t_j - t}{t_j - t_{j-1}} y_{t_{j-1}} + \frac{t - t_{j-1}}{t_j - t_{j-1}} y_{t_j} \quad \text{for all } t \in [t_{j-1}, t_j], j = 1, \dots, k,$$

satisfies the required conditions. This completes the induction. \square

Corollary 2.6. *Let k, n, P , and Q be as in Lemma 2.4, let $f : P \times [0, 1] \rightarrow \mathbb{C}^n$ and $\alpha : P \rightarrow \mathbb{C}^n$ be continuous maps, and assume that there is a pair of closed intervals J and J' such that $J \cap J' = \emptyset$, $J \cup J' \subset (0, 1)$, and the path $f_p := f(p, \cdot) : [0, 1] \rightarrow \mathbb{C}^n$ is \mathbb{C} -full on J and \mathbb{R} -full on J' for every $p \in P$. Then, there exists a continuous function $h : P \times [0, 1] \rightarrow \mathbb{C}^*$ such that $h(p, 0) = h(p, 1) = 1$ for all $p \in P$ and*

$$\int_0^1 h(p, s) f(p, s) ds = \alpha(p) \quad \text{for all } p \in P.$$

If in addition $\alpha(p) = \int_0^1 f(p, s) ds$ for all $p \in Q \subset P$, then there is such a function h with $h(p, s) = 1$ for all $p \in Q$ and $s \in [0, 1]$.

Furthermore, the function h can be chosen such that $|\Re(h) - 1| < \sigma$ in $P \times [0, 1]$ for any given number $\sigma > 0$; in particular, we can choose h so that $\Re(h) > 0$.

The final claim of the corollary is not needed here, but is included for possible future applications.

Proof. Consider the period map $\mathcal{P} : \mathcal{C}([0, 1], \mathbb{C}^n) \rightarrow \mathbb{C}^n$ given by

$$\mathcal{P}(g) = \int_0^1 g(s) ds \in \mathbb{C}^n, \quad g \in \mathcal{C}([0, 1], \mathbb{C}^n).$$

By [3, Lemma 2.1], there are continuous functions $g_1, \dots, g_N : [0, 1] \rightarrow \mathbb{C}$ ($N \geq n$), supported on J , such that the function $w : \mathbb{C}^N \times [0, 1] \rightarrow \mathbb{C}$ given by

$$w(\zeta, s) := \prod_{i=1}^N (1 + \zeta_i g_i(s)), \quad \zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N, \quad s \in [0, 1],$$

has

$$\frac{\partial}{\partial \zeta} \mathcal{P}(w(\zeta, \cdot) f(p, \cdot)) \Big|_{\zeta=0} : T_0 \mathbb{C}^N \cong \mathbb{C}^N \rightarrow \mathbb{C}^n \quad \text{surjective for every } p \in P.$$

(Here we need the maps f_p to be full.) In particular,

$$(2.5) \quad w(\cdot, s) = 1 \quad \text{for all } s \in [0, 1] \setminus J \supset \{0, 1\}.$$

By virtue of the implicit function theorem, this implies that for every neighbourhood U of 0 in \mathbb{C}^n , there is a number $\epsilon > 0$ with the following property: if $\beta : P \rightarrow \mathbb{C}^n$ is a continuous map with $|\mathcal{P}(f(p, \cdot)) - \beta(p)| < \epsilon$ for all $p \in P$, then there is a continuous map $\zeta_\beta : P \rightarrow U$ such that

$$\mathcal{P}(w(\zeta_\beta(p), \cdot) f(p, \cdot)) = \beta(p) \quad \text{for all } p \in P.$$

Furthermore, if $\mathcal{P}(f(p, \cdot)) = \beta(p)$ for all $p \in Q$, then we can choose the map ζ_β such that $\zeta_\beta(p) = 0$ (and hence $w(\zeta_\beta(p), \cdot) = 1$) for all $p \in Q$. Fix a number $0 < \sigma < 1$, let U be a neighbourhood of 0 in \mathbb{C}^n so small that

$$(2.6) \quad |\Re(w) - 1| < \sigma \quad \text{in } U \times [0, 1]$$

(hence $\Re(w) > 0$ there), and let $\epsilon > 0$ be a number satisfying the above condition.

Consider the continuous map $\gamma : P \rightarrow \mathbb{C}^n$ given by

$$(2.7) \quad \gamma(p) = -i(\alpha(p) - \mathcal{P}(f(p, \cdot))), \quad p \in P.$$

Lemma 2.4 furnishes us with a continuous function $x : P \times [0, 1] \rightarrow \mathbb{R}$, supported on $P \times J'$, such that

$$|\mathcal{P}(x(p, \cdot) f(p, \cdot)) - \gamma(p)| < \epsilon \quad \text{for all } p \in P,$$

and if $\gamma(p) = 0$ for all $p \in Q$, then $x(p, \cdot) = 0$ for all $p \in Q$. It turns out that the continuous function $\tilde{h} = 1 + ix : P \times [0, 1] \rightarrow 1 + i\mathbb{R} \subset \mathbb{C}^*$ satisfies

$$(2.8) \quad \tilde{h} = 1 \quad \text{in } P \times ([0, 1] \setminus J') \supset P \times (J \cup \{0, 1\})$$

and, in view of (2.7),

$$|\mathcal{P}(\tilde{h}(p, \cdot) f(p, \cdot)) - \alpha(p)| < \epsilon \quad \text{for all } p \in P.$$

Moreover, if $\mathcal{P}(f(p, \cdot)) = \alpha(p)$ for all $p \in Q$, then $\tilde{h}(p, \cdot) = 1$ for all $p \in Q$. Since $\tilde{h}f = f$ in $P \times J$ and the functions g_1, \dots, g_N are supported on J , the period dominating property of w provides a continuous map $\zeta_\alpha : P \rightarrow U$ such that

$$(2.9) \quad \mathcal{P}(w(\zeta_\alpha(p), \cdot) \tilde{h}(p, \cdot) f(p, \cdot)) = \alpha(p) \quad \text{for all } p \in P,$$

and if $\mathcal{P}(f(p, \cdot)) = \alpha(p)$ for all $p \in Q$, then $\zeta_\alpha(p) = 0$ for all $p \in Q$. The continuous function $h : P \times [0, 1] \rightarrow \mathbb{C}^*$ given by $h(p, \cdot) = w(\zeta_\alpha(p), \cdot)\tilde{h}(p, \cdot)$ satisfies the conclusion of the corollary. Indeed, conditions (2.5) and (2.8) ensure that $h(\cdot, 0) = h(\cdot, 1) = 1$. By (2.9), we have

$$\int_0^1 h(p, s)f(p, s) ds = \mathcal{P}(w(\zeta_\alpha(p), \cdot)\tilde{h}(p, \cdot)f(p, \cdot)) = \alpha(p) \quad \text{for all } p \in P.$$

Moreover, if $\alpha(p) = \int_0^1 f(p, s) ds$ ($= \mathcal{P}(f(p, \cdot))$) for all $p \in Q$, then it is ensured that $\tilde{h}(p, \cdot) = 1$ and $\zeta_\alpha(p) = 0$ for all $p \in Q$, and hence $h(p, \cdot) = 1$ for all $p \in Q$. Finally, (2.5), (2.6), (2.8), and the facts that $\zeta_\alpha(P) \subset U$ and $\Re(\tilde{h}) = 1$ in $P \times [0, 1]$ guarantee that $|\Re(h) - 1| < \sigma$ in $P \times [0, 1]$. \square

Proof of Lemma 2.2. Let v denote the only critical point of ρ in $L \setminus \mathring{K}$, and note that $v \in \mathring{L} \setminus K$. We distinguish cases depending on the Morse index of ρ at v .

Assume first that the Morse index of ρ at v equals 0. We then proceed as in the proof of [3, Lemma 4.3]. In this case a new connected component of the sublevel set $\{\rho \leq s\}$ appears when s passes the value $\rho(v)$, and hence a new connected and simply connected component K' of L appears. In particular, K is a strong deformation retract of $L \setminus K'$. Thus, Lemma 2.3 provides a continuous family of functions $\tilde{\varphi}_p : L \setminus K' \rightarrow \mathbb{C}^*$ ($p \in [0, 1]^k$) of class $\mathcal{A}(L \setminus K')$ which is as close to the family φ_p as desired in $[0, 1]^k \times K$ and has $(\tilde{\varphi}_p - \varphi_p)f_p\theta$ exact on K for all $p \in [0, 1]^k$. Moreover, we can choose $\tilde{\varphi}_p = \varphi_p$ for all $p \in [0, 1]^{k-1} \times \{0\}$ provided that φ_p is of class $\mathcal{A}(L)$ and nowhere vanishing on L for every $p \in [0, 1]^{k-1} \times \{0\}$. (Here we need the maps f_p to be full.) To finish it suffices to extend the family $\tilde{\varphi}_p$ ($p \in [0, 1]^k$) to K' as a continuous family of nowhere vanishing functions of class $\mathcal{A}(K')$, choosing $\tilde{\varphi}_p|_{K'} = \varphi_p|_{K'}$ for all $p \in [0, 1]^{k-1} \times \{0\}$ if for every $p \in [0, 1]^{k-1} \times \{0\}$ the map φ_p is of class $\mathcal{A}(L)$, vanishes nowhere on L , and satisfies $\int_C \varphi_p f_p \theta = F_p([C])$ for all closed curves $C \subset L$.

Assume that, on the contrary, the Morse index of ρ at v equals 1. In this case there is a smooth Jordan arc γ in $\mathring{L} \setminus \mathring{K}$, with its two endpoints in bK and otherwise disjoint from K , such that $S = K \cup \gamma \subset \mathring{L}$ is an admissible subset of M and a strong deformation retract of L . Denote by Ω the connected component of $\mathring{L} \setminus K$ intersecting γ . Note that Ω contains γ except for its endpoints.

Fix $p \in [0, 1]^k$. We claim that

$$(2.10) \quad \text{the real span of } f_p(\Omega) \text{ equals } \mathbb{C}^n = \mathbb{R}^{2n}.$$

Indeed, since the map $f_p : M \rightarrow \mathbb{C}^n$ is holomorphic and full, $f_p|_\Omega : \Omega \rightarrow \mathbb{C}^n$ is full as well, and hence if $\lambda : \mathbb{C}^n \rightarrow \mathbb{C}$ is a \mathbb{C} -linear functional and $\lambda \circ f_p|_\Omega = 0$, then $\lambda = 0$. Let $\mu : \mathbb{C}^n \rightarrow \mathbb{R}$ be an \mathbb{R} -linear functional with $\mu \circ f_p|_\Omega = 0$. Now, μ is the real part of a \mathbb{C} -linear functional $\lambda : \mathbb{C}^n \rightarrow \mathbb{C}$. Since f is holomorphic and $\Re \lambda \circ f_p|_\Omega = 0$, we have $\lambda \circ f_p|_\Omega = 0$, so $\lambda = 0$ and hence $\mu = 0$. This guarantees (2.10), and thus there are points $x_1, \dots, x_{2n} \in \Omega$ such that

$$\text{span}_{\mathbb{R}}\{f_p(x_1), \dots, f_p(x_{2n})\} = \mathbb{C}^n.$$

(Here, again, we need the maps f_p to be full.) Since f_p depends continuously on $p \in [0, 1]^k$, the same holds if we replace p by any point p' in a small neighbourhood of p in $[0, 1]^k$. Thus, compactness of $[0, 1]^k$ ensures the existence of a finite set $\Lambda \subset \Omega \subset L \setminus K$ such that

$$(2.11) \quad \text{span}_{\mathbb{R}}\{f_p(x) : x \in \Lambda\} = \mathbb{C}^n \quad \text{for all } p \in [0, 1]^k.$$

Since Ω is connected, we may deform the arc γ outside a neighbourhood of its endpoints so that $\Lambda \subset \gamma$. Since $\Lambda \cap K = \emptyset$, we have that Λ lies in the relative interior of γ , and since Λ is finite there is a (closed) sub-arc γ' in the relative interior of γ such that $\Lambda \subset \gamma'$. Condition (2.11) then ensures that the map f_p is \mathbb{R} -full on γ' for all $p \in [0, 1]^k$. On the other hand, the fullness of the holomorphic map $f_p : M \rightarrow \mathbb{C}^n$ implies that f_p is \mathbb{C} -full on every sub-arc of γ for all $p \in [0, 1]^k$. (Here, once again, we need the maps f_p to be full.) Corollary 2.6 then enables us to extend the family of functions $\varphi_p : K \rightarrow \mathbb{C}^*$ to a continuous family of functions $\phi_p : S = K \cup \gamma \rightarrow \mathbb{C}^*$ ($p \in [0, 1]^k$) of class $\mathcal{A}(S)$ such that

$$\int_C \phi_p f_p \theta = F_p([C]) \quad \text{for all closed curves } C \subset S \text{ and all } p \in [0, 1]^k.$$

Furthermore, if φ_p is of class $\mathcal{A}(L)$, vanishes nowhere on L , and satisfies

$$\int_C \varphi_p f_p \theta = F_p([C]) \quad \text{for all closed curves } C \subset L, \text{ for all } p \in [0, 1]^{k-1} \times \{0\},$$

then we can choose $\phi_p = \varphi_p|_S$ for all $p \in [0, 1]^{k-1} \times \{0\}$. Since S is a strong deformation retract of L , this reduces the proof to the noncritical case granted in Lemma 2.3. \square

Theorem 2.1 is thus proved.

3. Proofs

In this section, we prove the two main Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Fix an integer $k \geq 1$ and let $P = [0, 1]^{k-1}$. For $k = 1$, we identify P and $P \times \{0\}$ with $\{0\} \subset [0, 1]$ and $P \times [0, 1]$ with $[0, 1]$. Assume that we have continuous maps $u : P \times \{0\} \rightarrow \text{CMI}_{\text{full}}(M, \mathbb{R}^n)$, $G : P \times [0, 1] \rightarrow \mathcal{O}_{\text{full}}(M, \mathbb{Q}^{n-2})$, and $F : P \times [0, 1] \rightarrow H^1(M, \mathbb{R}^n)$ such that

$$G|_{P \times \{0\}} = \mathcal{G} \circ u \quad \text{and} \quad F|_{P \times \{0\}} = \text{Flux} \circ u;$$

that is, the following square commutes.

$$(3.1) \quad \begin{array}{ccc} P \times \{0\} & \xrightarrow{u} & \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \\ \downarrow & \nearrow \phi & \downarrow (\mathcal{G}, \text{Flux}) \\ P \times [0, 1] & \xrightarrow{(G, F)} & \mathcal{O}_{\text{full}}(M, \mathbb{Q}^{n-2}) \times H^1(M, \mathbb{R}^n) \end{array}$$

To complete the proof it suffices to show that the map (G, F) lifts to a continuous map $\phi : P \times [0, 1] \rightarrow \text{CMI}_{\text{full}}(M, \mathbb{R}^n)$ such that

$$(3.2) \quad u = \phi|_{P \times \{0\}}, \quad G = \mathcal{G} \circ \phi, \quad \text{and} \quad F = \text{Flux} \circ \phi;$$

that is, the two triangles in (3.1) commute. Write

$$u(p, 0) = u_p^0, \quad G(p, t) = G_p^t, \quad \text{and} \quad F(p, t) = F_p^t \quad \text{for all } (p, t) \in P \times [0, 1].$$

Since the square (3.1) commutes, we have

$$(3.3) \quad \mathcal{G}(u_p^0) = G_p^0 \quad \text{and} \quad \text{Flux}(u_p^0) = F_p^0 \quad \text{for all } p \in P.$$

Fix a holomorphic 1-form θ vanishing nowhere on M . The Gauss map assignment $\mathcal{G} : \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ is naturally factored as

$$\text{CMI}_{\text{full}}(M, \mathbb{R}^n) \xrightarrow{\psi} \mathcal{O}_{\text{full}}(M, \mathbf{A}_*) \xrightarrow{\pi_*} \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}),$$

where $\psi(u) = 2\partial u/\theta$ and $\pi : \mathbf{A}_* = \{z \in \mathbb{C}_*^n : z_1^2 + \cdots + z_n^2 = 0\} \rightarrow \mathbf{Q}^{n-2}$ is the restriction of the canonical projection $\pi : \mathbb{C}_*^n = \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$. Here, $\mathcal{O}_{\text{full}}(M, \mathbf{A}_*)$ denotes the space of all holomorphic maps $f : M \rightarrow \mathbf{A}_* \subset \mathbb{C}^n$ that are *full* in the sense that the \mathbb{C} -linear span of $f(M)$ is all of \mathbb{C}^n . Recall that every holomorphic map $g : M \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ lifts to a holomorphic map $f : M \rightarrow \mathbb{C}_*^n$ such that $g = \pi \circ f = \pi_*(f)$; moreover, g is full if and only if f is full, while $g(M) \subset \mathbf{Q}^{n-2}$ if and only if $f(M) \subset \mathbf{A}_*$. Also recall that a conformal minimal immersion $u \in \text{CMI}(M, \mathbb{R}^n)$ is recovered from $\psi(u)$ by the integral formula

$$u(x) = u(x_0) + \Re \int_{x_0}^x \psi(u)\theta, \quad x \in M,$$

for any base point $x_0 \in M$, while

$$\text{Flux}(u)([C]) = \Im \int_C \psi(u)\theta, \quad [C] \in H_1(M, \mathbb{Z}).$$

Since $\mathcal{G} = \pi_* \circ \psi$, the first condition in (3.3) ensures that the following square of continuous maps commutes.

$$(3.4) \quad \begin{array}{ccc} P \times \{0\} & \xrightarrow{\psi \circ u} & \mathcal{O}_{\text{full}}(M, \mathbf{A}_*) \\ \downarrow & \nearrow \varrho & \downarrow \pi_* \\ P \times [0, 1] & \xrightarrow{G} & \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}) \end{array}$$

Since $\pi_* : \mathcal{O}_{\text{full}}(M, \mathbf{A}_*) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ is a fibration (see [3, Lemma 5.1]), the map G lifts to a continuous map $\varrho : P \times [0, 1] \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{A}_*)$ such that the two triangles in (3.4) commute:

$$(3.5) \quad \psi \circ u = \varrho|_{P \times \{0\}} \quad \text{and} \quad G = \pi_* \circ \varrho.$$

Write $\varrho(p, t) = \varrho_p^t$ for all $(p, t) \in P \times [0, 1]$. Theorem 2.1 provides a continuous family of holomorphic functions $\varphi_p^t : M \rightarrow \mathbb{C}^*$, $(p, t) \in P \times [0, 1]$, such that

$$(3.6) \quad \int_C \varphi_p^t \varrho_p^t \theta = \mathbf{i}F_p^t([C]) \in \mathbf{i}\mathbb{R}^n \quad \text{for all closed curves } C \subset M, (p, t) \in P \times [0, 1].$$

Furthermore, since $\varrho_p^0 = 2\partial u_p^0/\theta$ by the first condition in (3.5) and $F_p^0 = \text{Flux}(u_p^0)$ by the second condition in (3.3) for all $p \in P$, we have that

$$\int_C \varrho_p^0 \theta = \mathbf{i}F_p^0([C]) \quad \text{for all closed curves } C \subset M \text{ and all } p \in P,$$

so we can choose the family of maps φ_p^t so that

$$(3.7) \quad \varphi_p^0 = 1 \quad \text{for all } p \in P.$$

Fix a point $x_0 \in M$. It follows from (3.6) that the map $\phi_p^t : M \rightarrow \mathbb{R}^n$ defined by

$$\phi_p^t(x) = u_p^0(x_0) + \Re \int_{x_0}^x \varphi_p^t \varrho_p^t \theta, \quad x \in M,$$

is a well-defined full conformal minimal immersion (note that $\varphi_p^t \varrho_p^t : M \rightarrow \mathbf{A}_* \subset \mathbb{C}^n$ is holomorphic and full, and $\Re(\varphi_p^t \varrho_p^t \theta)$ is exact). We claim that the continuous map $\phi : P \times [0, 1] \rightarrow \text{CMI}_{\text{full}}(M, \mathbb{R}^n)$ given by $\phi(p, t) = \phi_p^t$ for all $(p, t) \in P \times [0, 1]$ satisfies the conditions in (3.2). Indeed, in view of (3.7) and the first condition in (3.5), we have

$$\phi_p^0(x) = u_p^0(x) + \Re \int_{x_0}^x \varrho_p^0 \theta = u_p^0(x) + \Re \int_{x_0}^x 2\partial u_p^0 = u_p^0(x), \quad x \in M, p \in P,$$

so $u = \phi|_{P \times \{0\}}$. On the other hand,

$$\mathcal{G}(\phi_p^t) = \pi_*(2\partial\phi_p^t/\theta) = \pi_*(\varphi_p^t \varrho_p^t) = \pi_*(\varrho_p^t) = G_p^t, \quad (p, t) \in P \times [0, 1],$$

where we have used that $\mathcal{G} = \pi_* \circ \psi$, the fact that φ_p^t takes values in \mathbb{C}^* , and the second condition in (3.5). Therefore, $G = \mathcal{G} \circ \phi$. Finally, (3.6) directly implies that $\text{Flux}(\phi_p^t) = F_p^t$ for all $(p, t) \in P \times [0, 1]$, that is, $F = \text{Flux} \circ \phi$. \square

We now turn to the proof of Theorem 1.2. First we need the following h-principle, which easily implies Corollary 1.3.

Theorem 3.1. *Let M be an open Riemann surface, $n \geq 3$, Q be a closed subset of a contractible finite CW-complex P , and $G : M \times P \rightarrow \mathbf{Q}^{n-2}$ be a continuous map such that $G_p := G(\cdot, p) \in \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ for all $p \in P$. For any $\mathcal{O}(M)$ -convex compact set $K \subset M$ and any $\epsilon > 0$ there is a homotopy $G^t : M \times P \rightarrow \mathbf{Q}^{n-2}$, $t \in [0, 1]$, satisfying the following conditions.*

- (i) $G_p^t := G^t(\cdot, p) : M \rightarrow \mathbf{Q}^{n-2}$ lies in $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ for all $(p, t) \in P \times [0, 1]$.
- (ii) $G_p^t = G_p$ for all $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$.
- (iii) $|G_p^t(x) - G_p(x)| < \epsilon$ for all $x \in K$ and $(p, t) \in P \times [0, 1]$.
- (iv) $G_p^t \in \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ for all $(p, t) \in (P \setminus Q) \times (0, 1]$.

In particular, if in addition $G_p \in \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ for all $p \in Q$, then we have $G_p^t \in \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ for all $(p, t) \in P \times (0, 1]$.

Proof. Since $\mathcal{G} : \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ is a fibration by Theorem 1.1 and P is contractible, we can lift the given map $G : M \times P \rightarrow \mathbf{Q}^{n-2}$ by \mathcal{G} to a map $u : M \times P \rightarrow \mathbb{R}^n$ such that $u_p = u(\cdot, p) \in \text{CMI}_{\text{full}}(M, \mathbb{R}^n)$ for all $p \in P$. By the parametric h-principle [5, Theorem 6.1(a)], for any $\delta > 0$, there is a homotopy $u^t : M \times P \rightarrow \mathbb{R}^n$, $t \in [0, 1]$, with the following properties.

- (a) $u_p^t := u^t(\cdot, p) : M \rightarrow \mathbb{R}^n$ lies in $\text{CMI}_{\text{full}}(M, \mathbb{R}^n)$ for all $(p, t) \in P \times [0, 1]$.
- (b) $u_p^t = u_p$ for all $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$.
- (c) $|u_p^t(x) - u_p(x)| < \delta$ for all $x \in K$ and $(p, t) \in P \times [0, 1]$.
- (d) $u_p^t \in \text{CMI}_{\text{full}}^c(M, \mathbb{R}^n)$ for all $p \in (P \setminus Q) \times (0, 1]$.

Setting $G_p^t = \mathcal{G}(u_p^t)$ with $\delta > 0$ small enough defines a homotopy as desired. Note in particular that if $G_p \in \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ for all $p \in Q$, then we cannot assert that u_p is complete for $p \in Q$, but we have $\mathcal{G}(u_p) = G_p \in \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ for all $p \in Q$, and hence (b) and (d) ensure that $G_p^t \in \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ for all $(p, t) \in P \times (0, 1]$. \square

Proof of Theorem 1.2. (a) Applying Theorem 3.1 with P a singleton and Q empty shows that the inclusion $j : \mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2}) \hookrightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ induces a surjection of path components. Applying Theorem 3.1 with P a closed ball of dimension $k \geq 1$ and Q the boundary sphere of P shows that j induces a monomorphism at the level of π_{k-1} and an epimorphism at the level of π_k .

(b) It suffices to show that the spaces $\mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ and $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ are ANRs. First, $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ is an ANR, being open in $\mathcal{O}(M, \mathbf{Q}^{n-2})$, which is ANR by [16, Theorem 9] since \mathbf{Q}^{n-2} is an Oka manifold. Second, since $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ is an ANR, Theorem 3.1 and [5, Proposition 5.2] imply that $\mathcal{O}_{\text{full}}^c(M, \mathbf{Q}^{n-2})$ is an ANR.

(c) It is not difficult to adapt the general position theorem [10, Theorem 5.4] for maps into \mathbf{A}_* to maps into \mathbf{Q}^{n-2} using the fact that if P is a contractible finite CW-complex, then every continuous map $P \rightarrow \mathcal{O}(M, \mathbf{Q}^{n-2})$ lifts by the projection $\pi : \mathbf{A}_* \rightarrow \mathbf{Q}^{n-2}$, whose fibre \mathbb{C}^* is Oka, to a continuous map $P \rightarrow \mathcal{O}(M, \mathbf{A}_*)$, and conclude that the inclusion $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}) \hookrightarrow \mathcal{O}(M, \mathbf{Q}^{n-2})$ is a weak homotopy equivalence.³ By the basic Oka principle, the inclusion $\mathcal{O}(M, \mathbf{Q}^{n-2}) \hookrightarrow \mathcal{C}(M, \mathbf{Q}^{n-2})$ is also a weak homotopy equivalence. Finally, if M has finite topological type, then $\mathcal{C}(M, \mathbf{Q}^{n-2})$ is an ANR (see [16, Proposition 7] and the references in its proof). \square

³For the hyperquadric \mathbf{A} (as opposed to a more general cone A as in [10, Theorem 5.4]), the notion of nondegeneracy in [1] and [10] is equivalent to nonflatness by [2, Lemma 2.3]. To adapt [10, Theorem 5.4] to full maps in place of nonflat maps, in its proof, simply invoke the proof of [2, Theorem 3.1(a)] instead of the proof of [1, Theorem 2.3(a)] (the latter theorem is incorrectly referred to as Theorem 3.2(a) in [10]). Beware that fullness is called nondegeneracy in [2].

4. The fibre of the Gauss map assignment

As before, we let M be an open Riemann surface and $n \geq 3$. By Theorem 1.1, the Gauss map assignment $\mathcal{G} : \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ is a Serre fibration. As shown already, $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ has the weak homotopy type of $\mathcal{C}(M, \mathbf{Q}^{n-2})$. Moreover, \mathbf{Q}^{n-2} is simply connected: for $n = 3$, \mathbf{Q}^{n-2} is isomorphic to the Riemann sphere; for $n \geq 4$, we invoke the fact that a smooth hypersurface in $\mathbb{C}\mathbb{P}^{n-1}$ is simply connected. It follows that $\mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ is path connected, so the fibres of $\mathcal{G} : \text{CMI}_{\text{full}}(M, \mathbb{R}^n) \rightarrow \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2})$ all have the same weak homotopy type. In this section, we shall determine this homotopy type.

Recall the factorisation

$$\text{CMI}_{\text{full}}(M, \mathbb{R}^n) \xrightarrow{\psi} \mathcal{O}_{\text{full}}(M, \mathbf{A}_*) \xrightarrow{\pi_*} \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}),$$

of \mathcal{G} used in Section 3 in the proof of Theorem 1.1, where $\psi(u) = 2\partial u/\theta$, with θ being a nowhere-vanishing holomorphic 1-form on M , and π_* is induced by the projection π from the punctured null quadric \mathbf{A}_* in \mathbb{C}^n onto the hyperquadric \mathbf{Q}^{n-2} in $\mathbb{C}\mathbb{P}^{n-1}$. The projection $\pi : \mathbf{A}_* \rightarrow \mathbf{Q}^{n-2}$ is a fibre bundle with fibre \mathbb{C}^* . Note that \mathcal{G} and π_* are canonically defined, whereas ψ depends on the choice of θ .

In the proof of Theorem 1.1, we used the result that π_* is a fibration [3, Lemma 5.1]. In fact, the lemma shows that $\pi_* : \mathcal{O}(M, \mathbf{A}_*) \rightarrow \mathcal{O}(M, \mathbf{Q}^{n-2})$ is a fibration. Its fibre F_0 , well defined up to weak homotopy equivalence, is $\mathcal{O}(M, \mathbb{C}^*)$ or, by the basic Oka principle, $\mathcal{C}(M, \mathbb{C}^*)$. Hence, F_0 has the weak homotopy type of a countably infinite disjoint union of circles, unless M is the plane or the disc, in which case F_0 has the weak homotopy type of a circle. We also need the result that ψ is a weak homotopy equivalence. This follows, by an argument similar to the proof of [10, Theorem 5.6], from the parametric h-principle [10, Theorem 5.3] (the version with vanishing *real* periods) adapted to full maps in place of nonflat maps.

Let F be the fibre of \mathcal{G} and consider the following commuting diagram.

$$\begin{array}{ccccc} F & \hookrightarrow & \text{CMI}_{\text{full}}(M, \mathbb{R}^n) & \xrightarrow{\mathcal{G}} & \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}) \\ \downarrow \psi & & \downarrow \psi & & \parallel \\ F_0 & \hookrightarrow & \mathcal{O}_{\text{full}}(M, \mathbf{A}_*) & \xrightarrow{\pi_*} & \mathcal{O}_{\text{full}}(M, \mathbf{Q}^{n-2}) \end{array}$$

The associated long exact sequences of homotopy groups show that ψ induces a weak homotopy equivalence $F \rightarrow F_0$, so Theorem 1.7 is proved.

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Antonio Alarcón

Departamento de Geometría y Topología e Instituto de Matemáticas (IMAG),
Universidad de Granada, Campus de Fuentenueva s/n, E-18071 Granada, Spain

e-mail: alarcon@ugr.es

Finnur Lárusson

School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005,
Australia

e-mail: finnur.larusson@adelaide.edu.au