ELLIPTICITY AND HYPERBOLICITY
IN GEOMETRIC COMPLEX ANALYSIS

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ABSTRACT. The purpose of these notes is to introduce third year and honours students to the notions of elliptic and hyperbolic complex manifolds in the accessible setting of domains in the complex plane $\mathbb{C}$. The main theme is the contrasting complex analytic nature of $\mathbb{C}^*$, the plane with one point removed, and $\mathbb{C}^{**}$, the plane with two points removed. We start with a review of the several equivalent definitions and some of the basic properties of holomorphic functions. We then introduce the Kobayashi semidistance, show that $\mathbb{C}^{**}$ is Kobayashi hyperbolic using the Ahlfors-Schwarz lemma, and thus derive Picard's little theorem. Finally we explore the ellipticity (in the sense of Gromov) of $\mathbb{C}^*$. Using the solvability of the Cauchy-Riemann equation on an arbitrary domain X in $\mathbb{C}$, discussed but not proved in detail here, we show that every continuous map $X \to \mathbb{C}^*$ can be deformed to a holomorphic map.

CONTENTS

1. HOLOMORPHIC FUNCTIONS

We are all familiar with the characterisation of the field $\mathbb{R}$ of real numbers as the unique (up to an appropriate notion of isomorphism, of course) ordered field in which every nonempty set which is bounded above has a supremum. A *valued field* is a field F with a map $|\cdot| : F \to [0, \infty)$ called a *valuation* or an *absolute value* such that for all $x, y \in F$:

- $|x| = 0$ if and only if $x = 0$,
- $|x + y| \leq |x| + |y|$,
- $|xy| = |x||y|$.

The valuation, or the valued field itself, is said to be:

- *nontrivial* if $|x| \neq 1$ for some $x \neq 0$,
- *complete* if the metric $(x, y) \mapsto |x - y|$ on F is complete,
- *Archimedean* if for all $x, y \in F$, $x \neq 0$, there is $n \in \mathbb{N}$ with $|nx| > |y|$.

Clearly, $\mathbb{R}$ itself is a nontrivial complete Archimedean valued field. This is the structure on which real analysis is based. It is a theorem of Ostrowski that there is precisely one

other such field, namely the field $\mathbb{C}$ of complex numbers with the usual absolute value. It makes sense, then, to try to develop analysis over $\mathbb{C}$ as well.

The first step could be to copy the definition of the derivative in real analysis and say that a function $f : X \to \mathbb{C}$, defined on a subset X of $\mathbb{C}$ which might as well be a domain, that is, nonempty, open, and connected, is *complex differentiable* at a point $c \in X$ if the *derivative*

$$f'(c) = \lim_{z \to c \text{ in } \mathbb{C}} \frac{f(z) - f(c)}{z - c}$$

exists in $\mathbb{C}$. If f is complex differentiable at every point of X, then we say that f is *analytic* or *holomorphic* on X. As in real analysis, we can show that holomorphic functions on X form a $\mathbb{C}$-algebra, usually denoted $\mathcal{O}(X)$, and prove a product rule, a quotient rule, etc.

If we think of f as a pair (u, v) of real-valued functions of two real variables, that is, we forget about multiplication by i and treat $\mathbb{C}$ as $\mathbb{R}^2$ via the real-linear correspondence $x + iy \leftrightarrow (x, y)$, then the existence of $f'(c)$ easily implies that the partial derivatives of u and v exist at c and satisfy the *Cauchy-Riemann equations*

$$u_x = v_y, \qquad u_y = -v_x.$$

Moreover, $f' = u_x + iv_x$. Thus, if f is differentiable in the real sense, then complex differentiability of f at c means precisely that the real derivative of f at c, as a real linear map $\mathbb{R}^2 \to \mathbb{R}^2$, is in fact complex linear. Using the differential operators

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right), \qquad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right),$$

we get $f' = \partial f$ and the Cauchy-Riemann equations boil down to $\bar{\partial} f = 0$. Thus $\bar{\partial}$ is sometimes called the *Cauchy-Riemann operator*. The *Laplacian* is the operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

A function u is called *harmonic* if $\Delta u = 0$. Real and imaginary parts of holomorphic functions are harmonic. Note that if f is holomorphic and has no zeros, then

$$\Delta \log |f| = 2(\log(f\bar{f}))_{z\bar{z}} = 2\left(\frac{f'\bar{f}}{f\bar{f}} \right)_{\bar{z}} = 0$$

because f'/f is holomorphic, so $\log |f|$ is harmonic.

Now let $f : X \to \mathbb{C}$ be holomorphic, and assume in addition that f is continuously differentiable in the real sense (with some extra work this assumption can be avoided). If D is an open disc with boundary circle C such that $\bar{D} \subset X$, we can then apply Green's theorem and obtain *Cauchy's integral theorem*:

$$\oint_C f\,dz = \oint_C (u + iv)(dx + idy) = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy$$

$$= \iint_D ((-v)_x - u_y)dx\,dy + i\iint_D (u_x - v_y)dx\,dy = 0,$$

where the last equality follows from the Cauchy-Riemann equations.

If $z \in D$, we can apply Green's theorem as above to the path integral

$$\oint_{\partial D_\epsilon} \frac{f(\zeta)}{\zeta - z}d\zeta,$$

where D_ϵ is D with a closed disc of a small radius $\epsilon > 0$ centred at z removed, so the integrand is holomorphic on a neighbourhood of $\bar{D}_\epsilon$, and conclude that this integral vanishes, that is,

$$\oint_C \frac{f(\zeta)}{\zeta - z}d\zeta = \oint_{C_\epsilon} \frac{f(\zeta)}{\zeta - z}d\zeta,$$

where C_ϵ is the circle of radius ϵ centred at z. Now

$$\oint_{C_\epsilon} \frac{f(\zeta)}{\zeta - z}d\zeta = \oint_{C_\epsilon} \frac{f(z)}{\zeta - z}d\zeta + \oint_{C_\epsilon} \frac{f(\zeta) - f(z)}{\zeta - z}d\zeta.$$

As $\epsilon \to 0$, the second term goes to zero (why?), whereas the first term is independent of ϵ. Namely, using the parametrisation $\zeta(t) = z + \epsilon e^{it}$, $t \in [0, 2\pi]$, for C_ϵ,

$$\oint_{C_\epsilon} \frac{d\zeta}{\zeta - z} = \int_0^{2\pi} \frac{\zeta'(t)dt}{\zeta(t) - z} = \int_0^{2\pi} \frac{\epsilon i e^{it}dt}{\epsilon e^{it}} = 2\pi i.$$

This proves *Cauchy's integral formula*:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}d\zeta, \qquad z \in D.$$

Cauchy's integral formula has massive consequences. Let us list a few. First of all, we see that the values of f *inside* C are determined by its values *on* C. Second, if we denote the centre of C by a and write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} \frac{1}{1 - \dfrac{z - a}{\zeta - a}} = \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\zeta - a)^{n+1}}$$

for $z \in D$ and $\zeta \in C$, then Cauchy's integral formula gives

$$f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n \text{ for } z \in D, \text{ with } a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - a)^{n+1}}d\zeta.$$

(There is a convergence issue here, of course, that must and can be addressed.) In other words, a holomorphic function is complex differentiable infinitely often, and on any open disc centred at a whose closure lies in its domain, it is the sum of its Taylor series centred at a. This seems like a miracle: think of the vast gulf between once differentiable functions and real analytic functions in real analysis. The fact that holomorphic functions are so closely related to polynomials gives complex analysis a strong algebraic flavour.

Third, differentiating under the integral in Cauchy's integral formula (which can be justified, as in real analysis) yields the integral formulas

$$f^{(n)}(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta, \qquad z \in D,$$

for all the derivatives of f. As an immediate consequence we obtain the following result, which comes as a big surprise to most of us when we see it for the first time. By an *entire function* we mean a holomorphic function defined on the whole complex plane.

Theorem 1 (Liouville's Theorem). *A bounded entire function is constant.*

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and bounded, say $|f| \le M$. Let C_r denote the circle of radius $r > 0$ centred at the origin, with parametrisation $\zeta(t) = re^{it}$, $t \in [0, 2\pi]$. Let $z \in \mathbb{C}$. For $r > |z|$, we have

$$|f'(z)| = \left| \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{rM\,dt}{|re^{it} - z|^2}.$$

Now $\dfrac{r}{|re^{it} - z|^2} \to 0$ uniformly on $[0, 2\pi]$ as $r \to \infty$. Thus, $f' = 0$ and f is constant. $\square$

Liouville's theorem yields one of the many proofs that the complex field is algebraically closed.

Corollary 1 (Fundamental Theorem of Algebra). *Every nonconstant complex polynomial has a root.*

Proof. Let P be a nonconstant polynomial with no zeros. Then the reciprocal $1/P$ is entire. It is easily seen that $|P(z)| \to \infty$, so $1/P(z) \to 0$, as $z \to \infty$. Hence $1/P$ is bounded, so $1/P$, and thus P, is constant by Liouville's theorem, which is absurd. $\square$

Liouville's theorem says that the image of a nonconstant entire function f cannot be bounded. It is natural to ask, then, how small the image can be. Clearly, f need not be surjective: the complex exponential map takes $\mathbb{C}$ onto $\mathbb{C} \setminus \{0\}$. In the next section, we will show that the image cannot be any smaller than this. We will prove Picard's little theorem, which states that an entire function omitting at least two values must be constant.

2. HYPERBOLICITY: THE TWICE PUNCTURED PLANE

We are all familiar with the notion of a metric space. A *metric space* is a set X with a function $d : X \times X \to [0, \infty)$, called a *metric* or a *distance* on X, such that for all $x, y, z \in X$:

(1) $d(x, y) = 0$ iff $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \le d(x, z) + d(z, y)$ (the triangle inequality).

A function d satisfying properties (2) and (3) is called a *pseudodistance* or a *semidistance* on X. A semidistance is *nondegenerate* if it is a distance, that is, if it satisfies (1).

An important way to define a distance on a smooth manifold or a complex manifold is via a Riemannian metric or a Hermitian metric, respectively, on the tangent bundle of the manifold. In the case of a domain X in $\mathbb{C}$ this can be explained very simply. A *metric* on X is a smooth function $\sigma : X \to (0, \infty)$. The *length* of a smooth curve $\gamma : [0, 1] \to X$ with respect to σ is

$$\ell_\sigma(\gamma) = \int_0^1 |\gamma'(t)|\sigma(\gamma(t))\,dt.$$

With $\sigma = 1$ we get the usual Euclidean length of γ. Setting, for $x, y \in X$,

$$d_\sigma(x, y) = \inf_\gamma \ell_\sigma(\gamma),$$

where γ runs through all smooth curves $[0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, defines a distance on X (why?), called the *distance induced by* σ. With $\sigma = 1$ on $X = \mathbb{C}$ we get the usual Euclidean distance. For which other domains does $\sigma = 1$ induce the Euclidean distance? Don't assume that every distance is induced by a metric.

Let $f : X \to Y$ be holomorphic map and let σ be a metric on Y. The *pullback* or *preimage* of σ by f is the function $f^*\sigma = (\sigma \circ f)|f'| : X \to [0, \infty)$. The pullback is not a metric if f has critical points, but it can still be used to define a semidistance as above. When is the semidistance nondegenerate?

The *Poincaré metric* on the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is given by the formula

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

With this metric, $\mathbb{D}$ becomes a famous model of non-Euclidean geometry, in which the distance-minimising curves (*geodesics*) are circular arcs perpendicular to the boundary circle. We will not need this fact; you can read about it in Krantz's book [6]. What we do need is the following fundamental result on the relationship between the non-Euclidean geometry of $\mathbb{D}$ and complex analysis on $\mathbb{D}$.

Theorem 2 (Ahlfors-Schwarz Lemma, special case). *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then*

$$f^*\rho \le \rho.$$

Hence, for all $x, y \in \mathbb{D}$,

$$d_\rho(f(x), f(y)) \le d_{f^*\rho}(x, y) \le d_\rho(x, y).$$

Proof. We prove the first inequality and leave the double inequality as an exercise. For $0 < r < 1$, let $\mu_r : r\mathbb{D} \to \mathbb{D}$, $z \mapsto z/r$, and $\rho_r = \mu_r^*\rho$, $z \mapsto \dfrac{r}{r^2 - |z|^2}$. Fix r and let

$$v = \frac{f^*\rho}{\rho_r} = \frac{|f'|(r^2 - |z|^2)}{r(1 - |f|^2)} \qquad \text{on } r\mathbb{D}.$$

Then $v \ge 0$ is continuous on $r\mathbb{D}$, smooth except at the critical points of f, and $v(z) \to 0$ as $|z| \to r^-$. Hence v has a maximum at some point $a \in r\mathbb{D}$. We need to show that $v(a) \le 1$; then we let $r \to 1^-$ and the proof is complete.

If $v(a) = 0$, then we're done. If not, $\log v$ is smooth near a and has a maximum at a. Hence, by the second derivative test, the trace of the Hessian of $\log v$, that is, the Laplacian of $\log v$, is nonpositive at a. But, using the fact that $\log |f'|$ is harmonic near a, we calculate that

$$0 \ge \tfrac{1}{4}\Delta \log v = (\log v)_{z\bar{z}} = \cdots = (f^*\rho)^2 - \rho_r^2$$

at a, so $v(a) \le 1$. $\qquad\square$

We now come to the central concept of this section, the *Kobayashi semidistance* on a domain X in $\mathbb{C}$. We define it as the largest semidistance d_X on X such that for every holomorphic map $f : \mathbb{D} \to X$ and all $z, w \in \mathbb{D}$,

(S) $$d_X(f(z), f(w)) \le d_\rho(z, w).$$

We call (S) the *distance-decreasing property* or the *shrinking property*.

Does this definition make sense? Is there a *largest* semidistance satisfying (S)? Take $x, y \in X$. Let $x_0, x_1, \ldots, x_m \in X$ with $x_0 = x$ and $x_m = y$, and let $f_j : \mathbb{D} \to X$, $j = 1, \ldots, m$, be holomorphic with $f_j(z_j) = x_{j-1}$, $f_j(w_j) = x_j$ for some $z_j, w_j \in \mathbb{D}$. Let $d_X(x, y)$ be the infimum of $d_\rho(z_1, w_1) + \cdots + d_\rho(z_m, w_m)$ for all such sequences of points and maps. You can check that the function $d_X : X \times X \to [0, \infty)$ thus defined has the required properties, namely:

- d_X is a semidistance on X.

- d_X satisfies (S).
- If d is any other semidistance on X satisfying (S), then $d \leq d_X$.

Consider two examples. If $x, y \in \mathbb{C}$ and $0 < \epsilon < 1$, then $f : \mathbb{D} \to \mathbb{C}$, $f(z) = (y - x)z/\epsilon + x$ is holomorphic and $f(0) = x$, $f(\epsilon) = y$. Thus $d_{\mathbb{C}}(x, y) \leq d_\rho(0, \epsilon) \to 0$ as $\epsilon \to 0$. Hence the Kobayashi semidistance on $\mathbb{C}$ is identically zero. On the other hand, for $\mathbb{D}$, the Ahlfors-Schwarz lemma says that d_ρ satisfies (S). Also, taking f in (S) to be the identity map, we see that if d is a semidistance on $\mathbb{D}$ satisfying (S), then $d \leq d_\rho$. Hence, $d_{\mathbb{D}} = d_\rho$.

A domain X is said to be *Kobayashi hyperbolic* if d_X is nondegenerate, and X is said to be *Brody hyperbolic* if every holomorphic map $\mathbb{C} \to X$ is constant. Thus $\mathbb{D}$ is Kobayashi hyperbolic, while $\mathbb{C}$ is not. Also, by Liouville's theorem, every bounded domain is Brody hyperbolic. Convince yourself that both properties are preserved by passing to a subdomain. We leave the following result as an exercise.

Theorem 3. (a) *Let $f : X \to Y$ be holomorphic. Then*
$$d_Y(f(x), f(x')) \leq d_X(x, x') \qquad \text{for all } x, x' \in X.$$

(b) *A biholomorphism $X \to Y$ is an isometry with respect to d_X and d_Y.*

(c) *A Kobayashi hyperbolic domain is Brody hyperbolic.*

Because of (b), the Kobayashi semidistance, when it is nondegenerate, is often referred to as an *invariant metric*. There are several other invariant metrics of importance in complex analysis.

Picard's little theorem says that an entire function omitting at least two values must be constant. In other words, the twice punctured plane $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$ is Brody hyperbolic. (Note that it doesn't matter which two distinct points we remove from $\mathbb{C}$, the resulting domains are all the same up to biholomorphism.) This famous theorem has many proofs. Our approach is to show that $\mathbb{C}^{**}$ is Kobayashi hyperbolic. For this it suffices to find a nondegenerate semidistance d on $\mathbb{C}^{**}$ that satisfies the shrinking property (S). Then the largest such semidistance, that is, $d_{\mathbb{C}^{**}}$, must also be nondegenerate.

Let us look for a distance d on $\mathbb{C}^{**}$ satisfying (S) that is induced by a metric σ. We are then faced with the following question. How can we tell whether a metric σ induces a distance that satisfies (S)?

The answer involves the geometric notion of *curvature*. There are many notions of curvature in both real and complex differential geometry. Here we take the curvature of a metric $\sigma : X \to (0, \infty)$ on a domain X in $\mathbb{C}$ to be the function $\kappa_\sigma : X \to \mathbb{R}$ defined by the formula
$$\kappa_\sigma = -\frac{\Delta \log \sigma}{4\sigma^2} = -\frac{(\log \sigma)_{z\bar{z}}}{\sigma^2}.$$
It is easily calculated that the curvature of the Euclidean metric on $\mathbb{C}$ is zero and the curvature of the Poincaré metric on $\mathbb{D}$ is -1. Also, a straightforward computation shows that curvature behaves well with respect to pullbacks.

Theorem 4. *Let X and Y be domains in $\mathbb{C}$ and let σ be a metric on Y. Let $f : X \to Y$ be a holomorphic map with no critical points, so $f^*\sigma$ is a metric on X. Then*
$$\kappa_{f^*\sigma} = \kappa_\sigma \circ f.$$

We can now prove the full Ahlfors-Schwarz lemma by an easy adaptation of our previous proof.

Theorem 5 (Ahlfors-Schwarz Lemma). *Let X be a domain in $\mathbb{C}$ and $f : \mathbb{D} \to X$ be holomorphic. Let σ be a metric on X with curvature $\kappa_\sigma \leq -1$. Then*

$$f^*\sigma \leq \rho.$$

Hence, for all $x, y \in \mathbb{D}$,

$$d_\sigma(f(x), f(y)) \leq d_{f^*\sigma}(x, y) \leq d_\rho(x, y),$$

so the distance d_σ satisfies the shrinking property. Therefore X is Kobayashi hyperbolic.

Proof. As before, we define $v = f^*\sigma/\rho_r$. We run the previous argument to the final computation, which now looks like this:

$$(\log v)_{z\bar{z}} = (\log f^*\sigma)_{z\bar{z}} - (\log \rho_r)_{z\bar{z}} = -\kappa_{f^*\sigma}(f^*\sigma)^2 + \kappa_{\rho_r}\rho_r^2$$
$$= -(\kappa_\sigma \circ f)(f^*\sigma)^2 + \kappa_{\rho_r}\rho_r^2 \geq (f^*\sigma)^2 - \rho_r^2,$$

using Theorem 4 and the fact that $\kappa_{\rho_r} = -1$. $\qquad\square$

The Ahlfors-Schwarz lemma is a powerful result. For example, it directly implies that the Poincaré metric is the largest metric on $\mathbb{D}$ with curvature not exceeding -1 (just take f to be the identity map). That is a pleasant characterisation of this important metric. It also implies that $\mathbb{C}$ has no metric with curvature bounded above by a negative constant (how?).

In order to prove Picard's little theorem it remains to come up with a metric σ on $\mathbb{C}^{**}$ with curvature $\kappa_\sigma \leq -1$. With some experimentation we could probably find one, but Krantz has gone to the trouble already (see [6], Ch. 2, Sec. 2).

Lemma 1. *The metric*

$$z \mapsto \left[\frac{(1 + |z|^{1/3})^{1/2}}{|z|^{5/6}}\right]\left[\frac{(1 + |z - 1|^{1/3})^{1/2}}{|z - 1|^{5/6}}\right]$$

on $\mathbb{C}^{**}$ has curvature bounded above by a negative constant.

Once we have verified the lemma—the calculation is not too bad—we can easily modify the metric so as to make its curvature bounded above by -1 (how?).

Corollary 2 (Picard's Little Theorem). *An entire function omitting at least two values is constant.*

Let us summarise the three ingredients in our proof of this result.

- The Ahlfors-Schwarz lemma. It is a nontrivial result, expressing a connection between analysis and geometry.
- A metric on $\mathbb{C}^{**}$ with curvature not exceeding -1. It takes some ingenuity to come up with such a metric.
- The Kobayashi semidistance. Once you've got the right idea, the basic properties of the Kobayashi semidistance are relatively easy to prove.

Finally, you should now be able to prove the following result.

Theorem 6. *A domain in $\mathbb{C}$ is Brody hyperbolic if and only if it is Kobayashi hyperbolic if and only if it is $\mathbb{C}$ with at least two points removed.*

The equivalence of Brody hyperbolicity and Kobayashi hyperbolicity holds for all Riemann surfaces (one-dimensional complex manifolds), but can fail in higher dimensions. Hyperbolicity problems in higher-dimensional complex geometry have been intensively studied in recent years. Many deep questions remain unanswered.

3. Ellipticity: the punctured plane

We would now like to contrast the hyperbolicity of the twice punctured plane $\mathbb{C}^{**}$ with the very different behaviour of the once punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, referred to as ellipticity. Various notions of ellipticity for complex manifolds were introduced by Mikhail L. Gromov, winner of the 2009 Abel Prize, in a seminal paper [2] in 1989, and have been developed further since. Loosely speaking, ellipticity is about being the target of "many" holomorphic maps from $\mathbb{C}$: it can thus be viewed as "anti-hyperbolicity". Ellipticity of a domain is also about holomorphic maps from other domains into it having the same sort of flexibility that continuous maps have. In other words, it is about a tight relationship between topology and complex analysis. The following result is an example of what this means. There are much stronger results in a similar spirit that we won't describe here.

Theorem 7. *Let X be a domain in $\mathbb{C}$. Every continuous map $X \to \mathbb{C}^*$ can be deformed to a holomorphic map.*

More explicitly, the theorem says that if $f : X \to \mathbb{C}^*$ is continuous, then there is a continuous map $F : X \times [0, 1] \to \mathbb{C}^*$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) : X \to \mathbb{C}^*$ is holomorphic.

Note that the theorem fails badly if $\mathbb{C}^*$ is replaced by $\mathbb{C}^{**}$. Take for example $X = \mathbb{C}^*$. By Picard's little theorem, every holomorphic map $X \to \mathbb{C}^{**}$ is constant (why?), but there are plenty of continuous maps $X \to \mathbb{C}^{**}$ that cannot be deformed to a constant map. If you know a little bit of algebraic topology, you know that the set of homotopy classes of continuous maps $X \to \mathbb{C}^{**}$ is in bijective correspondence with the fundamental group of $\mathbb{C}^{**}$, which is the free group on two generators and thus infinite.

Note also that the theorem is easily proved for domains whose fundamental group is finitely generated. Let us only consider the example $X = \mathbb{C} \setminus \{a_1, \ldots, a_n\}$, where $a_1, \ldots, a_n$ are distinct points. The homotopy class of a continuous map $f : X \to \mathbb{C}^*$ is determined by how many times the image by f of a little circle around each puncture wraps around the origin. If we denote this winding number for a_j by $k_j \in \mathbb{Z}$, then f is homotopic to the holomorphic map $z \mapsto \prod_{j=1}^{n} (z - a_j)^{k_j}$. For general domains a more sophisticated approach is required.

The proof of Theorem 7 requires some preparations. From the point of view of partial differential equations, complex analysis is the study of the Cauchy-Riemann operator $\bar{\partial}$, whose kernel consists of the holomorphic functions. It is useful in many circumstances to be able to solve the equation $\bar{\partial} u = v$, where v is a given function, say smooth. Note that the solution u, if it exists, will not be uniquely determined: the sum of u and any holomorphic function is also a solution. When v has compact support in a domain in $\mathbb{C}$, which we might as well take to be $\mathbb{C}$ itself, a smooth solution on $\mathbb{C}$ can be given by the explicit integral formula

$$u(\zeta) = \frac{1}{\pi} \iint_{\mathbb{C}} \frac{v(z)}{\zeta - z} \, dx dy.$$

The solution u given by this formula is the convolution $v * \Phi$ of v with the fundamental solution $\Phi(z) = \dfrac{1}{\pi z}$ for $\bar{\partial}$ on $\mathbb{C}$ (if you know what this means). It is not very hard to show that $\bar{\partial} u = v$ using Green's theorem (see [1], §13). If v is smooth on a domain X in $\mathbb{C}$, but not of compact support, more work is required. The strategy is to approximate

v by functions v_n of compact support, solve $\bar{\partial} u_n = v_n$ by the integral formula, and adjust the solutions u_n by adding to them suitable holomorphic functions so that they converge on X to a function u with $\bar{\partial} u = v$. We omit the details but record the result.

Theorem 8. *Let X be a domain in $\mathbb{C}$. If $v : X \to \mathbb{C}$ is a smooth function, then there is a smooth function $u : X \to \mathbb{C}$ such that $\bar{\partial} u = v$.*

The proof of Theorem 7 uses ideas from sheaf cohomology without mentioning these words (see [1], §12).

Proof of Theorem 7. Let X be a domain in $\mathbb{C}$ and $f : X \to \mathbb{C}^*$ be continuous. Each point of X has a neighbourhood on which f has a continuous logarithm, so there is an open cover (U_α) of X such that $f = e^{2\pi i \lambda_\alpha}$ on U_α for some continuous $\lambda_\alpha : U_\alpha \to \mathbb{C}$ (we throw in the factor $2\pi i$ for convenience and suppress the index set through which α runs). Then, for all α, β, the function $n_{\alpha\beta} = \lambda_\alpha - \lambda_\beta : U_{\alpha\beta} = U_\alpha \cap U_\beta \to \mathbb{Z}$ is locally constant because it is continuous and takes values in $\mathbb{Z}$. Clearly, $n_{\alpha\beta} + n_{\beta\gamma} = n_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ (this is called a *cocycle condition*).

Suppose that we could find *holomorphic* functions $\mu_\alpha : U_\alpha \to \mathbb{C}$ such that $n_{\alpha\beta} = \mu_\alpha - \mu_\beta$ on $U_{\alpha\beta}$ for all α, β. (We already have such a *splitting* of the *cocycle* $(n_{\alpha\beta})$ by the *continuous* functions λ_α.) Then we would get a well-defined holomorphic function $g : X \to \mathbb{C}^*$ by setting $g = e^{2\pi i \mu_\alpha}$ on U_α, and the formula

$$F(x,t) = \exp\left(2\pi i [(1-t)\lambda_\alpha(x) + t\mu_\alpha(x)]\right) \qquad \text{for } (z,t) \in U_\alpha \times [0,1]$$

would give a well-defined continuous map $F : X \times [0,1] \to \mathbb{C}^*$ with $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$, as desired.

So how do we split the cocycle $(n_{\alpha\beta})$ holomorphically? First we split it smoothly. We choose a partition of unity (ϕ_α) subordinate to the open cover (U_α). This means that for each α we have a smooth function $\phi_\alpha : X \to [0,1]$ such that:

- The support $\overline{\{x \in X : \phi_\alpha(x) \neq 0\}}$ of ϕ_α is contained in U_α.
- Every point of X has a neighbourhood on which ϕ_α is identically zero for all but finitely many α.
- $\sum_\alpha \phi_\alpha = 1$ on X.

For each α, γ, we note that $n_{\alpha\gamma}\phi_\gamma$ is a well-defined smooth function on U_α (why?), so the sum $\nu_\alpha = \sum_\gamma n_{\alpha\gamma}\phi_\gamma$ is a well-defined smooth function on U_α. Also, by the cocycle condition on $(n_{\alpha\beta})$,

$$\nu_\alpha - \nu_\beta = \sum_\gamma (n_{\alpha\gamma} - n_{\beta\gamma})\phi_\gamma = \sum_\gamma n_{\alpha\beta}\phi_\gamma = n_{\alpha\beta}\sum_\gamma \phi_\gamma = n_{\alpha\beta} \qquad \text{on } U_{\alpha\beta}.$$

From this smooth splitting we can obtain a holomorphic splitting. Namely, since $n_{\alpha\beta}$ is locally constant, we have $\bar{\partial}\nu_\alpha - \bar{\partial}\nu_\beta = \bar{\partial}n_{\alpha\beta} = 0$ on $U_{\alpha\beta}$, so we get a well-defined smooth function $v : X \to \mathbb{C}$ defined as $\bar{\partial}\nu_\alpha$ on U_α for each α. By Theorem 8, there is a smooth function $u : X \to \mathbb{C}$ with $\bar{\partial} u = v$. Finally, set $\mu_\alpha = \nu_\alpha - u$. Then μ_α is holomorphic on U_α and $\mu_\alpha - \mu_\beta = (\nu_\alpha - u) - (\nu_\beta - u) = n_{\alpha\beta}$ on $U_{\alpha\beta}$. $\qquad \square$

The crucial ingredient in this proof is the fact that the $\bar{\partial}$-equation can be solved on every domain X in $\mathbb{C}$. There is a large and important class of complex manifolds, called *Stein manifolds*, on which the $\bar{\partial}$-equation can be solved, and our proof extends verbatim to the case when X is a Stein manifold. It is not as easy to see how the theorem could be extended from $\mathbb{C}^*$ to more general targets. It follows from Gromov's

work that $\mathbb{C}^*$ can be replaced by any complex Lie group and, more generally, by any complex manifold that is elliptic in the sense defined in [2], but then our proof must be replaced by a much more sophisticated argument. Loosely speaking, the resulting theorem expresses a kind of duality between Stein manifolds and elliptic manifolds. This idea has been made precise by means of abstract homotopy theory in the papers [7] and [8].

4. FURTHER READING

There is a vast number of books on complex analysis in one dimension. I particularly like Remmert's two volumes [10] and [11]. Ransford's lovely volume [9] offers a different approach. Krantz's book [6] is a very readable account of some geometric aspects of complex analysis, hyperbolicity in particular, mainly in one dimension. I used it as a source for these notes. Once you have mastered basic complex analysis, it is time to move to $\mathbb{C}^n$ with $n \geq 2$ (this subject is known by the old-fashioned name of *several complex variables*) and to Riemann surfaces (one-dimensional complex manifolds), after which you can tackle the general analytic and geometric theory of complex manifolds. As a first source for several complex variables, I recommend part one of the Kaup brothers' book [4]. The theory of Riemann surfaces lies at a crossroads of many mathematical subjects, including algebraic, analytic, and differential geometry, analysis, and number theory: note the subtitle of Jost's book [3]! Jost's book, Forster's [1], and Kirwan's [5] are all different but all excellent.

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