DIFFERENTIAL GEOMETRY

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Lecture notes for an honours/masters course at the University of Adelaide

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1. Introduction

The goal of this course is to acquire familiarity with the concept of a smooth manifold. Roughly speaking, a smooth manifold is a space on which we can do calculus. Manifolds arise in various areas of mathematics; they have a rich and deep theory with many applications, for example in physics.

The three main questions to be addressed in the course are:

(1) What is a smooth manifold?
(2) How (and what) can we differentiate and integrate on a manifold?
(3) What is this generalisation of calculus good for?

A preliminary answer to (1) is: a smooth manifold X is a topological space that *locally* looks enough like Euclidean space $\mathbb{R}^n$ (for some n) that we can "transport" calculus from $\mathbb{R}^n$ to X. *Globally,* X may be very different from $\mathbb{R}^n$.

Examples of manifolds include the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and the 2-torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + 3 = 4\sqrt{x^2 + y^2}\}.$$

We will make precise the sense in which S^2 and T^2 "locally look like" the plane $\mathbb{R}^2$.

Question (2) is too technical to talk about now: the answer involves so-called differential forms.

As for question (3), here are a few sample applications of the theory of manifolds.

(a) It provides a unified framework for Green's theorem, Gauss' theorem (a.k.a. the divergence theorem), and Stokes' theorem from multivariable calculus. These theorems turn out to be special cases of one vast generalisation of the fundamental theorem of calculus.

(b) It provides a mathematical framework for general relativity.

(c) The sphere and the torus look "different": it seems clear that they should not be thought of as the same surface. We will be able to make this precise and prove it, using a suitable notion of "sameness" for manifolds. In other words, we will be able to express in a mathematically rigorous way the apparent fact that the torus has a hole in the middle but the sphere does not.

(d) Let $U \subset \mathbb{R}^2$ be open and $f = (f_1, f_2) : U \to \mathbb{R}^2$ be a smooth map (that is, infinitely differentiable). Let us ask: Is f a gradient? In other words, is there a smooth function $u : U \to \mathbb{R}$ such that

$$f_1 = \frac{\partial u}{\partial x_1}, \quad f_2 = \frac{\partial u}{\partial x_2} \quad \text{on } U?$$

This is a system of partial differential equations. If u exists, then necessarily

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_1} = \frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial f_2}{\partial x_1}.$$

It turns out that every smooth f satisfying this condition is a gradient on U if and only if U satisfies a certain topological condition that we can define and study using the tools to be developed in this course, in particular the powerful tool called *cohomology.*

This is a prototypical example of the interplay between partial differential equations and topology that has become a major theme in modern mathematics.

2. Differentiation

2.1. **Review of the basics.** We are familiar with the complete ordered field $\mathbb{R}$ of real numbers. We let $\mathbb{R}^n$ denote the set of ordered n-tuples $(x_1, \ldots, x_n)$ of real numbers. This set is a vector space over $\mathbb{R}$ in the familiar way, with an inner product

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n \quad \text{for } x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n),$$

a norm

$$\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

satisfying the triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

and the Cauchy-Schwarz inequality

$$|x \cdot y| \leq \|x\| \|y\|,$$

and a metric

$$d(x, y) = \|x - y\|$$

turning $\mathbb{R}^n$ into a complete metric space with a topology that has a basis consisting of all open balls $B(a, r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$ centred at $a \in \mathbb{R}^n$ with radius $r > 0$. By a *function* we always mean a map with target (codomain) $\mathbb{R}$.

We will now briefly review the basics of differentiation, following chapter 2 of Spivak (see the list of references at the end of these notes), which is an excellent source for the details we have left out. So is Rudin, chapter 9.

Here is the fundamental definition of differential calculus. A function $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}$, is differentiable at $a \in U$ with derivative $c \in \mathbb{R}$ if $\lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h} = c$. Equivalently, $(f(a+h) - f(a) - ch)/h \to 0$ as $h \to 0$. Viewing $h \mapsto ch$ as a linear map $\mathbb{R} \to \mathbb{R}$, we generalise this definition as follows.

A map $f : U \to \mathbb{R}^m$, where U is an open subset of $\mathbb{R}^n$, is *differentiable* at $a \in U$ if there is a linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\|f(a + h) - f(a) - \lambda(h)\| / \|h\| \to 0 \text{ as } h \to 0 \text{ in } \mathbb{R}^n.$$

Then λ is uniquely determined and is called the *derivative* of f at a, denoted $Df(a)$ or $f'(a)$. The $m \times n$ matrix of $f'(a)$ with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is called the *Jacobian matrix* of f at a. We often blur the distinction between the derivative and the Jacobian matrix and also denote the latter by $Df(a)$ or $f'(a)$.

It is easy to show that a constant map is differentiable at every point with derivative zero and that a linear map is its own derivative at every point. Also, differentiability implies continuity.

Differentiability is preserved by composition of maps and there is a simple formula for the derivative of the composition.

Theorem 2.1 (Chain rule). *If U is open in $\mathbb{R}^n$, V is open in $\mathbb{R}^m$, $f : U \to \mathbb{R}^m$ is differentiable at $a \in U$, $f(U) \subset V$, and $g : V \to \mathbb{R}^k$ is differentiable at $f(a)$, then $g \circ f : U \to \mathbb{R}^k$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \circ f'(a).$$

We can now show that a map $f = (f_1, \ldots, f_m) : U \to \mathbb{R}^m$, $U \subset \mathbb{R}^n$, is differentiable at $a \in U$ if and only if each component function $f_i : U \to \mathbb{R}$ is, and $f'(a)$ is the $m \times n$ matrix whose i-th row is $f_i'(a)$. We can also prove the familiar formulas for the derivatives of a sum, a product, and a quotient. It follows, in Spivak's words, that we are now assured of the differentiability of those maps whose component functions are obtained by addition, multiplication, division, and composition from the projections $\pi_i : (x_1, \ldots, x_n) \mapsto x_i$ (which are linear transformations) and the functions that we can already differentiate by elementary calculus.

Next we come to the problem of actually calculating derivatives of functions of more than one variable. This is done using partial derivatives. If $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$, and $a \in U$, then the derivative of the function $t \mapsto f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n)$ at a_i, if it exists, is called the i-th *partial derivative* of f at a, denoted $D_i f(a)$. So partial derivatives can be computed by elementary calculus. If $D_i f(x)$ exists for every $x \in U$, then we have a function $D_i f : U \to \mathbb{R}$. The j-th partial derivative of this function at x, that is $D_j(D_i f)(x)$, if it exists, is denoted $D_{i,j} f(x)$.

Theorem 2.2. *If $D_{i,j} f$ and $D_{j,i} f$ are continuous on a neighbourhood of a, then $D_{i,j} f(a) = D_{j,i} f(a)$.*

Higher-order partial derivatives are defined in the obvious way. A function $U \to \mathbb{R}$, $U \subset \mathbb{R}^n$, is called *k times continuously differentiable*, or simply C^k, $k \geq 1$, if all its partial derivatives up to and including order k exist and are continuous on U. It is called *smooth* or C^∞ or *infinitely differentiable* if it has partial derivatives of all orders at every point of U (they are then continuous on U). When calculating partial derivatives of a smooth function, the order of differentiation is immaterial.

Theorem 2.3. (1) *If $f : U \to \mathbb{R}^m$, $U \subset \mathbb{R}^n$, is differentiable at $a \in U$, then the partial derivatives $D_j f_i(a)$ exist for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and the Jacobian matrix $f'(a)$ is the $m \times n$ matrix $(D_j f_i(a))$.*

(2) *If all $D_j f_i(x)$ exist for all x in a neighbourhood of a and if each function $D_j f_i$ is continuous at a, then f is differentiable at a.*

The converse of (1) fails. Even if $D_j f_i(a)$ exists for all i and j, we cannot conclude that f is differentiable at a: see exercise 2.1.

If f satisfies the hypothesis in (2), then f is said to be *continuously differentiable* at a. This is stronger than differentiability at a, that is, the converse of (2) fails in general. The function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$ is differentiable, but f' is not continuous at 0.

2.2. **The inverse function theorem.** It is an important and nontrivial fact that if the derivative of a map f at a point a is invertible, then f itself is invertible on a neighbourhood of a. We shall prove this using the contraction principle, which, as you recall, can also be used to prove Picard's existence theorem for ordinary differential equations. Spivak gives a different proof.

Recall that if (X, d) is a metric space, a map $\phi : X \to X$ is called a *contraction* if there is a number $c < 1$ such that $d(\phi(x), \phi(y)) \leq cd(x, y)$ for all $x, y \in X$. The contraction principle states that a contraction ϕ of a complete metric space X has a unique fixed point. In fact, for any $a \in X$, the iterates $\phi^n(a)$ converge to the fixed point as $n \to \infty$.

Theorem 2.4 (Inverse function theorem)**.** *Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ be C^k, $1 \leq k \leq \infty$. Suppose $a \in X$ and $f'(a)$ is invertible. Then there are open neighbourhoods U of a and V of $f(a)$ in $\mathbb{R}^n$ such that $f : U \to V$ is a bijection and the inverse $f^{-1} : V \to U$ is C^k.*

Note that invertibility of $f'(a)$ is not only sufficient but also necessary for the conclusion of the theorem: if the conclusion holds, then $f^{-1} \circ f = \mathrm{id}_U$ implies $(f^{-1})'(f(a)) \circ f'(a) = I$ by the chain rule, so $f'(a)$ is invertible.

Proof. (From Rudin, pp. 221–223.) Put $A = f'(a)$ and choose λ so that $2\lambda\|A^{-1}\| = 1$. Since f' is continuous at a (see Theorem 2.3), there is an open ball $U \subset X$, centred at a, such that $\|f' - A\| < \lambda$ on U.

Fix $y \in \mathbb{R}^n$ and define a map ϕ on X by $\phi(x) = x + A^{-1}(y - f(x))$. Note that $f(x) = y$ if and only if x is a fixed point of ϕ. Now $\phi' = I - A^{-1}f' = A^{-1}(A - f')$, so $\|\phi'\| < \frac{1}{2}$ on U. Hence, by the mean value theorem, $\|\phi(x) - \phi(x')\| \leq \frac{1}{2}\|x - x'\|$ for $x, x' \in U$, so ϕ has at most one fixed point in U, that is, $f(x) = y$ for at most one $x \in U$. This shows that f is injective on U.

Next let $V = f(U)$ and pick $y_0 \in V$. Then $y_0 = f(x_0)$ for some $x_0 \in U$. Let B be an open ball with centre x_0 and radius $r > 0$ so small that $\overline{B} \subset U$. We will show that $y \in V$ whenever $\|y - y_0\| < \lambda r$, proving that V is open.

So take y with $\|y - y_0\| < \lambda r$. With ϕ as above,

$$\|\phi(x_0) - x_0\| = \|A^{-1}(y - y_0)\| < \|A^{-1}\|\lambda r = \frac{r}{2}.$$

If $x \in \overline{B}$, it follows that

$$\|\phi(x) - x_0\| \leq \|\phi(x) - \phi(x_0)\| + \|\phi(x_0) - x_0\| < \frac{1}{2}\|x - x_0\| + \frac{r}{2} \leq r,$$

so $\phi(x) \in B$. Thus ϕ is a contraction of $\overline{B}$. Being a closed subset of $\mathbb{R}^n$, $\overline{B}$ is complete. By the contraction principle, ϕ has a fixed point $x \in \overline{B}$. Then $y = f(x) \in f(\overline{B}) \subset f(U) = V$.

At this point we have shown that there are open neighbourhoods U of a and V of $f(a)$ such that $f : U \to V$ is a bijection. So far, we have only used differentiability of f on X, invertibility of $f'(a)$, and continuity of f' at a.

Now let $y, y + k \in V$ with preimages $x, x + h \in U$. With ϕ as above,

$$\phi(x + h) - \phi(x) = h + A^{-1}(f(x) - f(x + h)) = h - A^{-1}k,$$

so $\|h - A^{-1}k\| \leq \frac{1}{2}\|h\|$. Therefore $\|A^{-1}k\| \geq \frac{1}{2}\|h\|$ and $\|h\| \leq 2\|A^{-1}\|\|k\| = \lambda^{-1}\|k\|$, so as $k \to 0$, $h \to 0$. Since f' is continuous at a, after shrinking U if necessary, we may assume that $f'(x)$ is invertible for all $x \in U$. Let $T = f'(x)^{-1}$. Since

$$f^{-1}(y + k) - f^{-1}(y) - Tk = h - Tk = -T(f(x + h) - f(x) - f'(x)h),$$

we have

$$\frac{\|f^{-1}(y + k) - f^{-1}(y) - Tk\|}{\|k\|} \leq \frac{\|T\|}{\lambda}\frac{\|f(x + h) - f(x) - f'(x)h\|}{\|h\|}.$$

As $k \to 0$, the right hand side goes to zero, so the left hand side does as well, showing that f^{-1} is differentiable at y with $(f^{-1})'(y) = f'(f^{-1}(y))^{-1}$. If f is C^1, this formula shows that $(f^{-1})'$ is continuous on V, so f^{-1} is C^1 as well. We can now prove by induction that if f is C^k, then f^{-1} is C^k on V. Namely, suppose this holds for $k-1$ and that f is C^k. Then f^{-1} is C^{k-1} by the induction hypothesis. Also, f' is C^{k-1}, so, since the entries of the inverse of an invertible matrix are smooth functions of the entries of the matrix, $(f')^{-1}$ is C^{k-1}. Hence our formula shows that $(f^{-1})'$ is C^{k-1}, so f^{-1} is C^k. (Note the repeated application of the chain rule.) □

If U and V are open sets in $\mathbb{R}^n$, then a C^k map $U \to V$ with a C^k inverse is called a C^k *diffeomorphism*. By a *diffeomorphism* we will mean a C^∞ diffeomorphism. A map $f : U \to V$ is called a *local diffeomorphism* at $a \in U$ if there are open neighbourhoods U' of a and V' of $f(a)$ such that $f : U' \to V'$ is a diffeomorphism. So the inverse function theorem implies that a smooth map with an invertible derivative at a point p is a local diffeomorphism at p.

A smooth bijection need not have a differentiable inverse: consider $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$.

The map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (e^x \cos y, e^x \sin y)$, is smooth with invertible derivative at every point, so it is locally injective, but it is not (globally) injective. What is f in terms of complex numbers?

The inverse function theorem has two equivalent formulations that are often used. It is a good exercise to show that each of the three results implies the other two.

Theorem 2.5 (Implicit function theorem)**.** *Suppose f is a C^k map, $1 \leq k \leq \infty$, from a neighbourhood of (x_0, y_0) in $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^m$ such that the derivative of the map $y \mapsto f(x_0, y)$ is invertible at y_0. Write $c = f(x_0, y_0)$. Then there are open neighbourhoods U of x_0 and V of y_0 and a C^k map $g : U \to V$ such that for every $(x, y) \in U \times V$, we have $f(x, y) = c$ if and only if $y = g(x)$.*

The theorem tells us that the level set of f through (x_0, y_0) is, near (x_0, y_0), the graph of a C^k function of x. In other words, for x close to x_0, there is a unique y close to y_0 such that $f(x, y) = f(x_0, y_0)$, and, if f is C^k, this y depends C^k-differentiably on x. The reader should draw a picture of this.

By the *rank* of a differentiable map f at a point a we mean the rank of the derivative $f'(a)$ as a linear transformation, that is, the dimension of its image. By basic linear algebra, this is the largest number r such that $f'(a)$ has an invertible $r \times r$ submatrix.

Theorem 2.6 (Rank theorem)**.** *Let $f : X \to \mathbb{R}^m$ be a C^k map, $1 \leq k \leq \infty$, where X is open in $\mathbb{R}^n$, with rank s at every point of X. For every $a \in X$, there are open neighbourhoods $U \subset X$ of a and V of $f(a)$ with $f(U) \subset V$, and C^k diffeomorphisms $\phi : U \to U'$ and $\psi : V \to V'$, where U' is open in $\mathbb{R}^n$ and V' is open in $\mathbb{R}^m$, such that $\psi \circ f \circ \phi^{-1}$ has the simple form*

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_s, 0, \ldots, 0) \qquad \textit{for all } (x_1, \ldots, x_n) \in U'.$$

Clearly, the largest rank a map f as above can have is $r = \min\{n, m\}$. If f has rank r at a, then f has rank r at every point in a neighbourhood W of a (why?), so the rank theorem applies. If $n \leq m$, so $r = n$ and $f'(a)$ is injective, then f is said to be an *immersion* on W. Then the rank theorem shows that f is locally injective on W. If $n \geq m$, so $r = m$ and $f'(a)$ is surjective, then f is said to be a *submersion* on W. Then the rank theorem shows that $f(W)$ is open.

If $f'(a)$ is surjective for a map $f : X \to \mathbb{R}^m$ as above, that is, f has rank m at a, then a is called a *regular point* of f. Otherwise, a is called a *critical point* of f. Note that the set of critical points is closed. If $b \in \mathbb{R}^m$ and $f^{-1}(b)$ contains no critical points (for example if b is not in the image of f), then b is called a *regular value* of f; otherwise, b is called a *critical value*. What does the rank theorem tell us about the local structure of the preimage of a regular value?

3. Smooth manifolds

3.1. **Charts and atlases.** We now want to extend the preceding theory from the setting of $\mathbb{R}^n$ to more general spaces X on which it still makes sense to talk about smooth functions. Since differentiability is a local notion, we can transport it from $\mathbb{R}^n$ to X if every point of X has a neighbourhood with an identification with an open subset of $\mathbb{R}^n$. We just need to make sure that two such identifications on overlapping neighbourhoods define the same notion of smoothness. This leads to the following definitions.

A *chart* on a topological space X is a homeomorphism $\phi : U \to U'$, where U is an open subset of X and U' is an open subset of $\mathbb{R}^n$ for some n (which may depend on ϕ). Here, $n \geq 0$ and we interpret $\mathbb{R}^0$ as the trivial vector space $\{0\}$. We sometimes speak of the chart (U, ϕ) and call U a *coordinate neighbourhood* in X. If $a \in U$, we may call ϕ a chart *at* a.

An *atlas* $\mathcal{A}$ on X is a set of charts on X such that

(1) the domains of the charts in $\mathcal{A}$ cover X, and
(2) any two charts $\phi : U \to U'$ and $\psi : V \to V'$ in $\mathcal{A}$ are *compatible*, meaning that the composition $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is smooth.

By (2) applied to ψ and ϕ in the opposite order, $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is in fact a diffeomorphism.

The set of atlases on a space X is partially ordered by inclusion. Every atlas $\mathcal{A}$ is contained in a unique maximal atlas (an atlas contained in no strictly larger atlas), namely the set of all charts compatible with every chart in $\mathcal{A}$ (exercise).

A *smooth manifold* (or, for us, simply a *manifold*) is a second countable Hausdorff space X with a maximal atlas $\mathcal{A}$. We also refer to a maximal atlas on X as a *smooth structure* on X. Usually we do not describe a maximal atlas explicitly, but rather specify it by a small, explicit atlas contained in it.

As an exercise, show that if X is connected, then the number n above is the same for all charts (whose source and target are nonempty). It is called the *dimension* of X. The connected components of a disconnected manifold can have different dimensions. If all the connected components of a manifold X have the same dimension n, we write $\dim X = n$ and sometimes refer to X as *pure-dimensional*. Note that a 0-dimensional manifold is nothing but a countable set with the discrete topology.

The empty set $X = \varnothing$ with its unique topology and the empty atlas satisfies the definition of a smooth manifold. We do not assign a dimension to the empty manifold (although it would make some sense to set $\dim \varnothing = -\infty$).

Most of the theory in these notes, although valid for all manifolds, is only of interest for manifolds of dimension at least 1. It is only in section 6.1 that 0-dimensional manifolds start to play a role. Until then, to avoid tedious trivialities, we will implicitly assume that our manifolds have dimension at least 1.

The notion of a C^k manifold can be defined by an obvious modification of the above. In this course, we will only be concerned with C^∞ manifolds.

Here are some examples of manifolds.

(1) $\mathbb{R}^n$ with the atlas consisting of nothing but the identity map $\mathbb{R}^n \to \mathbb{R}^n$. We call this manifold n-dimensional *Euclidean space*.
(2) Any open subset of a given manifold X (called an *open submanifold* of X).
(3) The product space $X \times Y$, where X and Y are manifolds. If $\phi : U \to U' \subset \mathbb{R}^n$ is a chart on X and $\psi : V \to V' \subset \mathbb{R}^m$ is a chart on Y, then we take the map $U \times V \to U' \times V' \subset \mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$, $(x, y) \mapsto (\phi(x), \psi(y))$, to be a chart on $X \times Y$.
(4) The n-dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ (see exercise 3.2).
(5) The n-dimensional real projective space $\mathbb{RP}^n$ (see Boothby, pp. 61–63).

Work out or look up the details! We shall view an open subset U of $\mathbb{R}^n$ as a manifold with the atlas consisting of nothing but the identity map $U \to U$ (this is the *standard* smooth structure on U).

Let X be a manifold. A function $f : X \to \mathbb{R}$ is said to be *smooth* if for every chart $\phi : U \to U'$ on X, the composition $f \circ \phi^{-1} : U' \to \mathbb{R}$ is smooth in the usual sense. Then f is continuous. The set $C^\infty(X)$ of smooth functions $X \to \mathbb{R}$ is an algebra over $\mathbb{R}$, a subalgebra of the $\mathbb{R}$-algebra $C^0(X)$ of continuous functions $X \to \mathbb{R}$.

More generally, if X and Y are manifolds, a continuous map $f : X \to Y$ is called *smooth* if for every chart $\phi : U \to U'$ on X and $\psi : V \to V'$ on Y, the composition $\psi \circ f \circ \phi^{-1}$ is smooth on $\phi(U \cap f^{-1}(V))$ in the usual sense. If X and Y are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, with the standard smooth structures, this definition agrees with our original notion of smoothness (exercise). Clearly, the composition of smooth maps is smooth. Thus, a smooth map $f : X \to Y$ induces a map $f^* : C^\infty(Y) \to C^\infty(X)$ by precomposition: $f^*(h) = h \circ f$. This map is a morphism of $\mathbb{R}$-algebras.

Shortly (once you've learned Whitney's embedding theorem, Theorem 3.4), you will be able to show that if X and Y are manifolds and $f : X \to Y$ is a map (a priori not necessarily continuous) such that $h \circ f$ is smooth on X for every smooth function h on Y, then f is smooth. The following question is hard, but you might like to think about it. Is every $\mathbb{R}$-algebra morphism $C^\infty(Y) \to C^\infty(X)$ given by precomposition with a map $X \to Y$ (which then must be smooth)?

A smooth map between manifolds with a smooth inverse is called a *diffeomorphism.* We think of diffeomorphic manifolds as the "same" manifold. Using charts, we can extend the definitions of an immersion, submersion, regular and critical points and values, rank, etc. to smooth maps between manifolds. Note that, trivially, a chart is a diffeomorphism.

3.2. **Submanifolds and embeddings.** Let Y be a closed subset of an n-dimensional manifold X. We say that Y is a k-dimensional *(closed) submanifold* of X if Y can be covered by charts $\phi : U \to U'$ such that

$$Y \cap U = \{x \in U : \phi_i(x) = 0 \text{ for } i = k+1, \dots, n\}.$$

This means that in suitable coordinates, Y lies in X the way the linear subspace $\mathbb{R}^k \times \{0\}^{n-k}$ lies in $\mathbb{R}^n$. Then Y is a manifold in its own right with an atlas consisting of the charts

$$(\phi_1, \dots, \phi_k) : U \cap Y \to \{x \in \mathbb{R}^k : (x, 0) \in U'\},$$

and the inclusion $Y \hookrightarrow X$ is smooth (check!). This smooth structure on Y is called the structure *induced* from X.

The following result provides many important examples of submanifolds.

Theorem 3.1. *Let X and Y be manifolds and $f : X \to Y$ be a smooth map. If $c \in Y$ is a regular value of f, then $f^{-1}(c)$ is a submanifold of X.*

As will be evident from the proof, if X and Y are pure-dimensional, then

$$\dim f^{-1}(c) = \dim X - \dim Y,$$

unless $f^{-1}(c)$ is empty.

Proof. (Sketch.) First, since f is continuous, $f^{-1}(c)$ is closed in X. Take $a \in f^{-1}(c)$ and let ϕ and ψ be charts at a and c respectively such that $\phi(a) = 0$ and $\psi(c) = 0$. Then $g = \psi \circ f \circ \phi^{-1}$ is a submersion at 0, so by the rank theorem, by possibly changing the charts, $g(x_1, \dots, x_n) = (x_1, \dots, x_m)$. Here, n denotes the dimension of X at a and m the dimension of Y at c. Then, near a, $f^{-1}(c)$ consists of those points x for which $\phi_i(x) = 0$ for $i = 1, \dots, m$. $\square$

As an exercise, use this result to show that the sphere S^2 and the torus T^2 from the introduction are smooth submanifolds of $\mathbb{R}^3$.

An *embedding* of a manifold X into a manifold Y is a map $f : X \to Y$ whose image $f(X)$ is a submanifold of Y such that $f : X \to f(X)$ is a diffeomorphism (then f is smooth as a map into Y). The next result gives what is usually the easiest way to verify that a given map is an embedding.

Recall that a continuous map $f : X \to Y$ between topological spaces is called *proper* if $f^{-1}(K)$ is compact for every compact subset K of Y. For "nice" (how nice?) topological spaces, such as manifolds, an equivalent condition that is often easy to check is that if (x_n) is a sequence in X converging to ∞_X, the point at infinity in the one-point compactification of X (this simply means that (x_n) eventually leaves every compact subset of X), then $f(x_n) \to \infty_Y$. A homeomorphism is always proper.

Theorem 3.2. *A map $f : X \to Y$ between manifolds is an embedding if and only if it is a proper injective immersion.*

Find an example of an injective immersion $\mathbb{R} \to \mathbb{R}^2$ with a closed image, which is not proper and not an embedding.

Proof. $\Rightarrow$: Suppose f is an embedding. Being a diffeomorphism onto a submanifold of Y, it is an injective immersion. Also, f is the composition of a homeomorphism $X \to f(X)$ and the inclusion $f(X) \hookrightarrow Y$, which is proper because $f(X)$ is closed in Y, so f is proper.

$\Leftarrow$: Being proper, f is closed, that is, if $E \subset X$ is closed, then $f(E)$ is closed in Y (exercise). In particular, the image $f(X)$ is closed in Y. This shows that the continuous bijection $f : X \to f(X)$ is in fact a homeomorphism, where $f(X)$ is given the subspace topology induced from Y.

Let $b \in f(X)$, say $b = f(a)$. Since f is an immersion, by the rank theorem, there are charts ϕ on a neighbourhood U of a and ψ on a neighbourhood V of b such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots)$. Since $f : X \to f(X)$ is a homeomorphism, by shrinking U and V if need be, we may assume that $f(U) = V \cap f(X)$. We see, then, that $f(X)$ is a submanifold of Y, given in the chart ψ by the equations $x_k = 0$ for $k > n$, and that f and its inverse are smooth. □

Corollary 3.1. *If $f : X \to Y$ is an injective immersion and X is compact, then f is an embedding.*

Proof. We need to verify that f is proper. If K is compact in Y, then K is closed in Y since Y is Hausdorff. Hence $f^{-1}(K)$ is closed in X since f is continuous, and therefore compact since X is compact. □

3.3. **Partitions of unity and Whitney's embedding theorem.** Partitions of unity are a powerful and much-used tool for a variety of constructions involving manifolds. Loosely speaking, they allow us to form global objects by gluing together local objects or, alternatively, decompose a global object into a sum of locally supported pieces.

Let X be a manifold with an open cover $\mathcal{U} = (U_i)_{i \in I}$. A *partition of unity subordinate to* $\mathcal{U}$ is a family $(\rho_j)_{j \in J}$ of smooth functions $\rho_j : X \to [0, 1]$ with compact supports such that

(1) for each $j \in J$, there is $i \in I$ such that $\operatorname{supp} \rho_j \subset U_i$,
(2) for every compact subset $K \subset X$, we have $K \cap \operatorname{supp} \rho_j \neq \varnothing$ for only finitely many $j \in J$, and
(3) $\sum\limits_{j \in J} \rho_j = 1$ on X.

Recall that the *support* of a function $f : X \to \mathbb{R}$ is the closure of the set of $x \in X$ with $f(x) \neq 0$. Note that the sum in (3) is well defined in that all but finitely many of its terms are zero when restricted to any compact set.

Theorem 3.3. *If $\mathcal{U}$ is an open cover of a manifold X, then there is a partition of unity subordinate to $\mathcal{U}$.*

Assuming this result, as an exercise, show that if we do not require the functions ρ_i to have compact supports, then there is a partition of unity with $J = I$ and $\operatorname{supp} \rho_i \subset U_i$ for each $i \in I$.

The proof of the theorem proceeds through a number of steps.[1] We first recall the function $f : \mathbb{R} \to \mathbb{R}$ defined by setting $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. This is an example of an infinitely differentiable function which is not real-analytic (it does not equal the sum of its Taylor series at 0). From this function, we can construct many other useful functions, for instance so-called *bump functions* $g(x) = c_1 f(x - a) f(b - x)$, for $a < b$ with $c_1 = e^{4/(b-a)}$, and *cutoff functions* $h(x) = c_2 \int_{-\infty}^{x} g$, where $c_2^{-1} = \int_a^b g$. These functions are smooth. Draw their graphs!

Now let $(U_i)_{i \in I}$ be an open cover of a manifold X. Let $K \subset X$ be compact and $B \subset X$ be closed (possibly empty) with $K \cap B = \varnothing$. Let $x_0 \in K$, say $x_0 \in U_i$. Let ϕ be a chart on an open neighbourhood $V \subset U_i \cap X \setminus B$ of x_0, and find $\epsilon > 0$ such that $\overline{B}(\phi(x_0), \epsilon) \subset \phi(V)$. Let h be a smooth cutoff function as constructed above with

[1]This exposition is borrowed from notes of Nicholas Buchdahl.

$h(t) = 0$ for $t < \epsilon^2/4$ and $h(t) = 1$ for $t > \epsilon^2$ and set $h_{x_0}(x) = 1 - h(\|\phi(x) - \phi(x_0)\|^2)$ for $x \in V$, so h_{x_0} is identically 1 in a neighbourhood of x_0 and identically 0 outside a larger neighbourhood. Smoothly extend h_{x_0} to all of X by setting it equal to 0 outside V. Note that $\operatorname{supp} h_{x_0}$ is compact and contained in U_i.

Repeat this construction for each $x \in K$ and obtain a family $(h_x)_{x \in K}$ of smooth functions on X. Let $W_x = h_x^{-1}(1)^\circ$ (the circle means interior): this is an open neighbourhood of x. Then $(W_x)_{x \in K}$ is an open cover of K, so there is a finite subcover consisting of, say, $W_{x_1}, \ldots, W_{x_m}$.

If X is compact, do the above with $K = X$ and $B = \varnothing$ and set $\rho_j = h_{x_j}/h$ for $j = 1, \ldots, m$, where $h = h_{x_1} + \cdots + h_{x_m}$. Theorem 3.3 is thereby proved. We leave the rest of the proof for X noncompact as an exercise.

The following application of partitions of unity is one of the earliest and most fundamental results in the theory of smooth manifolds. It says that the manifolds we have defined in an abstract fashion are in fact no more general than submanifolds of Euclidean space. It should be emphasised, though, that a manifold may not come to us equipped with any natural embedding into Euclidean space. The fact that an embedding into Euclidean space exists is often not really relevant.

Theorem 3.4 (Whitney's embedding theorem)**.** *An n-dimensional smooth manifold can be embedded as a submanifold of $\mathbb{R}^{2n}$.*

The rough idea of the proof is this: A manifold comes equipped with diffeomorphisms of small open subsets onto open subsets of Euclidean space. We can use a partition of unity to piece these together to form an embedding of the whole manifold into Euclidean space of higher dimension.

Proof. We will only prove that a *compact* n-dimensional manifold X embeds in $\mathbb{R}^m$ for *some* m. As above, find a finite cover of X by charts (U_i, ϕ_i), $i = 1, \ldots, k$, and smooth functions $h_i : X \to [0, 1]$ such that $\operatorname{supp} h_i \subset U_i$ and the interiors of the sets $h_i^{-1}(1)$, $i = 1, \ldots, k$, cover X. Define smooth maps $\psi_i = h_i \cdot \phi_i : X \to \mathbb{R}^n$ (defined as zero outside of $\operatorname{supp} h_i$) and $f = (h_1, \ldots, h_k, \psi_1, \ldots, \psi_k) : X \to \mathbb{R}^{k(n+1)}$. Then f is an immersion because at each point x of X, at least one of the maps h_i equals 1 on a neighbourhood of x, so on that neighbourhood, $\psi_i = \phi_i$ is an immersion. Also, f is injective. Namely, if $x \neq x'$ are points in X, then $h_i(x) = 1$ for some i. If also $h_i(x') = 1$, so $x, x' \in U_i$, then $\psi_i(x) = \phi_i(x) \neq \phi_i(x') = \psi_i(x')$, so $f(x) \neq f(x')$. On the other hand, if $h_i(x') \neq 1$, then clearly $f(x) \neq f(x')$. □

Another famous result of Whitney's using a similar technique is the fact that every closed subset of a manifold is the zero set of a nonnegative smooth function on the manifold (see Madsen-Tornehave, p. 224).

4. Tangent spaces

4.1. **Germs, derivations, and equivalence classes of paths.** We have defined what it means for a function $f : X \to \mathbb{R}$ defined on a manifold X to be smooth, but we have said nothing about how to actually differentiate it. Our next task is to start the development of differential calculus on manifolds. If X is an open subset of $\mathbb{R}^n$, then we differentiate f at a point $a \in X$ by calculating the partial derivatives of f at a. In other words, we compute the rate of change of f in each of the coordinate directions at a, that is, we take the derivatives $(f \circ \gamma_i)'(0)$ for the paths $\gamma_i(t) = a + te_i$ through a, where $e_i \in \mathbb{R}^n$ is the i-th standard basis vector with 1 as its i-th coordinate and all other coordinates 0. On a manifold we have no preferred directions or paths. What we can do is simply look at the rate of change $(f \circ \gamma)'(0)$ of f along *all* smooth paths $\gamma : (-\epsilon, \epsilon) \to X$ with $\gamma(0) = a$.

These rates of change only depend on the restriction of f to any neighbourhood of a, however small. This observation prompts us to define an equivalence relation on the set $\bigcup C^\infty(U)$, where U runs through all open neighbourhoods of a in X, by setting $g \sim h$ for $g \in C^\infty(U)$ and $h \in C^\infty(V)$ if there is a neighbourhood $W \subset U \cap V$ of a with $g = h$ on W. An equivalence class is called a *germ* of a smooth function at a. We sometimes write f_a for the germ at a of a smooth function f on an open neighbourhood of a. In a natural way the set of germs at a is an $\mathbb{R}$-algebra (exercise). It is denoted C_a^∞ or, if X needs to be mentioned, $C_{X,a}^\infty$. Note that the value at a of a germ at a is well defined.

Thus, every smooth path $\gamma : (-\epsilon, \epsilon) \to X$ with $\gamma(0) = a$ gives a well-defined map $D_\gamma : C_a^\infty \to \mathbb{R}$, $[f] \mapsto (f \circ \gamma)'(0)$. This map has two important properties that you should check:

(1) D_γ is linear (as a map of real vector spaces).
(2) D_γ satisfies the *Leibniz rule*: for all germs $f, g \in C_a^\infty$,

$$D_\gamma(fg) = D_\gamma(f)g(a) + f(a)D_\gamma(g).$$

Such a map is called a *derivation* on the $\mathbb{R}$-algebra C_a^∞. The derivations of C_a^∞ naturally form a vector space (exercise) denoted $\operatorname{Der} C_a^\infty$. A derivation always takes a constant function to zero (why?).

The next thing to note is that although there is a vast number of smooth paths through a in X, they do not give all that many derivations. More precisely, smooth paths γ_1 and γ_2 with $\gamma_1(0) = \gamma_2(0) = a$ give the same derivation if there is a chart ϕ at a with $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$ (exercise: this is then true for every chart at a). Namely, if $f \in C_a^\infty$, then, by the chain rule,

$$D_{\gamma_k}(f) = (f \circ \gamma_k)'(0) = \big((f \circ \phi^{-1}) \circ (\phi \circ \gamma_k)\big)'(0) = (f \circ \phi^{-1})'(\phi(a))\,(\phi \circ \gamma_k)'(0),$$

and these are the same for $k = 1, 2$. (Here, as often, we denote by the same symbol a germ at a and a representative for it on an unspecified open neighbourhood of a.) So we are led to declaring two smooth paths $\gamma_k : (-\epsilon_k, \epsilon_k) \to X$ with $\gamma_k(0) = a$, $k = 1, 2$, equivalent if $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$ for some (or every) chart ϕ at a. We would like this condition to mean that γ_1 and γ_2 have the same velocity or the same tangent vector at a. Since the notion of a tangent vector to a manifold has not been defined yet, we have the opportunity to make this true by fiat! So we *define* the *tangent space* to X at a, denoted T_aX, to be the set of equivalence classes of smooth paths through a with respect to the equivalence relation just defined.

The fundamental result that ties the above definitions together is as follows.

Theorem 4.1. *The map $T_aX \to \operatorname{Der} C_a^\infty$, $[\gamma] \mapsto D_\gamma$, is a bijection.*

We equip T_aX with the unique vector space structure that makes the map linear and refer to elements of T_aX as *tangent vectors* to X at a.

Before proceeding to the proof of the theorem, we need a lemma.

Lemma 4.1. *Let f be a smooth function on an open ball B centred at the origin 0 in $\mathbb{R}^n$. Then there are smooth functions $f_1, \dots, f_n$ on B such that $f_i(0) = D_i f(0)$ and*

$$f(x) = f(0) + x_1 f_1(x) + \cdots + x_n f_n(x) \quad \textit{for } x \in B.$$

Proof. By the fundamental theorem of calculus and the chain rule,

$$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx_1, \dots, tx_n)\, dt = \sum_{i=1}^n x_i \int_0^1 D_i f(tx_1, \dots, tx_n)\, dt,$$

so we simply let

$$f_i(x) = \int_0^1 D_i f(tx_1, \dots, tx_n)\, dt.$$

We need to invoke a suitable result about differentiability of integrals; see for instance Rudin, Theorem 9.42. □

Proof of Theorem 4.1. We have already seen that the map is well defined: equivalent paths give the same derivation. To check injectivity, suppose $[\gamma_1]$ and $[\gamma_2]$ are distinct tangent vectors at a. This means that for any chart (U, ϕ) at a, $(\phi \circ \gamma_1)'(0) \neq (\phi \circ \gamma_2)'(0)$, so for at least one of the component functions $\phi_i : U \to \mathbb{R}$ of ϕ, we have

$$D_{\gamma_1}(\phi_i) = (\phi_i \circ \gamma_1)'(0) \neq (\phi_i \circ \gamma_2)'(0) = D_{\gamma_2}(\phi_i).$$

This shows that $D_{\gamma_1} \neq D_{\gamma_2}$.

Surjectivity is the main part of the proof. Suppose D is a derivation on C_a^∞. Let (U, ϕ) be a chart at a. Let f be a smooth function on an open neighbourhood V of a (or the germ at a thereof). By shrinking U and V if need be, we may assume that $U = V$ and that $B = \phi(U)$ is an open ball centred at the origin 0 in $\mathbb{R}^n$ with $\phi(a) = 0$.

By Lemma 4.1 applied to the smooth function $g = f \circ \phi^{-1}$ on B, there are smooth functions $g_1, \dots, g_n$ on B with $g_i(0) = D_i g(0)$ and

$$g(x) = g(0) + x_1 g_1(x) + \cdots + x_n g_n(x) \quad \text{for } x \in B.$$

Thus, since D takes constants to zero and $\phi_i(a) = 0$,

$$\begin{aligned} D(f) \;&=\; D(g \circ \phi) = D\big(f(a) + \sum_i \phi_i (g_i \circ \phi)\big) = \sum_i D(\phi_i) g_i(0) \\ &= \sum_i D(\phi_i) D_i (f \circ \phi^{-1})(0) = D_\gamma(f), \end{aligned}$$

where $\gamma(t) = \phi^{-1}(c_1 t, \dots, c_n t)$, $c_i = D(\phi_i)$ (go through this calculation yourself). Thus, D is in the image of $T_a X$. □

More can be learned from this proof. We have just expressed the arbitrary derivation D on C_a^∞ as a linear combination of the derivations $f \mapsto D_i(f \circ \phi^{-1})(0)$, $i = 1, \dots, n$, where ϕ is any chart at a. We denote these derivations by $\left.\dfrac{\partial}{\partial \phi_i}\right|_a$. They are linearly independent — taking $f = \phi_j$, we get $D_i(\phi_j \circ \phi^{-1})(0) = \delta_{ij}$, because $\phi_j \circ \phi^{-1}$ is simply the projection $x \mapsto x_j$ — so they form a basis for $\operatorname{Der} C_a^\infty$. In particular, the dimension of $T_a X$ equals the dimension of X at a. For the record,

$$D = \sum_i D(\phi_i) \left.\frac{\partial}{\partial \phi_i}\right|_a \qquad \text{for every } D \in T_a X.$$

Let us consider the case where $X = \mathbb{R}^n$, as always with the standard smooth structure, so we can take the chart ϕ to be the identity map $\mathbb{R}^n \to \mathbb{R}^n$, or a translation thereof if we want it to take a particular point a to the origin. The basis for $\operatorname{Der} C_a^\infty$ just described consists of the derivations $f \mapsto D_i(f \circ \phi^{-1})(0) = D_i f(a)$, that is, of the partial derivatives $D_1, \dots, D_n$. Also, two smooth paths γ_1 and γ_2 through a are equivalent if and only if $\gamma_1'(0) = \gamma_2'(0)$. Since there clearly is a path γ with $\gamma(0) = a$ and $\gamma'(0) = v$ for every $v \in \mathbb{R}^n$ (say $\gamma(t) = tv+a$), we obtain an isomorphism $\iota : T_a\mathbb{R}^n \to \mathbb{R}^n$, $[\gamma] \mapsto \gamma'(0)$. In other words, $T_a\mathbb{R}^n$ naturally corresponds to the tangent space to $\mathbb{R}^n$ at a as you may have previously envisaged it as the n-dimensional vector space of velocity vectors of paths in $\mathbb{R}^n$ through a.

Finally, then, here is how we differentiate a smooth function $f : X \to \mathbb{R}$ at a point $a \in X$. We define the *derivative* (some say *differential*) of f at a to be the linear map $d_a f : T_a X \to \mathbb{R}$ that simply evaluates a derivation on the germ of f at a. In other words, if γ is a smooth path in X with $\gamma(0) = a$, then $d_a f([\gamma]) = (f \circ \gamma)'(0)$. Roughly speaking, the rates of change of f along all paths through a are built into the linear map $d_a f$.

Let us express $d_a f$ in terms of a basis for the dual space $T_a^* X$ of linear maps (or functionals) $T_a X \to \mathbb{R}$ (we call $T_a^* X$ the *cotangent space* to X at a). A natural choice of basis for $T_a^* X$ is the basis dual to the basis $\{\partial/\partial\phi_1, \dots, \partial/\partial\phi_n\}$ for $T_a X$ given by a chart ϕ at a. The dual basis consists of the *covectors* $d\phi_1, \dots, d\phi_n$ (more properly $d\phi_i(a)$) defined in the previous paragraph and characterised by the condition $d\phi_i(\partial/\partial\phi_j) = \delta_{ij}$ for $i, j = 1, \dots, n$. We have

$$d_a f = \sum_{i=1}^n d_a f\big(\frac{\partial}{\partial\phi_i}\big) d\phi_i(a) = \sum_{i=1}^n \frac{\partial f}{\partial \phi_i}\bigg|_a d\phi_i(a).$$

If $X = \mathbb{R}^n$ and we identify $T_a\mathbb{R}^n$ with $\mathbb{R}^n$ by the isomorphism ι above, then the composition $d_a f \circ \iota^{-1} : \mathbb{R}^n \to T_a\mathbb{R}^n \to \mathbb{R}$,

$$v \mapsto d_a f(\iota^{-1}(v)) = (f \circ \iota^{-1}(v))'(0) = \frac{d}{dt} f(tv+a)\big|_{t=0} = D_a f(v),$$

identifies $d_a f$ with the derivative $D_a f$ of f at a as defined in multivariable calculus. When ϕ is the identity map of $\mathbb{R}^n$, it is customary to write $x_1, \dots, x_n$ for the component functions of ϕ and write $\partial/\partial x_i$ for $\partial/\partial\phi_i$ and dx_i for $d\phi_i$.

4.2. **The derivative of a smooth map.** Now let $f : X \to Y$ be a smooth map between manifolds. We define the *derivative* (sometimes we say *differential* or *tangent map*) of f at a point $a \in X$ to be the linear map $d_a f : T_a X \to T_{f(a)} Y$ that takes a derivation D in $T_a X$ to the derivation $h \mapsto D(h \circ f)$ in $T_{f(a)} Y$. Equivalently, $d_a f$ takes an equivalence class $[\gamma]$ of smooth paths through a to the class $[f \circ \gamma]$ (check!). Sometimes we write $T_a f$ or f_* for $d_a f$ (if suppressing a does not lead to confusion) and $f^* h$ for $h \circ f$. Then the defining formula takes the appealing form

$$f_* D(h) = D(f^* h).$$

In other words, the derivative acts by precomposing derivations on C_a^∞ by the precomposition map $f^* : C_{f(a)}^\infty \to C_a^\infty$, $h \mapsto h \circ f$.

There are a few things that need to be checked here (roll up your sleeves!). First, if h is the germ at $f(a)$ of a smooth function $\tilde{h}$ on an open neighbourhood V of $f(a)$, then $\tilde{h} \circ f$ is a smooth function on the open neighbourhood $f^{-1}(V)$ of a and its germ $f^*(h)$ at a only depends on h and not on the choice of the representative $\tilde{h}$. Hence, the map

$f^* : C^\infty_{f(a)} \to C^\infty_a$ is well defined. Second, f^* is linear (in fact a morphism of $\mathbb{R}$-algebras), so if D is a derivation in T_aX, then $D \circ f^* : C^\infty_{f(a)} \to \mathbb{R}$ is linear. Third, $D \circ f^*$ satisfies the Leibniz rule, so it is a derivation. Finally, $d_af : D \mapsto D \circ f^*$ is linear. You should also verify that in case $Y = \mathbb{R}$, our definition agrees with the one given at the end of the previous section, using the natural identification $T_{f(a)}\mathbb{R} \to \mathbb{R}$.

The next question is how to compute the derivative d_af. What does it mean to "compute" a linear map? It means to calculate its matrix with respect to given bases for its source and target. We shall use the bases given by a chart ϕ at a and a chart ψ at $f(a)$. Say X is n-dimensional at a and Y is m-dimensional at $f(a)$. For $j = 1, \ldots, n$, we need to expand the tangent vector $d_af(\partial/\partial\phi_j)$ in terms of the basis vectors $\partial/\partial\psi_i$, $i = 1, \ldots, m$, for $T_{f(a)}Y$. The coefficient of $\partial/\partial\psi_i$ in that expansion is

$$d_af(\partial/\partial\phi_j)(\psi_i) = \partial/\partial\phi_j(\psi_i \circ f) = D_j(\psi_i \circ f \circ \phi^{-1})(\phi(a)).$$

This is nothing but the (i,j)-entry in the $m \times n$ Jacobian matrix of $\psi \circ f \circ \phi^{-1}$ at $\phi(a)$. In particular, when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ with the standard structures, so we can take ϕ and ψ to be the identity maps, our new definition of the derivative coincides with the old one from multivariable calculus. More explicitly, the tangent map of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ is given by the formula

$$df\Big(\sum_{j=1}^n c_j \frac{\partial}{\partial x_j}\Big) = \sum_{j=1}^n c_j df\Big(\frac{\partial}{\partial x_j}\Big) = \sum_{i=1}^m \sum_{j=1}^n c_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i}.$$

We now have a quick proof of the chain rule. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps between manifolds and $a \in X$, then

$$d_a(g \circ f) = d_{f(a)}g \circ d_af,$$

or more elegantly

$$(g \circ f)_* = g_* \circ f_*.$$

The proof goes like this. Precomposition maps acting on germs satisfy $(g \circ f)^* = f^* \circ g^*$, simply because for a germ h,

$$(g \circ f)^*h = h \circ (g \circ f) = (h \circ g) \circ f = (g^*h) \circ f = f^*(g^*h) = (f^* \circ g^*)h.$$

Hence, for every derivation D in T_aX,

$$(g \circ f)_*D = D \circ (g \circ f)^* = D \circ f^* \circ g^* = (f_*D) \circ g^* = g_*(f_*D) = (g_* \circ f_*)D.$$

This makes the chain rule look trivial! Still, it generalises the old chain rule from multivariable calculus, which was not trivial to prove. Can you see how the old chain rule can be derived from the new one?

We conclude this section by looking at the special case where Y is a closed submanifold of X and the map under consideration is the inclusion $i : Y \hookrightarrow X$. Let $a \in Y$ and let $\phi : U \to U' \subset \mathbb{R}^n$ be a chart on a neighbourhood U of a in X such that $Y \cap U = \{x \in U : \phi_i(x) = 0 \text{ for } i = k+1, \ldots, n\}$, so $\tilde\phi = (\phi_1, \ldots, \phi_k) : U \cap Y \to \{x \in \mathbb{R}^k : (x, 0) \in U'\}$ is a chart on Y. The $n \times k$ matrix of the derivative $d_ai : T_aY \to T_aX$ with respect to the bases given by these charts is the Jacobian matrix of the map $\phi \circ i \circ \tilde\phi^{-1}$. This map simply takes $(x_1, \ldots, x_k)$ to $(x_1, \ldots, x_k, 0, \ldots, 0)$, so its Jacobian matrix is the $k \times k$ identity matrix on top of the $(n-k) \times k$ zero matrix. In other words,

$$d_ai\Big(\frac{\partial}{\partial\tilde\phi_i}\Big) = \frac{\partial}{\partial\phi_i}, \qquad i = 1, \ldots, k.$$

In particular, $d_a i$ is injective, so it identifies T_aY with a k-dimensional linear subspace of T_aX.

More generally, a smooth map $f : X \to Y$ is an immersion if and only if $d_af : T_aX \to T_{f(a)}Y$ is injective for all $a \in X$. Similarly, f is a submersion if and only if d_af is surjective for all $a \in X$, and f is a local diffeomorphism if and only if d_af is bijective for all $a \in X$.

5. Differential forms and integration on manifolds

5.1. **Introduction.** We have seen how to differentiate a smooth real-valued function f on a manifold X and obtain a map $df : X \to \bigcup_{a\in X} T_a^*X$, $a \mapsto d_af$, called the *derivative* or *differential* of f. This map is an example of a *smooth differential form of degree* 1, or simply a 1-*form*, on X, which is defined as a map $\omega : X \to \bigcup_{a\in X} T_a^*X$ taking each $a \in X$ to an element of T_a^*X, which is smooth in the sense that if ϕ is a chart on X and we write $\omega = \sum \omega_i d\phi_i$, then the functions ω_i are smooth on the domain of ϕ. (Exercise: this smoothness condition only needs to be verified for a single chart at each point of X; then it holds for every chart.) We denote the vector space of smooth 1-forms on X by $\Omega^1(X)$. We will refer to smooth functions as 0-forms and from now on often write $\Omega^0(X)$ for $C^\infty(X)$. Differentiation yields a linear operator $d : \Omega^0(X) \to \Omega^1(X)$, called the *exterior derivative* (the word "exterior" distinguishing it from various other differentiation operators that exist in mathematics), satisfying the Leibniz rule

$$d(fg) = fdg + gdf.$$

To continue our generalisation of calculus to manifolds, we would now like know how to sensibly differentiate smooth 1-forms. Perhaps the derivative of a 1-form $\sum \omega_i d\phi_i$ should be something called a 2-form, involving the partial derivatives of the coefficients ω_i in a way that does not depend on the choice of a chart ϕ. It should then be possible to differentiate a smooth 2-form to get a 3-form, and so on. A fair bit of linear algebra is necessary in order to set up a formalism that accomplishes this in a natural and useful manner.

Another fundamental question gets answered at the same time: how do we generalise integration to manifolds? We are used to integrating functions over open sets in Euclidean space. Say we have a smooth function f on an n-dimensional manifold X and we want to integrate it, in the first instance, over the domain of a chart $\phi : U \to V \subset \mathbb{R}^n$ on X. Following our rule of thumb that charts are to be used to transfer calculus from $\mathbb{R}^n$ to X, we might try to define $\int_U f$ as $\int_V f \circ \phi^{-1}$, the latter integral being an ordinary Riemann or, if you like, Lebesgue integral. We could then define $\int_X f$ as $\sum \int \rho_i f$, where (ρ_i) is a partition of unity subordinate to a cover of X by charts.

The integral should not depend on the choice of chart. If we take another chart $\psi : U \to W$, how do $\int_V f \circ \phi^{-1}$ and $\int_W f \circ \psi^{-1}$ compare? The answer is provided by a change of variables, familiar from multivariable calculus. The map $\psi \circ \phi^{-1} : V \to W$ is a diffeomorphism and

$$\int_W f \circ \psi^{-1} = \int_V (f \circ \phi^{-1})\, |\det D(\psi \circ \phi^{-1})|.$$

We cannot expect the second factor in the right-hand integral to be identically 1, so our naive definition is inadequate. What we see, though, is that we can sensibly define the integral over X of an object that to each chart ϕ with domain U associates a smooth function f_ϕ on U such that if ψ is another chart with domain V, then

$$f_\phi = f_\psi \det D(\psi \circ \phi^{-1}) \circ \phi = f_\psi \det \left(\frac{\partial \psi_j}{\partial \phi_i} \right)$$

on $U \cap V$, *provided* the Jacobian determinant is everywhere positive: then

$$\int_{\phi(U\cap V)} f_\phi \circ \phi^{-1} = \int_{\psi(U\cap V)} f_\psi \circ \psi^{-1}.$$

It turns out that in the formalism alluded to above, this sort of object is nothing but a smooth differential form of degree n on X. The n-forms are the objects that can be integrated on an n-dimensional manifold, provided the manifold can be covered by charts with positive Jacobian determinants on all overlaps: such a manifold is said to be *orientable*.

In this chapter, we will define differential forms after briefly introducing the necessary linear algebra, and discuss how to differentiate and integrate them. In the next chapter we will tie the two operations together in Stokes' theorem, a vast generalisation of the fundamental theorem of calculus.

5.2. **A little multilinear algebra.** Let V be a real vector space. A map $\omega : V \times \cdots \times V \to \mathbb{R}$, where there are k copies of V, is called *multilinear* or *k-linear* if it is linear in each factor, that is,

$$\begin{aligned}\omega(v_1, \ldots, v_{i-1}, av + bw, v_{i+1}, \ldots, v_k) = a\omega(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k) \\ + b\omega(v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_k).\end{aligned}$$

We call ω *alternating* or *anti-symmetric* if $\omega(v_1, \ldots, v_k) = 0$ whenever there are $i \neq j$ with $v_i = v_j$. Equivalently, ω changes sign if we interchange any two of its arguments, or, again equivalently,

$$\omega(v_{\pi(1)}, \ldots, v_{\pi(k)}) = \operatorname{sgn}(\pi)\, \omega(v_1, \ldots, v_k)$$

for every permutation π of $\{1, \ldots, k\}$ (exercise). Recall that the sign of π is 1 if π is the composition of an even number of transpositions and -1 if π is the composition of an odd number of transpositions. An alternating k-linear map $V \times \cdots \times V \to \mathbb{R}$ is called a k-form on V. The k-forms on V make up a vector space usually denoted $\Lambda^k(V^*)$. Note that $\Lambda^1(V^*)$ is simply the dual space V^* of all linear functionals on V. By convention, $\Lambda^0(V^*) = \mathbb{R}$.

Suppose V is n-dimensional with basis vectors $e_1, \ldots, e_n$. If we have k vectors $v_i = \sum_{j=1}^{n} a_{ij} e_j$, $i = 1, \ldots, k$, and we plug them into a k-form ω on V, then, by linearity in each variable, we get

$$\omega(v_1, \ldots, v_k) = \sum_{j_1, \ldots, j_k = 1}^{n} a_{1j_1} \ldots a_{kj_k} \omega(e_{j_1}, \ldots, e_{j_k}).$$

If $k > n$, then the same basis vector must occur at least twice in each term on the right hand side. Hence, $\Lambda^k(V^*) = 0$ if $k > \dim V$.

There is a product on alternating forms that turns the direct sum $\bigoplus_{k=0}^{n} \Lambda^k(V^*)$, $n = \dim V$, into an associative algebra, called the *exterior algebra* or *alternating algebra* of

V^*, denoted $\Lambda(V^*)$. If $\omega \in \Lambda^p(V^*)$ and $\eta \in \Lambda^q(V^*)$, then we define the *wedge product* or *exterior product* $\omega \wedge \eta \in \Lambda^{p+q}(V^*)$ by the formula

$$\omega \wedge \eta(v_1, \dots, v_{p+q}) = \frac{1}{p!q!} \sum_{\pi} \operatorname{sgn}(\pi)\, \omega(v_{\pi(1)}, \dots, v_{\pi(p)})\, \eta(v_{\pi(p+1)}, \dots, v_{\pi(p+q)}),$$

where π runs through all permutations of $\{1, \dots, p+q\}$. The verification that $\Lambda(V^*)$ is an associative algebra is lengthy and will be omitted. The wedge product is not quite commutative, but satisfies

$$\omega \wedge \eta = (-1)^{pq}\, \eta \wedge \omega.$$

In particular, if $p = q = 1$, so ω and η are 1-forms, that is, linear functionals on V, then

$$\omega \wedge \eta(v, w) = \omega(v)\eta(w) - \omega(w)\eta(v).$$

The basis $\{e_1, \dots, e_n\}$ for V gives a dual basis $\{\alpha_1, \dots, \alpha_n\}$ for V^*, where $\alpha_i(e_j) = \delta_{ij}$. It may be shown that the k-forms $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}$, where $1 \le i_1 < \cdots < i_k \le n$, form a basis for $\Lambda^k(V^*)$. It follows that

$$\dim \Lambda^k(V^*) = \binom{n}{k} \text{ for } 0 \le k \le n, \text{ and } \dim \Lambda(V^*) = 2^n.$$

For the details we have omitted, see e.g. Madsen-Tornehave, chapter 2.

5.3. **Differential forms and the exterior derivative.** We can apply the previous section to the tangent space T_aX to an n-dimensional manifold X at a point $a \in X$. Let ϕ be a chart at a. We have discussed the basis $\{\partial/\partial\phi_1, \dots, \partial/\partial\phi_n\}$ for T_aX and the dual basis $\{d\phi_1, \dots, d\phi_n\}$ for the cotangent space T_a^*X (see section 4.1).

Generalising our definition of a 1-form in section 5.1, we define a *smooth differential form of degree* k, or simply a *k-form*, on X to be a map $\omega : X \to \bigcup_{a \in X} \Lambda^k(T_a^*X)$ taking each $a \in X$ to an element of $\Lambda^k(T_a^*X)$, such that if ϕ is a chart on X and we write

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \dots i_k} d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k},$$

then the functions $\omega_{i_1 \dots i_k}$ are smooth on the domain on ϕ. (As before, this smoothness condition only needs to be verified for a single chart at each point of X; then it holds for every chart.) We denote by $\Omega^k(X)$ the vector space of all k-forms on X. The wedge product applied pointwise to differential forms turns the direct sum $\Omega(X) = \bigoplus_{k=0}^{n} \Omega^k(X)$ into an associative algebra called the *exterior algebra* or *alternating algebra* of X.

A smooth map $f : X \to Y$ has a derivative $T_af : T_aX \to T_{f(a)}Y$ at each $a \in X$, which induces a dual map, called the *cotangent map* $T_a^*f : T_{f(a)}^*Y \to T_a^*X$, defined by precomposing linear functionals on $T_{f(a)}Y$ by T_af. In other words,

$$T_a^*f(\omega)(v) = \omega((T_af)(v))$$

for $\omega \in T_{f(a)}^*Y$ and $v \in T_aX$. This yields a linear map $f^* : \Omega(Y) \to \Omega(X)$ defined by taking $\omega \in \Omega^k(Y)$ to $f^*\omega \in \Omega^k(X)$ with

$$f^*\omega(a)(v_1, \dots, v_k) = \omega(f(a))(T_af(v_1), \dots, T_af(v_k)) \quad \text{for } v_1, \dots, v_k \in T_aX$$

if $k \ge 1$; for a 0-form h, that is, a function, $f^*h = h \circ f$. We call $f^*\omega$ the *pullback* of ω by f. As an exercise, check that f^* is in fact a morphism of algebras, that is,

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta.$$

If g is a smooth function on Y, then

$$f^* dg = d(f^* g) = d(g \circ f),$$

since for $a \in X$ and $v \in T_a X$,

$$(f^* dg)(a)(v) = d_{f(a)} g(d_a f(v)) = d_a(g \circ f)(v)$$

by the chain rule. In particular, when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, we have

$$f^* dx_i = df_i,$$

so the pullback by f of an arbitrary k-form on $\mathbb{R}^m$ is given by the formula

$$f^* \sum_{1 \le i_1 < \cdots < i_k \le m} \omega_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_{1 \le i_1 < \cdots < i_k \le m} (\omega_{i_1 \ldots i_k} \circ f)\, df_{i_1} \wedge \cdots \wedge df_{i_k}.$$

Now we shall extend the exterior derivative $d : \Omega^0(X) \to \Omega^1(X)$ to all differential forms. We do this first for forms on open sets X in Euclidean space. For a smooth function f on X and $k \ge 1$, we define

$$d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and then extend linearly to get $d : \Omega^k(X) \to \Omega^{k+1}(X)$. It is a good exercise to verify the following properties.

(1) If ω is a p-form and η is a q-form, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

(2) For every form ω, we have $d(d\omega) = 0$. Briefly, $d^2 = 0$.

(3) If U is open in $\mathbb{R}^n$, V is open in $\mathbb{R}^m$, $f : U \to V$ is smooth, and ω is a differential form on V, then

$$f^*(d\omega) = d(f^*\omega).$$

If ω is a k-form on a manifold X and (U, ϕ) is a chart on X, we define $d\omega$ on U by moving it to $\phi(U)$, taking the exterior derivative there as already defined, and then moving back to U. More precisely, we set

$$d\omega | U = \phi^* d(\phi^{-1})^*(\omega | U).$$

By property (3) above, this definition is independent of ϕ, so we have a well-defined $(k+1)$-form $d\omega$ on X. It is another good exercise to verify that the map $d : \Omega^k(X) \to \Omega^{k+1}(X)$ thus defined is linear, satisfies (1) and (2) above, and is equal to the exterior derivative as previously defined for $k = 0$. (These properties can in fact be shown to determine d uniquely.) Furthermore,

$$f^*(d\omega) = d(f^*\omega)$$

whenever ω is a differential form on the target of the smooth map f.

To conclude this section, let us verify that $d^2 = 0$ in the simplest nontrivial case, that is, for 0-forms on $\mathbb{R}^2$. We know that the 1-forms dx and dy form a basis for the $C^\infty(\mathbb{R}^2)$-module $\Omega^1(\mathbb{R}^2)$, and the 2-form $dx \wedge dy$ spans the 1-dimensional module $\Omega^2(\mathbb{R}^2)$ over $C^\infty(\mathbb{R}^2)$. Here, x and y denote the projections $\mathbb{R}^2 \to \mathbb{R}$ onto the first and second coordinate, respectively; that is, they denote the component functions of the chart $\mathrm{id}_{\mathbb{R}^2}$ that by itself constitutes the atlas that defines the standard smooth structure on $\mathbb{R}^2$. What we need from algebra are the fact that $\Omega(\mathbb{R}^2)$ with addition and the wedge product is an associative algebra, and the basic identities $dx \wedge dx = dy \wedge dy = 0$ and $dx \wedge dy = -dy \wedge dx$. For a smooth function f on $\mathbb{R}^2$, writing f_x for $\partial f/\partial x$ etc., we get

$$\begin{aligned} d(df) &= d(f_x dx + f_y dy) = d(f_x dx) + d(f_y dy) = df_x \wedge dx + df_y \wedge dy \\ &= (f_{xx} dx + f_{xy} dy) \wedge dx + (f_{yx} dx + f_{yy} dy) \wedge dy \\ &= f_{xy} dy \wedge dx + f_{yx} dx \wedge dy = (f_{yx} - f_{xy}) dx \wedge dy = 0, \end{aligned}$$

by the equality of mixed partials.

5.4. **Integration of differential forms on oriented manifolds.** Let ω be an n-form on an open set V in $\mathbb{R}^n$. Then $\omega = f dx_1 \wedge \cdots \wedge dx_n$ for a uniquely determined smooth function f on V. In fact, $f = \omega(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ (check!). We define

$$\int_V \omega = \int_V f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n,$$

if the Riemann or, if you like, Lebesgue integral on the right exists.

Let U be another open set in $\mathbb{R}^n$ and $\psi : U \to V$ be a diffeomorphism. As an exercise, show that

$$\psi^*(dx_1 \wedge \cdots \wedge dx_n) = \det(D\psi)\, dx_1 \wedge \cdots \wedge dx_n.$$

On the other hand, the formula for a change of variables from multivariable calculus tells us that if g is a function on V, say continuous with compact support, then

$$\int_U (g \circ \psi) \left| \det \left(\frac{\partial \psi_i}{\partial x_j} \right) \right| = \int_V g.$$

If you are familiar with the Lebesgue measure λ on $\mathbb{R}^n$, you will recognise that this formula says that

$$\psi^* \lambda = (\psi^{-1})_* \lambda = |\det(D\psi)|\, \lambda.$$

Thus, we can only conclude that

$$\begin{aligned} \int_U \psi^* \omega &= \int_U (f \circ \psi)\, \psi^*(dx_1 \wedge \cdots \wedge dx_n) = \int_U (f \circ \psi) \det \left(\frac{\partial \psi_i}{\partial x_j} \right) dx_1 \wedge \cdots \wedge dx_n \\ &= \int_U (f \circ \psi) \left| \det \left(\frac{\partial \psi_i}{\partial x_j} \right) \right| = \int_V f = \int_V \omega \end{aligned}$$

if ψ *preserves orientation* in the sense that its Jacobian determinant is positive at every point of U.

The fact that the form $dx_1 \wedge \cdots \wedge dx_n$ transforms by the Jacobian determinant and Lebesgue measure transforms by the absolute value of the Jacobian determinant complicates integration theory a little. It means that in order to be able to integrate n-forms over an n-dimensional manifold in a coordinate-independent way, we need an extra structure on the manifold, called an *orientation*.

An atlas $\mathcal{A}$ on a manifold X is said to be *oriented* if for every pair of charts (U, ϕ), (V, ψ) in $\mathcal{A}$, the diffeomorphism $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ preserves orientation, that is, has positive Jacobian determinant at every point of its domain. We say that X is *orientable* if it has an oriented atlas, belonging, of course, to the given smooth structure on X. Define an equivalence relation on the set of oriented atlases on X by declaring two such to be equivalent if their union is oriented; an equivalence class for this relation is called an *orientation* on X. Every equivalence class contains a unique maximal oriented atlas, namely the union of all the atlases in the class. An *oriented manifold* is a manifold with a choice of orientation.

A diffeomorphism $f : X \to Y$ between oriented manifolds is said to *preserve orientation* if for some (or, equivalently, every) oriented atlas $\mathcal{A}$ for Y belonging to the given orientation on Y, the atlas $\{\phi \circ f : \phi \in \mathcal{A}\}$ for X, which is oriented, belongs to the given orientation on X. This generalises the definition given above for a diffeomorphism between open subsets of $\mathbb{R}^n$, if we endow such a set with the standard orientation given by the atlas consisting of nothing but the identity map.

We will look at these concepts more closely in a minute, but first let us see how to integrate an n-form ω over an n-dimensional oriented manifold X. To avoid convergence issues, let us assume that ω has compact support. Take an oriented atlas $(\phi_i : U_i \to U_i')_{i \in I}$ for X (belonging to the given orientation). Let (ρ_i) be a partition of unity subordinate to the open cover (U_i) of X (recall from section 3.3 that if we do not require the functions ρ_i to have compact supports, then we can index them by I itself). Then

$$\omega = \sum_{i \in I} \rho_i \omega$$

and $\rho_i \omega$ has support in U_i for each $i \in I$. We define

$$\int_X \omega = \sum_{i \in I} \int_{U_i'} (\phi_i^{-1})^* (\rho_i \omega).$$

It is immediate that the map $\omega \mapsto \int_X \omega$ is linear.

Suppose we have another oriented atlas $(\psi_j : V_j \to V_j')_{j \in J}$ (belonging to the same orientation) and a partition of unity (σ_j) subordinate to (V_j). For each $i \in I$ and $j \in J$, the form $\rho_i \sigma_j \omega$ has support in $U_i \cap V_j$. Since the diffeomorphism $\psi_j \circ \phi_i^{-1} : \phi_i(U_i \cap V_j) \to \psi_j(U_i \cap V_j)$ preserves orientation, we have

$$\int_{U_i'} (\phi_i^{-1})^* (\rho_i \sigma_j \omega) = \int_{V_j'} (\psi_j^{-1})^* (\rho_i \sigma_j \omega),$$

so

$$\begin{aligned} \sum_{i \in I} \int_{U_i'} (\phi_i^{-1})^* (\rho_i \omega) &= \sum_{i \in I} \sum_{j \in J} \int_{U_i'} (\phi_i^{-1})^* (\rho_i \sigma_j \omega) = \sum_{i \in I} \sum_{j \in J} \int_{V_j'} (\psi_j^{-1})^* (\rho_i \sigma_j \omega) \\ &= \sum_{j \in J} \int_{V_j'} (\psi_j^{-1})^* (\sigma_j \omega). \end{aligned}$$

This shows that the definition of $\int_X \omega$ only depends on the orientation of X, but is independent of the choice of atlas and partition of unity.

As an exercise, show directly from the defining formula above that if X and Y are n-dimensional oriented manifolds, $f : X \to Y$ is a diffeomorphism preserving orientation, and ω is an n-form on Y with compact support, then

$$\int_X f^* \omega = \int_Y \omega.$$

We conclude this chapter with a closer examination of the notion of orientation. A *volume form* on an n-dimensional manifold X is a nowhere-vanishing n-form on X. Note that since $\Lambda^n(T_a^* X)$ is one-dimensional for every $a \in X$, if ω and η are volume forms on X, there is a unique smooth function f on X with no zeros such that $\omega = f\eta$. Thus, for example, the volume forms on $\mathbb{R}^n$ are precisely the forms $f dx_1 \wedge \cdots \wedge dx_n$, where f is a smooth function on $\mathbb{R}^n$ that is everywhere positive or everywhere negative.

Proposition 5.1. *A manifold is orientable if and only if it has a volume form.*

Proof. (This is a sketch: fill in the details!) $\Rightarrow$: Let $(\phi_i : U_i \to U_i')_{i \in I}$ be an oriented atlas for an orientable manifold X and (ρ_i) be a partition of unity subordinate to the open cover (U_i) of X. Then

$$\omega = \sum_{i \in I} \rho_i \phi_i^*(dx_1 \wedge \cdots \wedge dx_n)$$

is a volume form on X.

$\Leftarrow$: Let $(\phi_i : U_i \to U_i')_{i \in I}$ be any atlas on a manifold X with a volume form ω. We may assume that U_i are all connected (why?). If $\phi_i^*(dx_1 \wedge \cdots \wedge dx_n)$ is a negative multiple of ω on U_i, replace ϕ_1 by its negative $-\phi_1$. This yields an atlas (ϕ_i) such that $\phi_i^*(dx_1 \wedge \cdots \wedge dx_n)$ is a positive multiple of ω for all i. Such an atlas is oriented. $\square$

Let us call two volume forms ω and η on X *equivalent* if there is a positive function f with $\omega = f\eta$. This defines an equivalence relation on the set of all volume forms on X. If this set is not empty and X is connected, then there are precisely two equivalence classes, one containing all positive multiples of any volume form ω, and the other containing all negative multiples of ω.

Proposition 5.2. *There is a bijection between orientations of a manifold X and equivalence classes of volume forms on X. If X is connected and orientable, then X has precisely two orientations.*

Proof. (Sketch again.) The formula in the first half of the previous proof gives a well-defined map taking orientations on X to equivalence classes of volume forms on X. The second half of the proof describes the inverse of this map. $\square$

Finally, convince yourself that the integral of an n-form over a connected orientable n-dimensional manifold with respect to one of the orientations of the manifold is the negative of the integral with respect to the other.

6. Stokes' theorem

6.1. **Manifolds with boundary.** A manifold with boundary is locally modelled on the *half-space* $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}$ with its boundary $\partial\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = 0\}$ in the same way that a manifold is locally modelled on $\mathbb{R}^n$. We endow $\mathbb{H}^n$ with the subspace topology induced from $\mathbb{R}^n$, so a subset U of $\mathbb{H}^n$ is open if and only if $U = V \cap \mathbb{H}^n$ for some open subset V of $\mathbb{R}^n$. We say that a function $f : U \to \mathbb{R}$ is smooth if there is a smooth function $g : V \to \mathbb{R}$ on some such V with $g|U = f$.

Let X be a second countable Hausdorff space with a closed subset (possibly empty) denoted ∂X. A *chart* on $(X, \partial X)$ is a homeomorphism $\phi : U \to U'$, where U is an open subset of X and U' is an open subset of the half-space $\mathbb{H}^n$ for some n, such that

$$\phi(U \cap \partial X) = U' \cap \partial\mathbb{H}^n.$$

Charts $\phi : U \to U'$ and $\psi : V \to V'$ are *compatible* if $\psi \circ \phi^{-1}$ is smooth on the open subset $\phi(U \cap V)$ of $\mathbb{H}^n$ (with the notion of smoothness defined in the previous paragraph). As before, an *atlas* on $(X, \partial X)$ is defined to be a set of mutually compatible charts whose domains cover X, every atlas is contained in a unique maximal atlas, and a maximal atlas is called a *smooth structure* on $(X, \partial X)$. The pair $(X, \partial X)$ endowed with a smooth structure is called a *smooth manifold with boundary.* For brevity, we often write X for $(X, \partial X)$ and refer to X as a manifold with boundary. The subset ∂X is called the *boundary* of X. If the number $n \geq 1$ above is the same for all charts (as is the case if X is connected), then we call it the *dimension* of X. The open subset $X \setminus \partial X$, called the *interior* of X, is a manifold in the ordinary sense with an atlas consisting of all the charts $\phi : U \to U'$ on $(X, \partial X)$ with $U \cap \partial X = \varnothing$.

If $\dim X \geq 2$, then ∂X is an $(n-1)$-dimensional manifold in its own right with an atlas consisting of a map $(\phi_2, \ldots, \phi_n) : U \cap \partial X \to U' \cap \partial \mathbb{H}^n$ for each chart $\phi : U \to U'$ on $(X, \partial X)$ with $U \cap \partial X \neq \varnothing$ (check this, noting that $\partial \mathbb{H}^n$ may be identified with $\mathbb{R}^{n-1}$). If $\dim X = 1$, then ∂X is a discrete subset of X, so we can consider it as a 0-dimensional manifold.

Note that the concept of a manifold with boundary generalises the concept of a manifold (as defined in section 3.1). A manifold is a manifold with boundary whose boundary is empty. We will continue to reserve the word "manifold" for the notion defined in section 3.1, and be careful to say "manifold with boundary" whenever a nonempty boundary is allowed.

So far we have implicitly assumed that our manifolds had dimension at least 1, although most of what we have said makes sense, usually in a completely trivial way, for 0-dimensional manifolds, which, as noted in section 3.1, are nothing but countable sets with the discrete topology. Now, though, 0-dimensional manifolds have become important as boundaries of 1-dimensional manifolds. A connected manifold Y of dimension $n = 0$ is nothing but a one-point set $\{a\}$. An n-form, that is, a 0-form, on Y is a function $\omega : Y \to \mathbb{R}$, that is, just a real number $\omega(a)$. An orientation of Y simply determines whether the integral $\int_Y \omega$ is $\omega(a)$ or $-\omega(a)$. Let us call the first choice the *positive orientation* of Y and the second the *negative orientation*.

Examples of manifolds with boundary are provided by products $X \times Y$, where X is a manifold with boundary and Y is a manifold (without boundary). Then $\partial(X \times Y) = \partial X \times Y$. More examples are given by Propositions 6.1 and 6.2 below. The product of two manifolds with nonempty boundaries is not a manifold with boundary in a natural way: rather, it is something called a manifold with corners; we will not discuss this concept.

The theory of manifolds, as developed in these notes so far, can now be extended to manifolds with boundary. This is mostly straightforward, but still a fair amount of work to do in detail. The remainder of this section touches on some aspects of this.

Let $(X, \partial X)$ be a manifold with boundary. The tangent space $T_a X$ for $a \in X \setminus \partial X$ is defined as before; so is $T_a \partial X$ for $a \in \partial X$. There are two equivalent ways to define $T_a X$ for $a \in \partial X$. We can talk about smooth functions on open neighbourhoods of a in X, define the $\mathbb{R}$-algebra $C^\infty_{X,a}$ of germs of these as before, and take $T_a X$ to be the vector space of derivations on $C^\infty_{X,a}$. Alternatively, we can use smooth paths $\gamma : [0, \epsilon) \to X$ or $\gamma : (-\epsilon, 0] \to X$, $\epsilon > 0$, with $\gamma(0) = a$. To say that γ is smooth at a means that in some (or, equivalently, every) chart (U, ϕ) on $(X, \partial X)$ at a, $\phi \circ \gamma$ extends smoothly to $(-\delta, \epsilon)$ or $(-\epsilon, \delta)$, as the case may be, for some $\delta > 0$. As before, we call two such paths γ_1 and γ_2 equivalent if $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$, and define the tangent space $T_a X$ to be the set of equivalence classes. The class of a path $[0, \epsilon) \to X$ is called an *inward* vector, while the class of a path $(-\epsilon, 0] \to X$ is called an *outward* vector. It may be verified that the tangent space $T_a \partial X$ is in a natural way embedded as a linear subspace of $T_a X$ of codimension 1 (exercise). As such, $T_a \partial X$ consists of those tangent vectors to X at a that are both inward and outward.

The following results are analogous to Theorem 3.1 and can be proved in a similar way, using the rank theorem.

Proposition 6.1. *Let X be a manifold, $f : X \to \mathbb{R}$ be smooth, and $c \in \mathbb{R}$ be a regular value of f. Then $f^{-1}((-\infty, c])$ is a submanifold of X with boundary $f^{-1}(c)$.*

Taking $f = \|\cdot\|^2 : \mathbb{R}^n \to \mathbb{R}$, we see that the closed ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is a manifold with boundary $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

Proposition 6.2. *Let X be a manifold with boundary, Y be a manifold, and $f : X \to Y$ be a smooth map. If $c \in Y$ is a regular value of f and a regular value of $f|\partial X : \partial X \to Y$, then $f^{-1}(c)$ is a submanifold of X with boundary $\partial(f^{-1}(c)) = f^{-1}(c) \cap \partial X$.*

An orientation of a manifold X with boundary ∂X is defined as before by a volume form. If $n = \dim X \geq 2$, then an orientation can also be defined by an oriented atlas (the reader should look into why this is problematic when $\dim X = 1$).

If $n = \dim X \geq 2$ and $\mathcal{A}$ is an oriented atlas on $(X, \partial X)$, then the atlas $\mathcal{A}'$ on ∂X consisting of all the maps $\tilde{\phi} = (\phi_2, \ldots, \phi_n) : U \cap \partial X \to U' \cap \partial \mathbb{H}^n$, where $\phi : U \to U'$ is in $\mathcal{A}$ with $U \cap \partial X \neq \varnothing$, is oriented. To see this, let $\phi : U \to U'$ and $\psi : V \to V'$ be two charts in $\mathcal{A}$, and suppose $a \in U \cap V \cap \partial X$. Consider the diffeomorphism $g = \psi \circ \phi^{-1} : W \to W'$ between the open sets $W = \phi(U \cap V) = U' \cap \phi(V)$ and $W' = \psi(U \cap V) = V' \cap \psi(U)$ in $\mathbb{H}^n$. We have $g(W \cap \partial \mathbb{H}^n) = W' \cap \partial \mathbb{H}^n$, so $g_1 = 0$ on $W \cap \partial \mathbb{H}^n$. Also, $g_1 \leq 0$, so $\partial g_1/\partial x_1 \geq 0$ on $W \cap \partial \mathbb{H}^n$. Setting $\tilde{g} = \tilde{\psi} \circ \tilde{\phi}^{-1} : W \cap \partial \mathbb{H}^n \to W' \cap \partial \mathbb{H}^n$, the Jacobian matrix of g at $\phi(a)$ is

$$J(g) = \begin{bmatrix} \partial g_1/\partial x_1 & 0 \\ * & J(\tilde{g}) \end{bmatrix}.$$

Since $\det J(g)$ and $\partial g_1/\partial x_1$ are positive, so is $\det J(\tilde{g})$. This shows that $\mathcal{A}'$ is oriented; in particular, ∂X is orientable. We endow ∂X with the orientation defined by $\mathcal{A}'$ and call it the *induced orientation* on ∂X.

If $(X, \partial X)$ is 1-dimensional with orientation defined by a volume form ω, then the induced orientation on a point $a \in \partial X$ is defined to be positive if $\omega(v) \geq 0$ for every outward tangent vector $v \in T_a X$ and negative otherwise.

6.2. **Statement and proof of Stokes' theorem.** We can now state and prove the vast generalisation of the fundamental theorem of calculus known as Stokes' theorem.

Theorem 6.1 (Stokes' theorem)**.** *Let X be an oriented n-dimensional manifold with boundary ∂X with the induced orientation. Let ω be a smooth differential form on X of degree $n-1$ with compact support. Then*

$$\int_X d\omega = \int_{\partial X} \omega.$$

The right-hand side really means $\int_{\partial X} i^*\omega$, where $i : \partial X \hookrightarrow X$ is the inclusion. When $\partial X = \varnothing$, we interpret the right-hand side as zero.

Proof. Let (ρ_i) be a partition of unity subordinate to an open cover (U_i) of X by coordinate neighbourhoods.[2] Then

$$\int_X d\omega = \sum_i \int_X \rho_i \, d\omega = \sum_i \int_{U_i} d(\rho_i \omega),$$

[2] The proof of existence of partitions of unity is easily adapted to manifolds with boundary.

the reason for the second equality being that $\sum_i d\rho_i = 0$ since $\sum_i \rho_i = 1$. Also,

$$\int_{\partial X} \omega = \sum_i \int_{\partial X} \rho_i \omega = \sum_i \int_{U_i \cap \partial X} \rho_i \omega.$$

Thus we only need to prove Stokes' theorem for forms with support in a coordinate neighbourhood, that is, we only need to prove Stokes' theorem for differential forms with compact support in $\mathbb{H}^n$ with its standard orientation, which induces the standard orientation on the boundary $\partial\mathbb{H}^n$ viewed as $\mathbb{R}^{n-1}$.

Assume then that ω is an $(n-1)$-form with compact support in $\mathbb{H}^n$. Remember that this means that ω is the restriction to $\mathbb{H}^n$ of a smooth $(n-1)$-form on an open set in $\mathbb{R}^n$ containing $\mathbb{H}^n$.

The case $n = 1$ follows directly from the fundamental theorem of calculus: ω is a smooth function of compact support on $(-\infty, 0]$, and

$$\int_{\mathbb{H}^1} d\omega = \int_{-\infty}^{0} \omega'(x)\, dx = \omega(0) = \int_{\partial\mathbb{H}^1} \omega.$$

When $n \geq 2$, write

$$\omega = \sum_{i=1}^{n} f_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

with smooth functions $f_1, \ldots, f_n$, where the hat marks a missing term. Then

$$d\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_n.$$

For $i \geq 2$, since f_i has compact support in $\mathbb{H}^n$, for every $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n \in \mathbb{R}$ with $c_1 \leq 0$,

$$\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i}(c_1, \ldots, c_{i-1}, t, c_{i+1}, \ldots, c_n)\, dt = 0$$

by the fundamental theorem of calculus, so

$$\int_{\mathbb{H}^n} \frac{\partial f_i}{\partial x_i}\, dx_1 \ldots dx_n = 0.$$

The pullback of ω by the inclusion $i : \partial\mathbb{H}^n \hookrightarrow \mathbb{H}^n$ is $i^*\omega = f_1\, dx_2 \wedge \cdots \wedge dx_n$, so

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= \int_{\mathbb{H}^n} \frac{\partial f_1}{\partial x_1}\, dx_1 \ldots dx_n = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1}\, dx_1 \right) dx_2 \ldots dx_n \\ &= \int_{\mathbb{R}^{n-1}} f_1(0, x_2, \ldots, x_n)\, dx_2 \ldots dx_n = \int_{\partial\mathbb{H}^n} \omega, \end{aligned}$$

once again using the fundamental theorem of calculus. □

Note that in the above proof we invoked Fubini's theorem twice. One of its versions, sufficient for our purposes, states that the integral of a compactly supported continuous function on a product of intervals can be computed as an iterated integral and the order of integration does not matter.

Corollary 6.1. *Let X be an oriented n-dimensional manifold (without boundary). If ω is a compactly supported $(n-1)$-form on X, then*

$$\int_X d\omega = 0.$$

As mention in the introduction, Green's theorem and Gauss' theorem (a.k.a. the divergence theorem) from multivariable calculus are special cases of Stokes' theorem. Let us look at the former and leave the latter as an exercise. Suppose X is a 2-dimensional manifold embedded in $\mathbb{R}^2$ with the standard orientation, with boundary ∂X endowed with the induced orientation, which is the counterclockwise orientation with X to the left as ∂X is traversed in the positive direction. For example, by Proposition 6.1, we could take $X = f^{-1}((-\infty, c])$ and $\partial X = f^{-1}(c)$, where $c \in \mathbb{R}$ is a regular value of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$. Let $\omega = g\,dx + h\,dy$ be a smooth 1-form on X. This means that g and h are smooth functions on an open neighbourhood of X. Then Stokes' theorem implies that

$$\int_{\partial X} g\,dx + h\,dy = \int_X d\omega = \int_X \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right) dx \wedge dy.$$

6.3. **Topological applications of Stokes' theorem.** [3] Stokes' theorem is a powerful tool in differential topology. This section illustrates its usefulness.

Suppose M is an $(n+1)$-dimensional manifold with boundary $\partial M = X$, so X is an n-dimensional manifold, and Y is another n-dimensional manifold. Further, let all three manifolds be compact and oriented such that X has the orientation induced from M. If $f : X \to Y$ is a smooth map, let us ask whether f extends to a smooth map $F : M \to Y$. Suppose it does. If ω is an n-form on Y, then $d\omega = 0$, being an $(n+1)$-form on the n-dimensional manifold Y, so by Stokes' theorem,

$$\int_X f^*\omega = \int_{\partial M} F^*\omega = \int_M d(F^*\omega) = \int_M F^*(d\omega) = 0.$$

In particular, consider the case when $X = Y$ and f is the identity id_X. Then $F : M \to X$ is a smooth map with $F|X = \mathrm{id}_X$. We call such a map F a smooth *retraction* of M onto its boundary X. If F exists, we conclude that $\int_X \omega = 0$ for all n-forms ω on X, which is absurd (why?). We have proved the following result.

Proposition 6.3. *If M is a compact orientable manifold with boundary ∂M, then there is no smooth retraction $M \to \partial M$.*

From this we obtain a famous theorem.

Theorem 6.2 (Brouwer's fixed point theorem, smooth version)**.** *Let B^n be the closed unit ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ in $\mathbb{R}^n$. Every smooth map $B^n \to B^n$ has a fixed point.*

Proof. Suppose $f : B^n \to B^n$ is a smooth map without a fixed point. For each $x \in B^n$, draw a ray from $f(x)$ through x and let $g(x)$ be the point where the ray hits the boundary $\partial B^n = S^{n-1}$. The ray is well defined because $f(x) \neq x$. This defines a map $g : B^n \to S^{n-1}$ with $g(x) = x$ for $x \in S^{n-1}$. Derive a formula for g and convince yourself that g is smooth. By Proposition 6.3, such a map cannot exist. □

Let X and Y be manifolds. We say that smooth maps $g, h : X \to Y$ are smoothly *homotopic* if there is a smooth map $f : I \times X \to Y$, where $I = [0, 1]$, such that $f(0, \cdot) = g$ and $f(1, \cdot) = h$. Then f is called a smooth *homotopy* from g to h, or between g and h. A map homotopic to a constant map is called *null-homotopic*. A manifold X is called *contractible* if id_X is null-homotopic. A contractible manifold is connected (exercise).

[3] This section is mostly borrowed from notes from a course given by Robert J. Zimmer at the University of Chicago in autumn 1987.

As an example, $\mathbb{R}^n$ is contractible with a homotopy from a constant map to the identity map given by the formula $(t,x) \mapsto tx$. More generally, an open subset X of $\mathbb{R}^n$ is called *star-shaped* if there is a point $a \in X$ such that X contains the line segment from a to x for every $x \in X$. Then $(x,t) \mapsto ta + (1-t)x$ defines a homotopy from id_X to the constant map a. Clearly, if X is convex, then X is star-shaped. For examples of non-contractible manifolds, see Corollary 6.2.

Suppose X is oriented and let I have the standard orientation. Then $I \times X$ is an oriented manifold with boundary $\partial(I \times X)$, which is the disjoint union of X_0 and X_1, where we write $X_t = \{t\} \times X$ for $t \in I$. With the induced orientation on the boundary, X_1 has the same orientation as X, and X_0 has the opposite orientation.

Smooth maps $g, h : X \to Y$ yield a smooth map $f = g \cup h : X_0 \cup X_1 = \partial(I \times X) \to Y$. We see that g and h are homotopic if and only if f extends to a smooth map $I \times X \to Y$. If Y is oriented and X and Y are compact of the same dimension n, then we know from our considerations above that if this happens, then for every n-form ω on Y we have

$$0 = \int_{\partial(I \times X)} f^*\omega = \int_{X_0 \cup X_1} f^*\omega = -\int_X g^*\omega + \int_X h^*\omega.$$

Let us record this result.

Proposition 6.4. *Let X and Y be compact oriented n-dimensional manifolds and $g, h : X \to Y$ be smooth maps. If g and h are smoothly homotopic, then*

$$\int_X g^*\omega = \int_X h^*\omega$$

for every n-form ω on Y.

Corollary 6.2. *A compact orientable manifold X (with $\dim X \geq 1$ and without boundary) is not contractible.*

Proof. Suppose X is contractible, that is, id_X is homotopic to a constant map $c : X \to X$. Then

$$\int_X \omega = \int_X \mathrm{id}_X^*\omega = \int_X c^*\omega = 0$$

for every n-form ω on X, which is absurd. $\square$

We conclude this section by proving the fundamental theorem of algebra. We identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ via the map $x+iy \mapsto (x,y)$. Let p be a complex polynomial of degree $n \geq 1$, viewed as a map $\mathbb{C} \to \mathbb{C}$. We will show that p has a zero. We may asume that p is monic. Write

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0.$$

For $t \in I$, let

$$f_t(z) = z^n + t\sum_{j=0}^{n-1} a_j z^j,$$

so $f_0(z) = z^n$ and $f_1 = p$. As an exercise, show that for r sufficiently large, f_t does not vanish on $S(r) = \{z \in \mathbb{C} : |z| = r\}$ for any $t \in I$ (the proof of this elementary fact requires no complex analysis). Then $g_t = f_t/|f_t|$ gives a homotopy of smooth maps $S(r) \to S^1$, so by Proposition 6.4,

$$\int_{S(r)} g_0^*\omega = \int_{S(r)} g_1^*\omega$$

for every 1-form ω on S^1. Here we endow $S(r)$ with one of the two possible orientations; which one does not matter.

If p has no zeros, then $g_1 = p/|p|$ extends to a smooth map $h_1 : D(r) \to S^1$, defined by the same formula, where $D(r)$ denotes the closed disc $\{z \in \mathbb{C} : |z| \leq r\}$ with boundary $S(r)$. Thus, by our considerations at the beginning of this section,

$$\int_{S(r)} g_0^*\omega = \int_{S(r)} g_1^*\omega = 0$$

for every 1-form ω on S^1. Take

$$\omega = \operatorname{Im} \frac{dz}{z} = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

or, to be precise, let ω be the pullback of this form by the inclusion $S^1 \hookrightarrow \mathbb{C} \setminus \{0\}$ (using complex notation for differential forms is too tempting to resist). Why do you think this form is usually denoted $d\theta$? Now $g_0(z) = z^n/r^n$, so

$$g_0^*\omega = \operatorname{Im} \frac{d(z^n/r^n)}{z^n/r^n} = \operatorname{Im} \frac{nz^{n-1}dz}{z^n} = n \operatorname{Im} \frac{dz}{z}$$

and

$$\int_{S(r)} \operatorname{Im} \frac{dz}{z} = 0.$$

The final step is to calculate this integral. We do that with the help of the smooth map $\gamma : [0, 2\pi] \to S(r)$, $\gamma(t) = re^{it}$. We have

$$\gamma^* \operatorname{Im} \frac{dz}{z} = \operatorname{Im} \frac{d(re^{it})}{re^{it}} = \operatorname{Im} \frac{ie^{it}dt}{e^{it}} = dt.$$

Since γ restricts to a diffeomorphism $(0, 2\pi) \to S(r)\backslash\{r\}$, which we may assume preserves orientation, we get

$$0 = \int_{S(r)} \operatorname{Im} \frac{dz}{z} = \int_{[0,2\pi]} dt = 2\pi,$$

which is absurd. Thus, the fundamental theorem of algebra is proved.

7. Cohomology

7.1. **De Rham cohomology.** In homological algebra, a *complex* of modules over a ring R is a sequence of R-modules and R-linear maps

$$\ldots \xrightarrow{d_{-2}} E_{-1} \xrightarrow{d_{-1}} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} \ldots$$

such that $d_k \circ d_{k-1} = 0$ for every $k \in \mathbb{Z}$. Then Im $d_{k-1} \subset$ Ker d_k, so we can form the quotient module

$$H^k(E, d) = \text{Ker } d_k / \text{Im } d_{k-1},$$

called the k-th *homology group* of the complex.

We have already seen an example of a complex. Let X be an n-dimensional manifold (with or without boundary). As before, let $\Omega^k(X)$ be the real vector space of smooth differential forms of degree k on X. For $k > n$, this is the trivial vector space; also for $k < 0$ by convention. The complex

$$\ldots \longrightarrow 0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(X) \longrightarrow 0 \longrightarrow \ldots$$

is called the *de Rham complex* of X. Forms in the kernel of the exterior derivative d are called *closed*, and forms in the image of d are called *exact*. The k-th homology group of the de Rham complex of X,

$$H^k_{dR}(X) = \{\text{closed } k\text{-forms on } X\}/\{\text{exact } k\text{-forms on } X\},$$

is called the k-th *de Rham cohomology group* of X. The vector spaces $H^k_{dR}(X)$ for $k = 0, \ldots, n$ (they are trivial for other values of k) contain much useful information about the manifold X in a relatively accessible algebraic format. (It is traditional to call these vector spaces groups; they are of course abelian groups with respect to addition.)

The 0-th de Rham cohomology group $H^0_{dR}(X)$ is easily understood. It consists of all closed 0-forms on X, that is, all smooth functions $u : X \to \mathbb{R}$ such that $du = 0$. These are the locally constant functions on X, that is, the functions that are constant on each connected component of X. Thus, $H^0_{dR}(X) \cong \mathbb{R}^m$, where m is the number of connected components of X (possibly infinite).

Moving to degree 1, it is easy to see that $H^1_{dR}(I) = 0$ for every interval $I \subset \mathbb{R}$. Namely, every 1-form on I is closed and of the form $f dx$, where f is a smooth function on I. The fundamental theorem of calculus provides a smooth antiderivative F for f on I, and $f dx = dF$ is exact.

Proposition 7.1. *Let X be a compact orientable manifold (without boundary) of dimension n. Then $H^n_{dR}(X) \neq 0$.*

Proof. We may assume that X is connected (why, and where is this assumption used in this proof?). Fix an orientation on X. If η is an $(n-1)$-form on X, then $\int_X d\eta = \int_{\partial X} \eta = 0$ by Stokes' theorem, since $\partial X = \varnothing$. Hence, we have a well-defined linear map

$$H^n_{dR}(X) \to \mathbb{R}, \quad [\omega] \mapsto \int_X \omega.$$

If ω is a volume form on X for the chosen orientation, then $\int_X \omega \neq 0$, so this is not the zero map. $\square$

Now let X and Y be manifolds and $f : X \to Y$ be a smooth map. The pullback map $f^* : \Omega(Y) \to \Omega(X)$ gives a *morphism of complexes*

$$\begin{tikzcd}
\ldots \arrow[r] & \Omega^k(Y) \arrow[r, "d"] \arrow[d, "f^*"] & \Omega^{k+1}(Y) \arrow[r] \arrow[d, "f^*"] & \ldots \\
\ldots \arrow[r] & \Omega^k(X) \arrow[r, "d"] & \Omega^{k+1}(X) \arrow[r] & \ldots
\end{tikzcd}$$

from the de Rham complex of Y to that of X. This means that the vertical arrows are linear maps and all the squares commute, that is, $d \circ f^* = f^* \circ d$, as we know from section 5.3. A morphism of complexes always induces linear maps between the homology groups of the complexes. In our case, we see that if ω is a closed form on Y, then the pullback $f^*\omega$ on X is also closed, and if ω is exact, then $f^*\omega$ is also exact (check). Thus, pullback by f induces a linear map

$$f^* : H^k_{dR}(Y) \to H^k_{dR}(X)$$

for each $k \in \mathbb{Z}$. If $g : Y \to Z$ is another smooth map, then it is easily checked that $(g \circ f)^* = f^* \circ g^*$. Also, the identity map of X clearly induces the identity map of $H^k_{dR}(X)$. These facts can be summarised by saying that for each k, the assignment to a manifold X of its k-th de Rham cohomology group $H^k_{dR}(X)$ and to a smooth map $f : X \to Y$ of the induced map $f^* : H^k_{dR}(Y) \to H^k_{dR}(X)$ is a *functor* from the category of smooth manifolds and smooth maps to the category of real vector spaces and linear maps. It is a *contravariant* functor, meaning that it reverses the direction of arrows: this is why the homology groups of the de Rham complex are called **co**homology groups.

It follows that if $f : X \to Y$ is a diffeomorphism with inverse $g : Y \to X$, then each map $f^* : H^k_{dR}(Y) \to H^k_{dR}(Y)$ is a linear isomorphism with inverse g^*, since $f^* \circ g^* = (g \circ f)^* = \mathrm{id}^*_X = \mathrm{id}_{H^k_{dR}(X)}$ and similarly, $g^* \circ f^* = \mathrm{id}_{H^k_{dR}(Y)}$. Thus, the de Rham cohomology groups of a manifold X are diffeomorphism invariants of X. Later we will learn that they are in fact topological invariants.

7.2. **Cohomology calculations.** There are many intricate methods for calculating de Rham cohomology groups of manifolds, so as to obtain topological and differential-geometric information about them. In this section we consider some fairly simple but important examples. The following theorem is fundamental and has many interesting consequences, as we shall see. We will use its proof as an opportunity to present some further ideas from homological algebra.

Theorem 7.1. *Let X be a manifold and $I \subset \mathbb{R}$ be a nonempty interval. The projection $p : X \times I \to X$ induces an isomorphism $p^* : H^k_{dR}(X) \to H^k_{dR}(X \times I)$ for each $k \geq 0$. For every $s \in I$, the map $i_s : X \to X \times I$, $x \mapsto (x, s)$, induces the same map $i^*_s : H^k_{dR}(X \times I) \to H^k_{dR}(X)$, and this map is the inverse of p^*.*

Proof. This is clear for $k = 0$, since p induces a bijection between the connected components of $X \times I$ and those of X, and i_s induces the inverse bijection regardless of the choice of s.

Assume $k \geq 1$. Fix $s \in I$ and let $i = i_s$. Then $p \circ i = \mathrm{id}_X$, so $i^* \circ p^*$ is the identity on $H^k_{dR}(X)$. We need to show that $p^* \circ i^*$ is the identity on $H^k_{dR}(X \times I)$.

A common method for showing that two morphisms of complexes induce the same map of homology groups is to establish a *homotopy* between the morphisms. Here, a homotopy consists of linear maps $h_k : \Omega^k(X \times I) \to \Omega^{k-1}(X \times I)$ for each k, as shown in the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & \Omega^{k-1}(X \times I) \arrow[r, "d"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "p^*\circ i^*"] & \Omega^{k}(X \times I) \arrow[r, "d"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "p^*\circ i^*"] \arrow[dl, "h_k"] & \Omega^{k+1}(X \times I) \arrow[r] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "p^*\circ i^*"] \arrow[dl, "h_{k+1}"] & \dots \\
\dots \arrow[r] & \Omega^{k-1}(X \times I) \arrow[r, "d"] & \Omega^{k}(X \times I) \arrow[r, "d"] & \Omega^{k+1}(X \times I) \arrow[r] & \dots
\end{tikzcd}$$

such that

$$\mathrm{id} - p^* \circ i^* = d \circ h_k + h_{k+1} \circ d$$

on $\Omega^k(X \times I)$. If we have such a homotopy and $\omega \in \Omega^k(X \times I)$ is closed, then

$$\omega - p^* i^* \omega = dh_k(\omega) + h_{k+1}(d\omega) = dh_k(\omega),$$

so ω and $p^* i^* \omega$ differ by an exact form and represent the same element of $H^k_{dR}(X \times I)$.

It remains to construct the linear maps $h_k : \Omega^k(X \times I) \to \Omega^{k-1}(X \times I)$. Every form $\omega \in \Omega^k(X \times I)$ can be decomposed in a unique way as $\omega_1 + dt \wedge \omega_0$, where $\omega_1 \in \Omega^k(X \times I)$ and $\omega_0 \in \Omega^{k-1}(X \times I)$ do not involve dt (first uniquely decompose ω in a chart; then use a partition of unity to get a global decomposition). Here, t is the standard coordinate on I. Note that $p^*i^*\omega = \omega_1(x,s)$. Define

$$h_k(\omega)(x,t) = \int_s^t \omega_0(x,r)\,dr.$$

More precisely, in a chart on X, write ω_0 as a sum of forms $f(x,t)dx_{i_1}\wedge\cdots\wedge dx_{i_{k-1}}$ and replace $f(x,t)$ by $\int_s^t f(x,r)\,dr$.

Split the exterior derivative d on $X\times I$ as $d = d_X + d_I$, where d_X involves differentiation with respect to coordinates on X only and d_I is differentiation with respect to t. Then $d_I\omega = dt\wedge\partial_t\omega$, where $\partial_t\omega$ is obtained by differentiating the coefficients of ω with respect to t, and

$$\begin{aligned} d\omega &= d_X\omega + dt\wedge\partial_t\omega = d_X\omega_1 - dt\wedge d_X\omega_0 + dt\wedge\partial_t\omega_1 \\ &= d_X\omega_1 + dt\wedge(\partial_t\omega_1 - d_X\omega_0). \end{aligned}$$

Thus,

$$\begin{aligned} dh_k(\omega) + h_{k+1}(d\omega) &= d\int_s^t \omega_0(x,r)\,dr + \int_s^t \big(\partial_r\omega_1(x,r) - d_X\omega_0(x,r)\big)\,dr \\ &= \int_s^t d_X\omega_0(x,r)\,dr + dt\wedge\omega_0(x,t) \\ &\quad +\omega_1(x,t) - \omega_1(x,s) - \int_s^t d_X\omega_0(x,r)\,dr \\ &= \omega - \omega_1(x,s) = \omega - p^*i^*\omega, \end{aligned}$$

and the proof is complete. □

Repeated applications of this result, together with our observations in the previous section about the de Rham cohomology of intervals and about de Rham cohomology being a diffeomorphism invariant, give the following corollary.

Corollary 7.1. *Let X be a manifold diffeomorphic to $\mathbb{R}^n$. Then*

$$H^k_{dR}(X) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

Hence, every closed form on X of degree at least 1 is exact.

Every point in a manifold X has a neighbourhood U diffeomorphic to an open ball in $\mathbb{R}^n$, which in turn is diffeomorphic to $\mathbb{R}^n$ itself. Therefore, if ω is a closed form on X of degree at least 1, then the restriction $\omega|U$ is exact. This yields one version of the famous Poincaré lemma.

Corollary 7.2 (Poincaré lemma)**.** *A differential form of degree at least 1 is closed if and only if it is locally exact.*

The next consequence of Theorem 7.1 is that de Rham cohomology does not distinguish homotopic maps.

Corollary 7.3. *Let X and Y be manifolds and $f, g : X \to Y$ be smooth maps. If f and g are smoothly homotopic, then $f^* = g^* : H^k_{dR}(Y) \to H^k_{dR}(X)$ for all k.*

Proof. By definition, if f and g are smoothly homotopic, then there is a smooth map $F : X \times [0,1] \to Y$ such that $f = F \circ i_0$ and $g = F \circ i_1$, where $i_s : X \to X \times [0,1]$, $x \mapsto (x, s)$. By Theorem 7.1, $i_0^* = i_1^* : H^k_{dR}(X \times [0,1]) \to H^k_{dR}(X)$, so $f^* = i_0^* \circ F^* = i_1^* \circ F = g^*$. □

A smooth map $f : X \to Y$ is called a smooth *homotopy equivalence* if it has a *homotopy inverse*, that is, there is a smooth map $g : Y \to X$ such that $g \circ f$ is homotopic to id_X, and $f \circ g$ is homotopic to id_Y. Clearly, a diffeomorphism is a homotopy equivalence. It follows from Corollary 7.3 that to induce isomorphisms of de Rham cohomology groups, f only needs a homotopy inverse, not an actual inverse. We leave the proof as an exercise.

Corollary 7.4. *If $f : X \to Y$ is a smooth homotopy equivalence between manifolds, then the induced linear maps $f^* : H^k_{dR}(Y) \to H^k_{dR}(X)$ are isomorphisms for all k.*

Since a constant map between manifolds induces the zero map on de Rham cohomology in degree at least 1, we obtain one more corollary.

Corollary 7.5. *Let X be a contractible manifold. Then*

$$H^k_{dR}(X) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

It should be mentioned that a contractible manifold of dimension n need not be diffeomorphic to $\mathbb{R}^n$. Examples of this are not easy to construct. There are 3-dimensional examples (noncompact, without boundary) called *Whitehead manifolds.* They are complements in the 3-sphere of certain complicated closed sets. There are no examples in dimension 2.

We do need to see some non-contractible examples. We conclude this section by computing the de Rham cohomology of the spheres $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, $n \geq 1$. This is a fair bit of work. First we consider the circle S^1. Of course it is only $H^1_{dR}(S^1)$ that is in question. Consider the map from the proof of Proposition 7.1,

$$H^1_{dR}(S^1) \to \mathbb{R}, \quad [\omega] \mapsto \int_{S^1} \omega,$$

where S^1 is given the usual counterclockwise orientation. As explained in that proof, this map is not the zero map, so it is surjective. We will show that it is also injective, from which we can conclude not only that

$$H^1_{dR}(S^1) \cong \mathbb{R},$$

but also that a 1-form ω on S^1 is exact if and only if $\int_{S^1} \omega = 0$.

So suppose ω is a 1-form on S^1 with $\int_{S^1} \omega = 0$. Consider the map $f : \mathbb{R} \to S^1$, $t \mapsto (\cos t, \sin t)$. (If you have studied covering spaces, you will recognise f as the universal covering map of S^1.) Now $f^*\omega$ is exact, so there is a smooth function g on $\mathbb{R}$ with $f^*\omega = dg$. For each $x \in \mathbb{R}$, f restricts to an orientation-preserving diffeomorphism $(x, x + 2\pi) \to S^1 \setminus \{f(x)\}$, so

$$g(x + 2\pi) - g(x) = \int_x^{x+2\pi} dg = \int_x^{x+2\pi} f^*\omega = \int_{S^1} \omega = 0.$$

This shows that g is periodic with period 2π, so there is a smooth function h on S^1 with $g = h \circ f$. Then $f^*\omega = dg = df^*h = f^*dh$. Since f is a local diffeomorphism, this yields $\omega = dh$, so ω is exact.

Now we outline the case of the n-dimensional sphere S^n, $n \geq 2$, using a different method, leaving the verifications of various claims as exercises. The complement in S^n of a single point is diffeomorphic to $\mathbb{R}^n$. The complement of two points in S^n is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$, which in turn is diffeomorphic to $S^{n-1} \times \mathbb{R}$ and in particular connected (this fails for $n = 1$). Write $S^n = U_1 \cup U_2$, where U_1 and U_2 are the complements of the "north pole" $(0, \ldots, 0, 1)$ and the "south pole" $(0, \ldots, 0, -1)$ respectively. Then $V = U_1 \cap U_2$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$, where S^{n-1} may be identified with the "equator" $\{x \in S^n : x_{n+1} = 0\} \subset V$.

Let ω be a closed k-form on S^n, $k \geq 1$. On each U_i, ω is exact, say $\omega|U_i = d\eta_i$, where η_i is a $(k-1)$-form on U_i. Then $d(\eta_1 - \eta_2) = 0$ on V. Suppose $k = 1$. Then $\eta_1 - \eta_2$ is a 0-form, that is, a smooth function on V with vanishing differential. Since V is connected, $\eta_1 - \eta_2 = c$ on V for some constant c. Then we have a well-defined smooth function η on S^n defined as η_1 on U_1 and $\eta_2 + c$ on U_2, and $d\eta = \omega$, so ω is exact. This shows that $H^1_{dR}(S^n) = 0$ for $n \geq 2$.

Now suppose $k \geq 2$. Since the difference $\eta_1 - \eta_2$ is closed on V, and hence on S^{n-1}, it represents a class in $H^{k-1}_{dR}(S^{n-1})$. This class is independent of the choice of η_1 and η_2. Moreover, if ω is exact, say $\omega = d\eta$ for some $(k-1)$-form η on S^n, then we can take $\eta_j = \eta|U_j$, so $\eta_1 - \eta_2$ is zero on V and represents the zero class in $H^{k-1}_{dR}(S^{n-1})$. Thus we have a map

$$H^k_{dR}(S^n) \to H^{k-1}_{dR}(S^{n-1}),$$

which is easily seen to be linear. We claim that this map is an isomorphism. As for injectivity, suppose the map takes the class of ω to zero. This means that the form $\eta_1 - \eta_2$ is exact on S^{n-1}. By Theorem 7.1, it is also exact on V, say $\eta_1 - \eta_2 = d\xi$ for a $(k-2)$-form ξ on V. Let ρ_i, $i = 1, 2$, form a partition of unity subordinate to the open cover $\{U_1, U_2\}$ of S^n and set $\xi_1 = \rho_2\xi$, $\xi_2 = -\rho_1\xi$. Then ξ_1, ξ_2 are smooth $(k-2)$-forms on U_1 and U_2 respectively, and $\xi = (\rho_1 + \rho_2)\xi = \xi_1 - \xi_2$ on V. Since $\eta_1 - d\xi_1 = \eta_2 - d\xi_2$ on V, we obtain a well-defined $(k-1)$-form η on S^n with $\eta|U_i = \eta_i - d\xi_i$. Then $d\eta = \omega$, so ω is exact.

As for surjectivity, suppose θ is a closed $(k-1)$-form on S^{n-1}. Pull it back to V and, using a partition of unity as above, find $(k-1)$-forms θ_i on U_i, $i = 1, 2$, such that $\theta = \theta_1 - \theta_2$ on V. We obtain a well-defined closed k-form on S^n defined as $d\theta_i$ on U_i, whose class in $H^k_{dR}(S^n)$ maps to the class of θ in $H^{k-1}_{dR}(S^{n-1})$.

We now have the following facts about the de Rham groups of spheres.

(1) Since S^n is connected, $H^0_{dR}(S^n) = \mathbb{R}$ for $n \geq 1$.
(2) $H^1_{dR}(S^1) = \mathbb{R}$.
(3) $H^k_{dR}(S^n)$ is isomorphic to $H^{k-1}_{dR}(S^{n-1})$ for $k \geq 2$ and $n \geq 2$.
(4) $H^1_{dR}(S^n) = 0$ for $n \geq 2$.

Putting all this information together, we obtain the following result.

Proposition 7.2. *The de Rham cohomology groups of the n-dimensional sphere S^n, $n \geq 1$, are:*

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Some textbooks prove this proposition using the so-called *Mayer-Vietoris sequence*, which is a much-used tool for calculating cohomology groups (see e.g. Conlon).

We can now verify that, as mentioned in the introduction, the 2-sphere S^2 and the 2-torus T^2 are not diffeomorphic. Since $H^1_{dR}(S^2) = 0$, it suffices to show that $H^1_{dR}(T^2) \neq 0$. The proof illustrates the power of the very general idea of functoriality.

Let us accept that T^2 is diffeomorphic to $S^1 \times S^1$. Consider the smooth maps $i : S^1 \to S^1 \times S^1$, $x \mapsto (x, s)$, for some fixed $s \in S^1$, and $p : S^1 \times S^1$, $(x, y) \mapsto x$. They induce maps

$$H^1_{dR}(S^1) \xrightarrow{p^*} H^1_{dR}(S^1 \times S^1) \xrightarrow{i^*} H^1_{dR}(S^1).$$

Now $p \circ i = \mathrm{id}_{S^1}$, so $i^* \circ p^* = (p \circ i)^*$ is the identity on $H^1_{dR}(S^1) = \mathbb{R}$. Hence, the cohomology group in the middle cannot be zero. (In fact, $H^1_{dR}(T^2) = \mathbb{R}^2$, as can be shown using the Mayer-Vietoris sequence.)

If we are not willing to accept that T^2 is diffeomorphic to $S^1 \times S^1$, then we can work explicitly with the particular torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + 3 = 4\sqrt{x^2 + y^2}\}$$

from the introduction, and use the maps $i : S^1 \to T^2$, $(u, v) \mapsto (u, v, 0)$, and $p : T^2 \to S^1$, $(x, y, z) \mapsto (x, y)/\|(x, y)\|$.

7.3. **Čech cohomology and de Rham's theorem.** Let ω be a closed 1-form on a manifold X. By the Poincaré lemma, ω is locally exact, so there is an open cover $\mathcal{U} = (U_i)_{i \in I}$ of X and smooth functions f_i on U_i such that $\omega|U_i = df_i$ for each $i \in I$. On each intersection $U_{i_0 i_1} = U_{i_0} \cap U_{i_1}$, we have $d(f_{i_0} - f_{i_1}) = 0$, so the function $a_{i_0 i_1} = f_{i_0} - f_{i_1}$ is locally constant. On each triple intersection $U_{i_0 i_1 i_2} = U_{i_0} \cap U_{i_1} \cap U_{i_2}$, we have

$$a_{i_0 i_1} + a_{i_1 i_2} = (f_{i_0} - f_{i_1}) + (f_{i_1} - f_{i_2}) = f_{i_0} - f_{i_2} = a_{i_0 i_2}.$$

(Note that if $U_{i_0 i_1 i_2} = \varnothing$, then this condition is trivially satisfied and imposes no restriction.) A family $(a_{i_0 i_1})_{i_0, i_1 \in I}$ of locally constant functions $a_{i_0 i_1} : U_{i_0 i_1} \to \mathbb{R}$ satisfying $a_{i_0 i_1} + a_{i_1 i_2} = a_{i_0 i_2}$ on $U_{i_0 i_1 i_2}$ for all $i_0, i_1, i_2 \in I$ is called a 1-*cocycle* of locally constant real-valued functions with respect to the open cover $\mathcal{U}$. These cocycles form a real vector space $Z^1(\mathcal{U}, \mathbb{R})$. If ω is exact, say $\omega = df$ where $f : X \to \mathbb{R}$ is smooth, and we take $f_i = f|U_i$, then $(a_{i_0 i_1})$ is the zero vector in $Z^1(\mathcal{U}, \mathbb{R})$.

Let us note two simple properties of a cocycle $(a_{i_0 i_1})$. Taking $i_0 = i_1$, we obtain $a_{i_0 i_0} + a_{i_0 i_2} = a_{i_0 i_2}$ on $U_{i_0 i_2}$ for all $i_0, i_2 \in I$. Hence, $a_{ii} = 0$ on U_i for all $i \in I$. Therefore, taking $i_0 = i_2$, we get $a_{i_0 i_1} + a_{i_1 i_0} = a_{i_0 i_0} = 0$ on $U_{i_0 i_1}$, so $a_{i_0 i_1} = -a_{i_1 i_0}$.

Let us next investigate the dependence of $(a_{i_0 i_1})$ on the choice of the open cover $\mathcal{U}$ and the functions f_i. Suppose we take another open cover $\mathcal{V} = (V_j)_{j \in J}$ of X and functions g_j on V_j with $\omega|V_j = dg_j$ for each $j \in J$, and let $b_{j_0 j_1} = g_{j_0} - g_{j_1}$ on $V_{j_0 j_1} = V_{j_0} \cap V_{j_1}$. Take a common refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$, that is, an open cover $\mathcal{W} = (W_k)_{k \in K}$ of X with maps $\sigma : K \to I$ and $\tau : K \to J$ such that $W_k \subset U_{\sigma(k)}$ and $W_k \subset V_{\tau(k)}$ for every $k \in K$. For example, we could take $K = I \times J$ and $W_{(i,j)} = U_i \cap V_j$. There is a linear map $\sigma^* : Z^1(\mathcal{U}, \mathbb{R}) \to Z^1(\mathcal{W}, \mathbb{R})$ taking $(a_{i_0 i_1})_{i_0, i_1 \in I}$ to $(a_{\sigma(k_0)\sigma(k_1)}|W_{k_0 k_1})_{k_0, k_1 \in K}$, and a similarly defined map $\tau^* : Z^1(\mathcal{V}, \mathbb{R}) \to Z^1(\mathcal{W}, \mathbb{R})$.

On $W_k \subset U_{\sigma(k)} \cap V_{\tau(k)}$, we have $d(f_{\sigma(k)} - g_{\tau(k)}) = \omega - \omega = 0$, so $c_k = f_{\sigma(k)} - g_{\tau(k)}|W_k$ is a locally constant function. The family $(c_k)_{k \in K}$ is called a 0-*cochain* of locally constant real-valued functions with respect to $\mathcal{W}$. Clearly, it yields a 1-cocycle $(c_{k_0 k_1})$ with $c_{k_0 k_1} = c_{k_0} - c_{k_1}$. Cocycles obtained in this way from 0-cochains are called 1-*coboundaries*, and they form a subspace $B^1(\mathcal{W}, \mathbb{R})$ of $Z^1(\mathcal{W}, \mathbb{R})$. (More explicitly, a 1-coboundary with respect to $\mathcal{W}$ is a 1-cocycle of the special form $(c_{k_0} - c_{k_1})_{k_0,k_1 \in K}$, where each c_k is a locally constant function $W_k \to \mathbb{R}$.) Now

$$a_{\sigma(k_0)\sigma(k_1)} - b_{\tau(k_0)\tau(k_1)} = f_{\sigma(k_0)} - f_{\sigma(k_1)} - g_{\tau(k_0)} + g_{\tau(k_1)} = c_{k_0} - c_{k_1} = c_{k_0 k_1},$$

on $W_{k_0 k_1}$, so

$$\sigma^*(a) - \tau^*(b) \in B^1(\mathcal{W}, \mathbb{R}).$$

Thus, the cocycles a and b obtained from ω through different choices of an open cover and anti-derivatives are equal modulo coboundaries after passing to a common refinement of the covers.

This motivates the following definitions. If $\mathcal{U}$ is an open cover of X, then the quotient space

$$\check{H}^1(\mathcal{U}, \mathbb{R}) = Z^1(\mathcal{U}, \mathbb{R})/B^1(\mathcal{U}, \mathbb{R})$$

is called the first *Čech cohomology group* of X with respect to $\mathcal{U}$ with real coefficients. If $\mathcal{W} = (W_k)_{k \in K}$ is a refinement of $\mathcal{U} = (U_i)_{i \in I}$ with $\sigma : K \to I$ such that $W_k \subset U_{\sigma(k)}$ for all $k \in K$, then $\sigma^* : Z^1(\mathcal{U}, \mathbb{R}) \to Z^1(\mathcal{W}, \mathbb{R})$ takes coboundaries to coboundaries and therefore induces a map

$$\sigma^* : \check{H}^1(\mathcal{U}, \mathbb{R}) \to \check{H}^1(\mathcal{W}, \mathbb{R}).$$

This map only depends on $\mathcal{U}$ and $\mathcal{W}$. Any map $\sigma : K \to I$ with $W_k \subset U_{\sigma(k)}$ for all $k \in K$ induces the same map $\sigma^* : \check{H}^1(\mathcal{U}, \mathbb{R}) \to \check{H}^1(\mathcal{W}, \mathbb{R})$ (exercise).

Let $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ be open covers of X. We say that classes $\alpha \in \check{H}^1(\mathcal{U}, \mathbb{R})$ and $\beta \in \check{H}^1(\mathcal{V}, \mathbb{R})$ are equivalent if there is a common refinement $\mathcal{W} = (W_k)_{k \in K}$ of $\mathcal{U}$ and $\mathcal{V}$ with maps $\sigma : K \to I$ and $\tau : K \to J$ such that $\sigma^*\alpha = \tau^*\beta$. This defines an equivalence relation on the union of the Čech cohomology groups $\check{H}^1(\mathcal{U}, \mathbb{R})$ for all open covers $\mathcal{U}$ of X. The set of equivalence classes is called the first *Čech cohomology group* of X with coefficients in $\mathbb{R}$, denoted $\check{H}^1(X, \mathbb{R})$. This set is, in a natural way, a real vector space: the vector space structure is uniquely determined by the requirement that the restriction of the quotient map to $\check{H}^1(\mathcal{U}, \mathbb{R}) \to \check{H}^1(X, \mathbb{R})$ be linear for every open cover $\mathcal{U}$ of X.

With these definitions, we have a well-defined map $\Phi : H^1_{dR}(X) \to \check{H}^1(X, \mathbb{R})$, which is easily seen to be linear.

Theorem 7.2 (De Rham's theorem)**.** *Let X be a manifold. The map*

$$\Phi : H^1_{dR}(X) \to \check{H}^1(X, \mathbb{R})$$

is a linear isomorphism.

Proof. Remember how Φ acts. Take a class α in $H^1_{dR}(X)$, find a closed 1-form ω representing α, and choose an open cover $(U_i)_{i \in I}$ of X and antiderivatives f_i for ω on U_i. Then $\Phi(\alpha)$ is the class in $\check{H}^1(X, \mathbb{R})$ represented by the 1-cocycle $(a_{ij}) \in Z^1(\mathcal{U}, \mathbb{R})$ with $a_{ij} = f_i - f_j$ on $U_i \cap U_j$.

To define a map Ψ in the other direction, we take a class β in $\check{H}^1(X, \mathbb{R})$ and represent it by a 1-cocycle $(a_{ij}) \in Z^1(\mathcal{U}, \mathbb{R})$ with respect to some open cover $(U_i)_{i \in I}$ of X. Find a partition of unity (ρ_i) subordinate to (U_i). Then

$$f_i = \sum_{\nu \in I} a_{i\nu} \rho_\nu$$

is a well-defined smooth function on U_i with

$$f_i - f_j = \sum_{\nu \in I} (a_{i\nu} - a_{j\nu})\rho_\nu = \sum_{\nu \in I} a_{ij}\rho_\nu = a_{ij} \sum_{\nu \in I} \rho_\nu = a_{ij}$$

on U_{ij}. Also,

$$df_i - df_j = \sum_{\nu \in I} (a_{i\nu} - a_{j\nu})d\rho_\nu = a_{ij} \sum_{\nu \in I} d\rho_\nu = a_{ij}\, d \sum_{\nu \in I} \rho_\nu = 0$$

on U_{ij}, so there is a well-defined smooth 1-form ω on X with $\omega|U_i = df_i$. We take $\Psi(\beta)$ to be the class of ω in $H^1_{dR}(X)$. It is now easily verified that Ψ is well defined and that Φ and Ψ are inverse to each other. $\square$

The definition of the first Čech cohomology group of X did not refer to the smooth structure of X in any way. It only used open covers of X and locally constant functions on open subsets of X. These are of course determined by the topology of X (indeed, our definition of Čech cohomology applies to arbitrary topological spaces). It follows that a homeomorphism between manifolds induces an isomorphism of their first Čech cohomology groups. Hence, by de Rham's theorem, their first de Rham cohomology groups will be isomorphic as well.

Corollary 7.6. *The first de Rham cohomology group of a manifold is a topological invariant.*

Higher Čech cohomology groups can be defined and shown to be isomorphic to the corresponding de Rham cohomology groups, so homeomorphic manifolds in fact have isomorphic de Rham cohomology groups in every degree.

8. Exercises

In addition to the exercises in this section, the notes leave numerous details to be checked by the reader and should be read with paper and pencil at hand. More exercises can be found in the references.

2.1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that:

(a) f is continuous at (0,0),

(b) both partial derivatives $D_1 f$ and $D_2 f$ exist at (0,0), but

(c) f is not differentiable at (0,0).

(d) Thus, f cannot be continuously differentiable. Verify this directly.

2.2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map with $f(tx) = tf(x)$ for every $x \in \mathbb{R}^n$ and every $t \in \mathbb{R}$. Show that f is linear. *Hint.* Use the definition of differentiability.

2.3. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Prove that f is infinitely differentiable at 0 with $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Do this in all detail, using L'Hôpital's rule. Here, $f^{(n)}$ denotes the n-th derivative of f (and $f^{(0)}$ is understood to be f itself). It follows that f is *not* equal to the sum of its Taylor series at 0, so f is not real-analytic.

2.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that f is not injective. *Hint.* Think about the statement: it seems intuitively obvious, but it isn't trivial to prove. Say $D_1 f$ is nowhere zero on some open disc D (what if it isn't?). Apply the inverse function theorem to the map $g : D \to \mathbb{R}^2$, $g(x, y) = (f(x, y), y)$.

2.5. Let U be a connected open set in $\mathbb{R}^n$ and $f : U \to U$ be a continuously differentiable map with $f \circ f = f$. Show that f has constant rank on $f(U)$.

3.1. Let $\mathcal{A}$ be an atlas on a topological space X. Let $\mathcal{B}$ be the set of those charts on X that are compatible with every chart in $\mathcal{A}$. Show that:

(a) $\mathcal{A} \subset \mathcal{B}$.

(b) $\mathcal{B}$ is an atlas on X.

(c) $\mathcal{B}$ is the largest atlas containing $\mathcal{A}$, that is, if $\mathcal{C}$ is an atlas on X and $\mathcal{A} \subset \mathcal{C}$, then $\mathcal{C} \subset \mathcal{B}$.

(d) $\mathcal{B}$ is a *maximal* atlas, that is, if $\mathcal{C}$ is an atlas on X and $\mathcal{B} \subset \mathcal{C}$, then $\mathcal{B} = \mathcal{C}$.

(e) $\mathcal{B}$ is the unique maximal atlas containing $\mathcal{A}$.

3.2. Consider the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. Since $\mathbb{R}^n$ is a second countable Hausdorff space, so is S^n with the induced topology. We turn S^n into an n-dimensional manifold by defining an atlas $\mathcal{A}$ on it with $2n+2$ charts $(U_i^{\pm}, h_i^{\pm})$, $i = 1, \ldots, n+1$, where

$$U_i^+ = \{x \in S^n : x_i > 0\}, \qquad U_i^- = \{x \in S^n : x_i < 0\},$$

and $h_i^{\pm} : U_i^{\pm} \to D = \{y \in \mathbb{R}^n : \|y\| < 1\}$ takes x to $(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1})$, where the hat means that x_i is omitted. (Draw a picture for S^1 and S^2!)

Prove that $\mathcal{A}$ really is an atlas on S^n, that is:

(a) the maps $h_i^{\pm}$ are homeomorphisms,

(b) the domains of the charts in $\mathcal{A}$ cover S^n, and

(c) any two of the charts in $\mathcal{A}$ are compatible.

This is the *standard* smooth structure on S^n. It is a deep and remarkable result of John Milnor, published in 1956, that S^7, viewed as a topological space, has other smooth structures, not diffeomorphic to the standard one. Much research has been done since on such questions.

3.3. Let $\mathcal{A}$ be the standard maximal atlas on $\mathbb{R}$ containing the atlas $\{\mathrm{id}_{\mathbb{R}}\}$ and $\mathcal{A}'$ be the maximal atlas containing the atlas $\{\phi\}$, where ϕ is the homeomorphism $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$. Show that $\mathcal{A}$ and $\mathcal{A}'$ are not the same smooth structure, but that they are diffeomorphic.

3.4. Show that an open ball in $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$ itself.

3.5. Let X and Y be manifolds such that X is compact and Y is connected. Show that any submersion $X \to Y$ is surjective. *Hint.* Start by using the rank theorem to show that a submersion is open, that is, maps an open set onto an open set.

3.6. Let X be a compact manifold and $f : X \to \mathbb{R}$ be a smooth function. Show that f has at least two critical points.

3.7. Show that the torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + 3 = 4\sqrt{x^2 + y^2}\}$$

is a submanifold of $\mathbb{R}^3$, using the theorem about the inverse image of a regular value.

3.8. Show that the set

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + x^3 y^4 = z^2 + 1\}$$

is a 2-dimensional submanifold of $\mathbb{R}^3$.

3.9. Give an example to show that if $f : X \to Y$ is a smooth map and $c \in Y$ is a critical value of f, then the preimage $f^{-1}(c)$ need not be a submanifold of X.

3.10. Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map, then the graph $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = f(x)\}$ is an n-dimensional submanifold of $\mathbb{R}^{n+m}$.

3.11. Let U and V be open subsets of a manifold X with $\overline{U} \subset V$. Show that if $f : V \to \mathbb{R}$ is a smooth function, then there is a smooth function $g : X \to \mathbb{R}$ such that $g = f$ on U. *Hint.* Use partitions of unity.

3.12. Let Y be a closed submanifold of a manifold X. Let $f : Y \to \mathbb{R}$ be a smooth function. Show that f extends to a smooth function on X, that is, there is a smooth function $g : X \to \mathbb{R}$ such that $g|Y = f$. *Hint.* Use partitions of unity.

3.13. Let X be a manifold. Show that there exists a smooth function $f : X \to [0, \infty)$ which is proper, meaning that $f^{-1}[0, c]$ is compact in X for every $c > 0$. *Hint.* Use partitions of unity.

3.14. Let $a_1, \ldots, a_m$ be distinct points in a manifold X. Let $c_1, \ldots, c_m$ be real numbers. Show that there exists a smooth function $f : X \to \mathbb{R}$ such that $f(a_i) = c_i$ for $i = 1, \ldots, m$.

3.15. Recall that an *ideal* in a ring R (commutative and with a unity) is an additive subgroup $\mathfrak{a}$ of R such that if $r \in R$ and $a \in \mathfrak{a}$, then $ra \in \mathfrak{a}$. An ideal $\mathfrak{a}$ is *maximal* if $\mathfrak{a} \neq R$ and the only ideal properly containing $\mathfrak{a}$ is R itself. Equivalently, the quotient ring $R/\mathfrak{a}$ is a field.

(a) Let X be a manifold. Show that for every $a \in X$, the set

$$\mathfrak{m}_a = \{f \in C^\infty(X) : f(a) = 0\}$$

is a maximal ideal in $C^\infty(X)$.

(b) Let $\mathcal{M}$ be the set of all maximal ideals in $C^\infty(X)$. By (a), there is a well-defined map

$$\mu : X \to \mathcal{M}, \quad a \mapsto \mathfrak{m}_a.$$

Show that μ is injective. Assuming X is compact, show that μ is bijective.

So we can recover a compact manifold (at least as a set) from the algebraic structure of its ring of smooth functions!

3.16. Show that if X and Y are manifolds, Y is connected, and $f : X \to Y$ is a map, not assumed to be continuous, such that $h \circ f$ is smooth on X for every smooth function h on Y, then f is smooth. *Hint.* Use Whitney's embedding theorem.

3.17. (Challenge problem.) Classify all 1-dimensional manifolds (up to diffeomorphism). *Hint.* See Conlon, section 1.6.

4.1. Let X and Y be manifolds such that X is connected. Show that if $f : X \to Y$ is a smooth map and $d_a f = 0$ for all $a \in X$, then f is constant.

4.2. Consider the $\mathbb{R}$-algebra C_0^∞ of germs of smooth functions at the origin in $\mathbb{R}^n$, $n \geq 1$. Define $\mathfrak{m} = \{f \in C_0^\infty : f(0) = 0\}$. Show that $\mathfrak{m}$ is the unique maximal ideal in C_0^∞.

4.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$ and $f^{(n)}(0) \neq 0$. Show that there is $\epsilon > 0$ and a smooth function $h : (-\epsilon, \epsilon) \to \mathbb{R} \setminus \{0\}$ such that $f(x) = x^n h(x)$ for $|x| < \epsilon$.

4.4. Let X be an n-dimensional manifold and $a \in X$. Show that the $\mathbb{R}$-algebras $C^\infty_{X,a}$ and $C^\infty_{\mathbb{R}^n,0}$ are isomorphic. Thus there is really only one algebra of germs of smooth functions in each dimension.

4.5. Recall that a *zero divisor* in a ring is an element $a \neq 0$ such that $ab = 0$ for some $b \neq 0$. A ring is called an *integral domain* if it has no zero divisors. Is $C^\infty_{\mathbb{R}^n,0}$ an integral domain?

4.6. Let X be a manifold and $f : X \to \mathbb{R}$ be a smooth function. We have given two descriptions of the derivative $d_a f$ of f at a point $a \in X$, as the linear map $d_a^1 f : T_a X \to \mathbb{R}$ that acts on derivations by $D \mapsto D(f_a)$, and as the linear map $d_a^2 f : T_a X \to T_{f(a)}\mathbb{R}$ that acts on derivations by $D \mapsto D \circ f^*$. Show that these two descriptions are equivalent in the sense that $d_a^1 f = \iota \circ d_a^2 f$, where ι is the isomorphism $T_{f(a)}\mathbb{R} \to \mathbb{R}$, $[\gamma] \mapsto \gamma'(0)$.

5.1. The ordered bases (dx, dy, dz) for $\Omega^1(\mathbb{R}^3)$, $(dy \wedge dz, dz \wedge dx, dx \wedge dy)$ for $\Omega^2(\mathbb{R}^3)$, and $dx \wedge dy \wedge dz$ for $\Omega^3(\mathbb{R}^3)$, viewed as modules over the ring $A = C^\infty(\mathbb{R}^3) = \Omega^0(\mathbb{R}^3)$, yield linear isomorphisms $\Psi_k : A^{r_k} \to \Omega^k(\mathbb{R}^3)$, $r_1 = r_2 = 3$, $r_3 = 1$.

Show that under these isomorphisms, the exterior derivative corresponds to the differential operators grad, curl, and div from multivariable calculus. More precisely, for a function f, a 1-form η, and a 2-form ω,

$$df = \Psi_1(\operatorname{grad} f), \quad d\eta = \Psi_2(\operatorname{curl}(\Psi_1^{-1}\eta)), \quad d\omega = \Psi_3(\operatorname{div}(\Psi_2^{-1}\omega)).$$

Recall that (with subscripts denoting partial derivatives)

$$\operatorname{grad} f = (f_x, f_y, f_z), \operatorname{curl}(f, g, h) = (h_y - g_z, f_z - h_x, g_x - f_y), \operatorname{div}(f, g, h) = f_x + g_y + h_z.$$

5.2. Recall how we defined the exterior derivative of a differential form on an open set U in $\mathbb{R}^n$. For a smooth function f on U, we defined

$$df = \sum_{i=1}^n D_i f\, dx_i \quad \text{and} \quad d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

The second equation uniquely determines the linear map $d : \Omega^k(U) \to \Omega^{k+1}(U)$, $k = 1, \ldots, n$. Prove the following three properties of d.

(1) If ω is a p-form and η is a q-form, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

(2) For every form ω, we have $d(d\omega) = 0$. Briefly, $d^2 = 0$.

(3) If U is open in $\mathbb{R}^n$, V is open in $\mathbb{R}^m$, $f : U \to V$ is smooth, and ω is a differential form on V, then

$$f^*(d\omega) = d(f^*\omega).$$

5.3. Recall how we extended the definition of the exterior derivative from open sets in Euclidean space to arbitrary manifolds. If ω is a k-form on a manifold X and (U, ϕ) is a chart on X, we defined

$$d\omega|U = \phi^* d(\phi^{-1})^*(\omega|U).$$

(a) Show that this definition is independent of the chart ϕ, so we have a well-defined map $d : \Omega^k(U) \to \Omega^{k+1}(U)$ for $k = 0, \ldots, n$.

(b) Show that d is linear and satisfies properties (1) and (2) in exercise 5.2.

(c) Show that for $k = 0$, d as defined here is equal to d as defined previously (in section 4.1 of the lectures).

(d) Show that $f^*(d\omega) = d(f^*\omega)$ whenever ω is a differential form on the target of the smooth map f.

5.4. Find a smooth 2-form ω on $\mathbb{R}^4$ such that $\omega \wedge \omega \neq 0$.

5.5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Show that

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = \det(Df)\, dx_1 \wedge \cdots \wedge dx_n.$$

5.6. Show that if X and Y are n-dimensional oriented manifolds, $f : X \to Y$ is a diffeomorphism preserving orientation, and ω is an n-form on Y with compact support, then $\int_X f^*\omega = \int_Y \omega$.

5.7. Let X be a manifold covered by two orientable open subsets whose intersection is connected. Show that X is orientable.

5.8. Show that the n-sphere S^n is orientable. (See exercise 3.2 or 5.7.)

6.1. View the compact interval $[0, 1]$ as a manifold with boundary embedded in $\mathbb{R}$ with the standard smooth structure. Show that $[0, 1]$ carries no oriented atlas. (Still, $[0, 1]$ has a volume form, such as dx, so it is orientable; this is the anomaly mentioned in section 6.1.)

6.2. Let X be a manifold with nonempty boundary ∂X. Show that there is a smooth function $f : X \to [0, \infty)$ such that $\partial X = f^{-1}(0)$.

6.3. If f and g are smooth functions on a compact interval $[a, b]$, then the formula for *integration by parts* states that

$$\int_a^b f'(x)g(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\, dx.$$

State and prove a generalisation of this formula to differential forms on a manifold with boundary.

6.4. Let X be a manifold, $\gamma : [a, b] \to X$ be a smooth path in X, and ω be a smooth 1-form on X. We define the *path integral* of ω along γ by the formula

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega.$$

Here we endow $[a, b]$ with the standard orientation defined by the volume form dx.

(a) Prove that if $\psi : [a,b] \to [a,b]$ is an orientation-preserving diffeomorphism — what does this mean in concrete terms? — then

$$\int_{\gamma\circ\psi} \omega = \int_{\gamma} \omega.$$

(b) Prove that if f is a smooth function on X, then

$$\int_{\gamma} df = f(\gamma(b)) - f(\gamma(a)).$$

6.5. Show that a contractible manifold is connected.

6.6. Let X and Y be manifolds. We have defined smooth maps $f, g : X \to Y$ to be smoothly homotopic if there is a smooth map $F : [0,1] \times X \to Y$ with $F(0,\cdot) = f$ and $F(1,\cdot) = g$. Prove that this defines an equivalence relation on the set of all smooth maps $X \to Y$. Transitivity is subtle! Start by proving that there is a smooth map $G : [0,1] \times X \to Y$ and $\epsilon > 0$ with $G(t,\cdot) = f$ for $t \leq \epsilon$ and $G(t,\cdot) = g$ for $t \geq 1 - \epsilon$.

6.7. Suppose $f, g : X \to Y$ are smoothly homotopic smooth maps. Show that if $h : W \to X$ is a smooth map, then $f \circ h$ and $g \circ h$ are smoothly homotopic. Show that if $k : Y \to Z$ is a smooth map, then $k \circ f$ and $k \circ g$ are smoothly homotopic.

6.8. Suppose X and Y are manifolds such that either Y is contractible, or X is contractible and Y is connected. Show that any two smooth maps $X \to Y$ are smoothly homotopic.

6.9. Let X be a compact connected orientable manifold. Let $f : X \to X$ be a smooth map. Show that if f is smoothly homotopic to the identity, then f is surjective.

7.1. Show that:

(a) if differential forms ω and η are closed, then $\omega \wedge \eta$ is closed.

(b) if one of ω and η is closed and the other is exact, then $\omega \wedge \eta$ is exact.

(c) the pullback by a smooth map of a closed form is closed.

(d) the pullback by a smooth map of an exact form is exact.

7.2. Let U be an open subset of $\mathbb{R}^2$. Formulate a necessary and sufficient condition in terms of de Rham cohomology for every smooth map $f = (f_1, f_2) : U \to \mathbb{R}^2$ with $\partial f_1/\partial x_2 = \partial f_2/\partial x_1$ to be the gradient of a smooth function on U.

7.3. Let $B^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ be the closed unit disc in $\mathbb{R}^2$ viewed as a manifold with boundary $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$. Let the circle S^1 have the usual counterclockwise orientation. Let X be a manifold. A *loop* in X is a smooth map $\gamma : S^1 \to X$. Say γ is *contractible* if it extends to a smooth map $B^2 \to X$. The *loop integral* of a 1-form ω on X along γ is

$$\int_{\gamma} \omega = \int_{S^1} \gamma^*\omega.$$

Prove that $\int_{\gamma} \omega = 0$ for every exact 1-form ω on X. Prove that if γ is contractible, then $\int_{\gamma} \omega = 0$ for every closed 1-form ω on X.

7.4. Let X be a compact orientable submanifold of $\mathbb{R}^n$, which is not just a single point. Show that there is no smooth retraction $\mathbb{R}^n \to X$.

7.5. Let X be an n-dimensional compact oriented manifold. Show that the map

$$H^k_{dR}(X) \times H^{n-k}_{dR}(X) \to \mathbb{R}, \quad ([\omega], [\eta]) \mapsto \int_X \omega \wedge \eta,$$

is well defined for $k = 0, \ldots, n$.

The map is clearly bilinear and thus induces a linear map $H^k_{dR}(X) \to H^{n-k}_{dR}(X)^*$. The *Poincaré duality theorem* states that this map is an isomorphism.

7.6. Let $f : X \to Y$ be a smooth homotopy equivalence between manifolds. Prove that the induced maps $f^* : H^k_{dR}(Y) \to H^k_{dR}(X)$ are isomorphisms for all k.

7.7. Calculate the de Rham cohomology groups of $\mathbb{R}^n \setminus \{0\}$ for each $n \geq 1$.

7.8. Let X and Y be manifolds. Show that for every k,

$$\dim H^k_{dR}(X) \leq \dim H^k_{dR}(X \times Y).$$

7.9. For a manifold X, let $\Omega^k_c(X)$ be the vector space of smooth k-forms on X with compact support. The k-th de Rham cohomology group of X with compact supports is the quotient space

$$H^k_c(X) = \frac{\{\omega \in \Omega^k_c(X) : d\omega = 0\}}{d\Omega^{k-1}_c(X)}.$$

(Here we take $\Omega^{-1}_c(X) = 0$.) Calculate $H^k_c(\mathbb{R})$ for $k = 0, 1$.

7.10. Consider the open cover $\mathcal{U}$ of the n-sphere S^n, $n \geq 1$, consisting of the complement of the north pole and the complement of the south pole. Calculate the Čech cohomology group $\check{H}^1(\mathcal{U}, \mathbb{R})$. Compare it with the de Rham cohomology group $H^1_{dR}(S^n)$.

7.11. Let $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{W} = (W_k)_{k \in K}$ be open covers of a topological space, such that $\mathcal{W}$ refines $\mathcal{U}$, meaning that for every $k \in K$, there is $i \in I$ with $W_k \subset U_i$. By the Axiom of Choice, there is a map $\sigma : K \to I$ such that $W_k \subset U_{\sigma(k)}$ for all $k \in K$. Suppose $\rho : K \to I$ is another such map. Show that $\rho^* = \sigma^* : \check{H}^1(\mathcal{U}, \mathbb{R}) \to \check{H}^1(\mathcal{W}, \mathbb{R})$.

9. References

Boothby, W. M. *An introduction to differentiable manifolds and Riemannian geometry.* 513.83 B725i

Bröcker, T. and Jänich, K. *Introduction to differential topology.* 514.76 B865e

Conlon, L. *Differentiable manifolds: a first course.* 514.76 C752d

Lee, J. M. *Introduction to smooth manifolds.* 514.76 L478i

Madsen, I. H. and Tornehave, J. *From calculus to cohomology.* 515.142 M183f

Milnor, J. W. *Topology from the differentiable viewpoint.* 513.83 M659t

Rudin, W. *Principles of mathematical analysis.* 517 R91

Spivak, M. *Calculus on manifolds.* 517.2 S761

Warner, F. W. *Foundations of differentiable manifolds and Lie groups.* 514.76 W281f