MAM4

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Phase-type (*PH*-type) distributions

Consider a Markov chain with $n+1$ states, initial probability vector $(\boldsymbol{\alpha}, \alpha_{n+1})$ and transition rate matrix

$$Q = \begin{bmatrix} S & \boldsymbol{S}^0 \\ \mathbf{0} & 0 \end{bmatrix},$$

where S is a non-singular $n \times n$ matrix with

$$\begin{aligned} S_{ii} &< 0, \\ S_{ij} &\geq 0 \quad \text{for all } i \neq j \text{ and} \\ \boldsymbol{S}^0 &\geq 0, \end{aligned}$$

where $\boldsymbol{S}^0$ is an n-vector of rates such that

$$S\boldsymbol{e} + \boldsymbol{S}^0 = \mathbf{0}.$$

Markovian Arrival Processes (*MAP*s)

The *MAP*

- Allows other than negative exponentially distributed inter-arrival times.
- Preserves the Markov property for state transitions.
- Has a convenient matrix form representation.

Consider a two state process and associate with probability $p_{i,i}(1)$ an arrival at the end of a sojourn in state i for $i \in \{1, 2\}$, which immediately returns the process to state i. With probability $p_{i,j}(0) = 1 - p_{i,i}(1)$, there is no arrival but a transition to the state j where $j \neq i$. Hence we may write

$$D_0 = \begin{bmatrix} -\lambda_1 & p_{1,2}(0)\lambda_1 \\ p_{2,1}(0)\lambda_2 & -\lambda_2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} p_{1,1}(1)\lambda_1 & 0 \\ 0 & p_{2,2}(1)\lambda_2 \end{bmatrix}.$$

It is a natural extension to allow transitions between states to correspond to arrivals as well, which generalises to the *MAP* and in the two state case has the following representation

$$D_0 = \begin{bmatrix} -\lambda_1 & p_{1,2}(0)\lambda_1 \\ p_{2,1}(0)\lambda_2 & -\lambda_2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} p_{1,1}(1)\lambda_1 & p_{1,2}(1)\lambda_1 \\ p_{2,1}(1)\lambda_2 & p_{2,2}(1)\lambda_2 \end{bmatrix},$$

where $\sum_j p_{i,j}(.) = 1$ for each $i \in \{1, 2\}$.

PH-type distributions can be used to model arrival processes by using them as inter-event times in a renewal process.

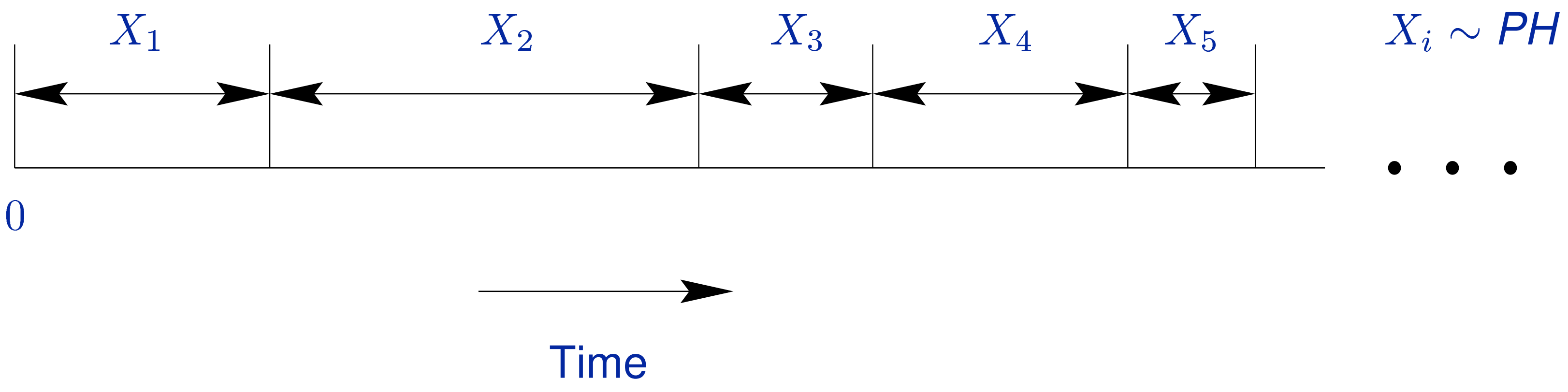


Figure 1: *PH-renewal process.*

The resultant *PH*-renewal process represented by $(\boldsymbol{\alpha}, S)$ has the following *MAP* representation

$$D_0 = S \text{ and } D_1 = \boldsymbol{S}^0 \boldsymbol{\alpha}.$$

In order to display the capability of the *MAP* it is useful to observe some realisations of point processes.

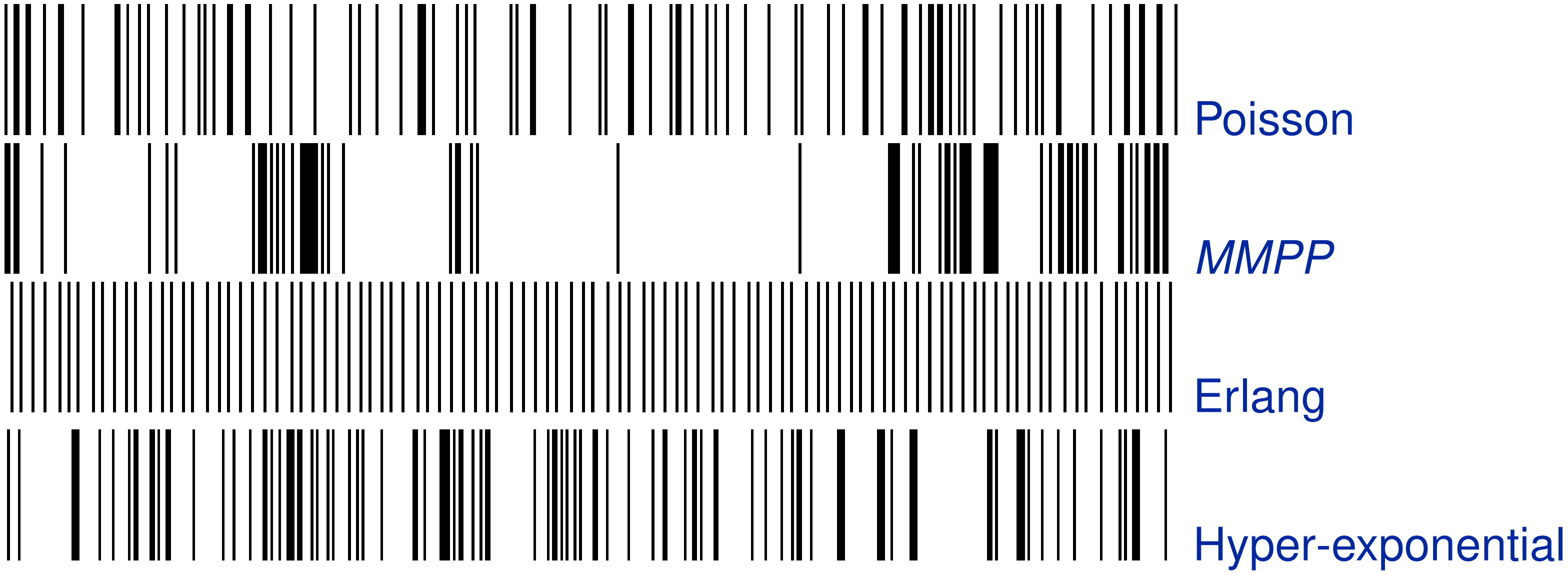
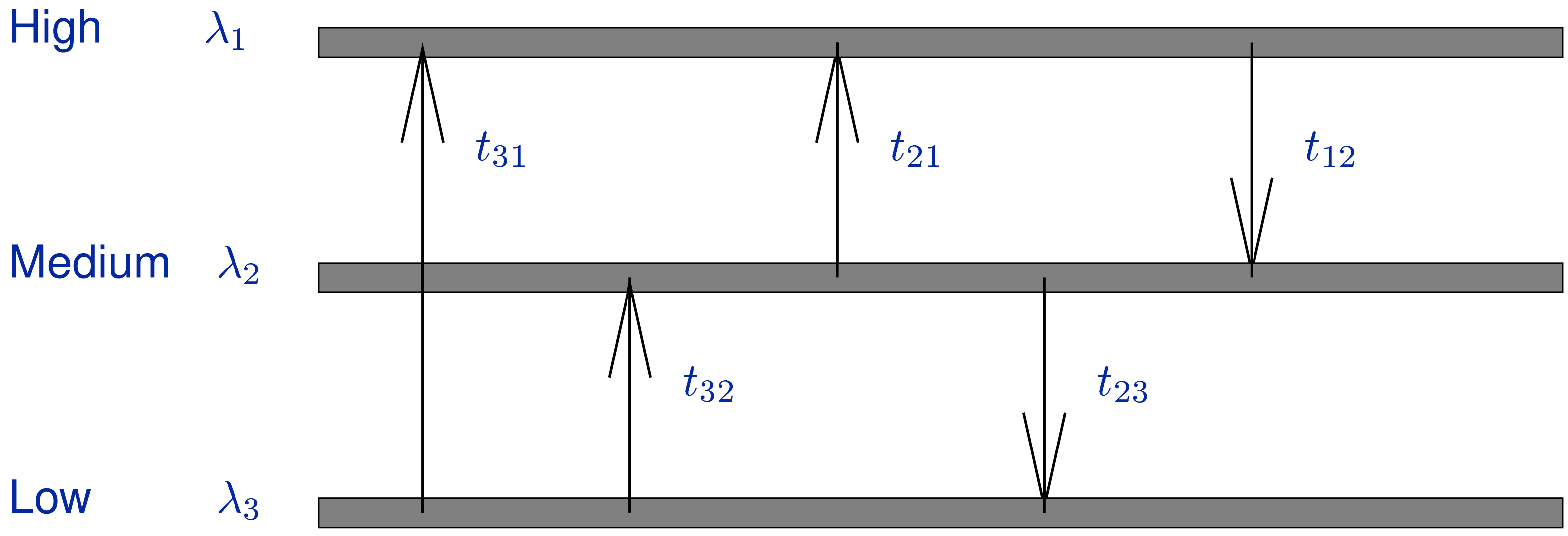
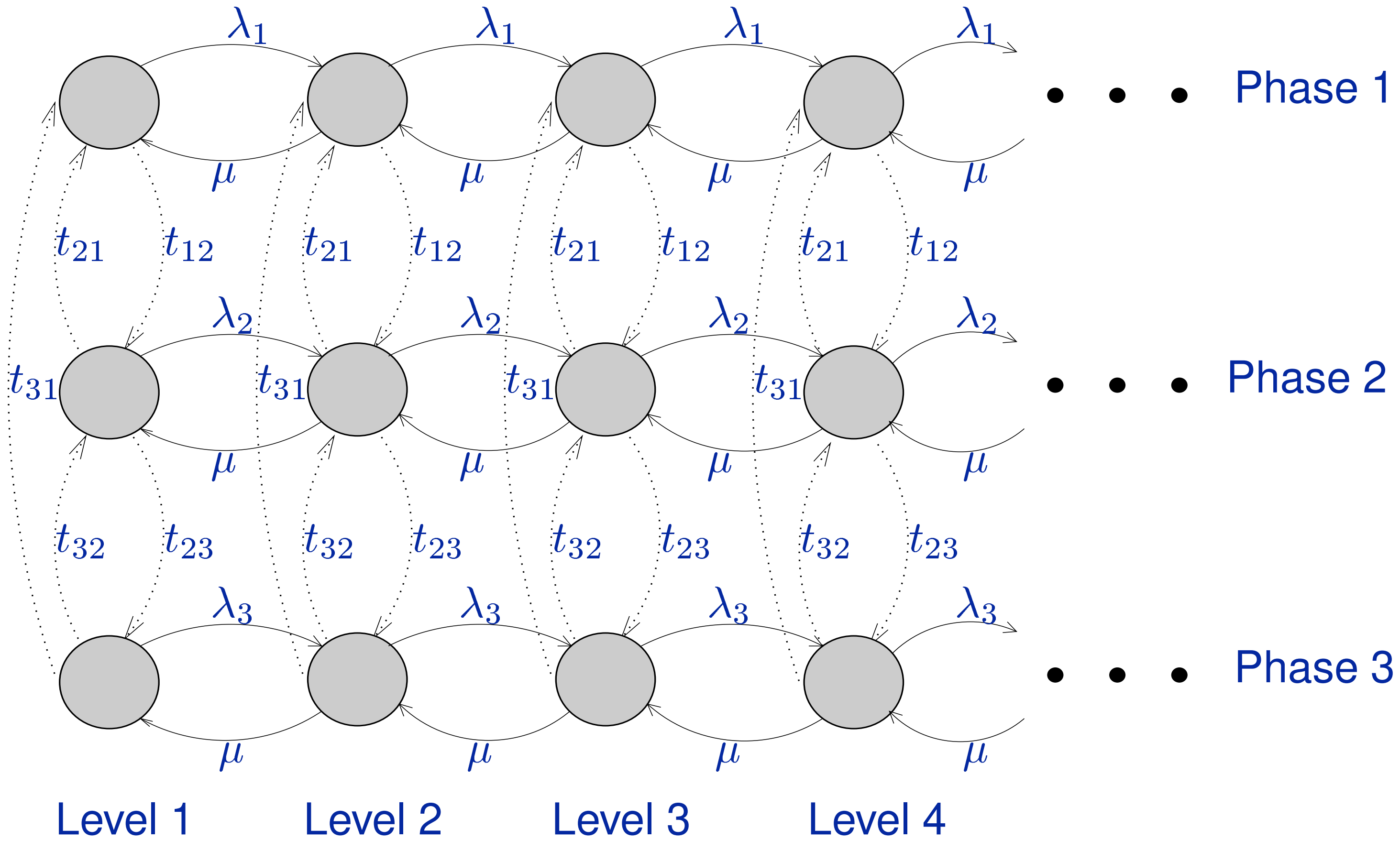


Figure 2: *Point processes of arrivals*

Consider an *M/M/1* queue in a random environment, where the arrival rate changes only as depicted and the service rate remains constant at rate μ.





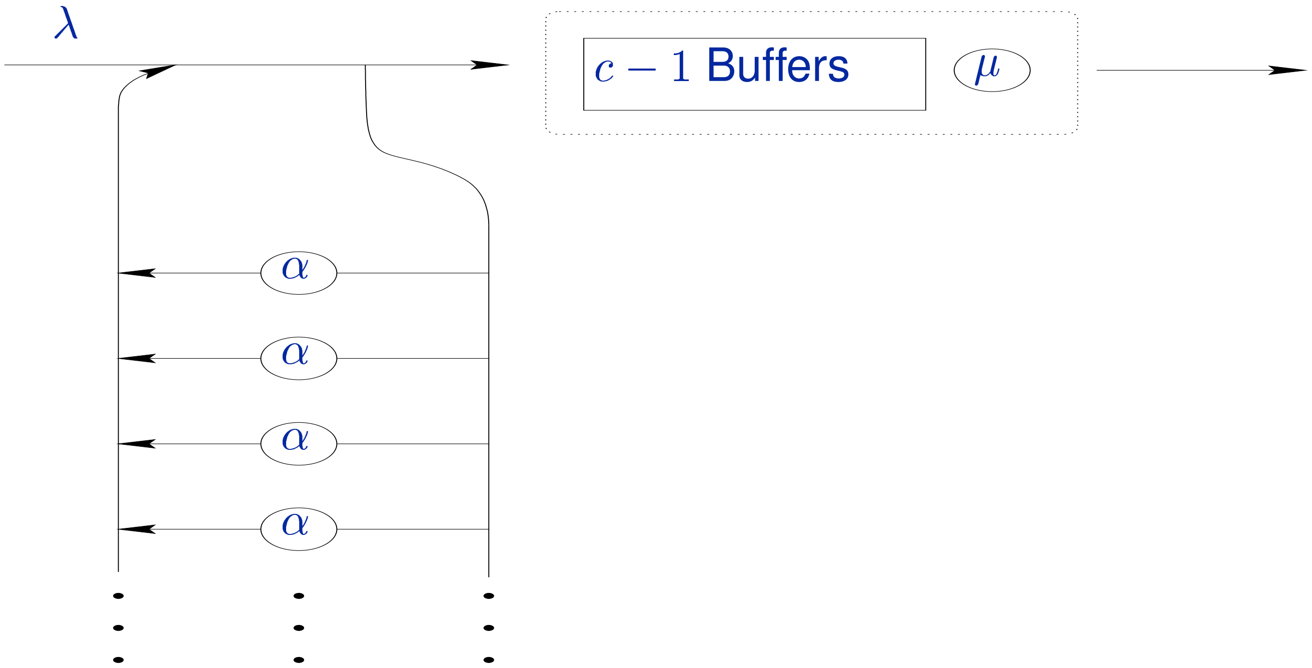
λ_1
λ_2
λ_3
μ
t_{21}
t_{12}
t_{31}
t_{32}
t_{23}
Phase 1
Phase 2
Phase 3
Level 1
Level 2
Level 3
Level 4

$$
Q = \begin{bmatrix}
\begin{bmatrix} \alpha & t_{12} & 0 \\ t_{21} & \beta & t_{23} \\ t_{31} & t_{32} & \gamma \end{bmatrix} & \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \cdots \\
\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} & \begin{bmatrix} \delta & t_{12} & 0 \\ t_{21} & \epsilon & t_{23} \\ t_{31} & t_{32} & \eta \end{bmatrix} & \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} & \\
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} & \begin{bmatrix} \delta & t_{12} & 0 \\ t_{21} & \epsilon & t_{23} \\ t_{31} & t_{32} & \eta \end{bmatrix} & \\
\vdots & & & \ddots
\end{bmatrix}
$$

$$
\begin{aligned}
\alpha &= -(t_{12} + \lambda_1), \\
\beta &= -(t_{21} + \lambda_2 + t_{23}), \\
\gamma &= -(t_{31} + t_{32} + \lambda_3), \\
\delta &= \alpha - \mu, \\
\epsilon &= \beta - \mu \quad \text{and} \\
\eta &= \gamma - \mu.
\end{aligned}
$$

Note that this is also the Q matrix for an *MMPP/M/1* queue, where the arrival (birth) process is the Markov Modulated Poisson Process.

Another example: Consider an *M/M/1/*$(c-1)$ queue where arrivals finding the buffer full, go away and retry after an exponentially distributed length of time with parameter α represented diagrammatically as



If we let N_1 be the number of customers in the queue and N_2 be the number of customers in retrial, we can establish a two dimensional Markov process description using a block Q matrix as follows. We first chose N_1 to be the phase, which in this case will give us a finite block size. If we had chosen N_2 to be the phase, this would not have been the case.

The following transitions and rates apply

$$
\begin{aligned}
(n,i) &\rightarrow (n,i+1) && \text{at rate } \lambda \text{ if } 0 \le i \le c-1, \\
(n,i) &\rightarrow (n,i-1) && \text{at rate } \mu \text{ if } 1 \le i \le c, \\
(n,i) &\rightarrow (n+1,i) && \text{at rate } \lambda \text{ if } i \ge c, \\
(n,i) &\rightarrow (n-1,i) && \text{at rate } n\alpha \text{ if } n \ge 1, \quad 0 \le i \le c-1.
\end{aligned}
$$

From these transition rates we may construct a Q matrix as follows

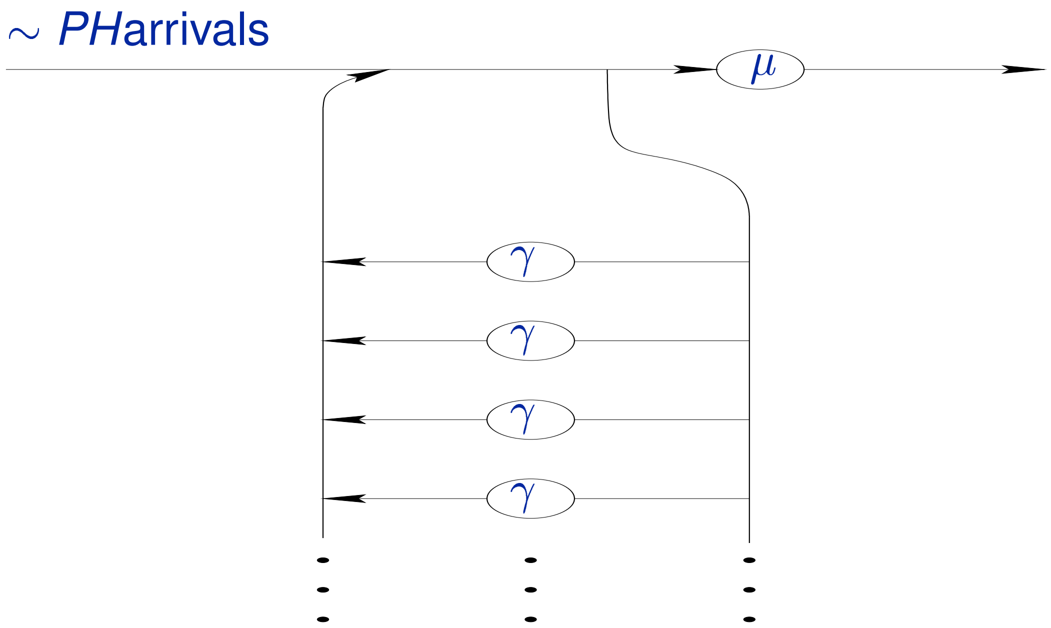
$$Q = \begin{bmatrix} B_1 & A_{00} & \mathbf{0} & \cdots & & \\ A_{20} & A_{10} & A_{01} & \mathbf{0} & \cdots & \\ \mathbf{0} & A_{21} & A_{11} & A_{02} & \mathbf{0} & \cdots \\ \vdots & \mathbf{0} & A_{22} & A_{12} & A_{03} & \ddots \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix},$$

This Q matrix is an example of a level-dependent *QBD* because as the level increases, the rate of retrial increases due to the number in retrial. This is reflected in the entries of the matrices A_{2i} and A_{1i}.

Let's consider the matrices A_{2i}

$$A_{20} = \begin{bmatrix} 0 & \alpha & 0 & \cdots & 0 \\ & 0 & \alpha & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & 0 & \alpha \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \quad A_{2i} = (i+1)A_{20} \text{ for all } i \geq 1,$$

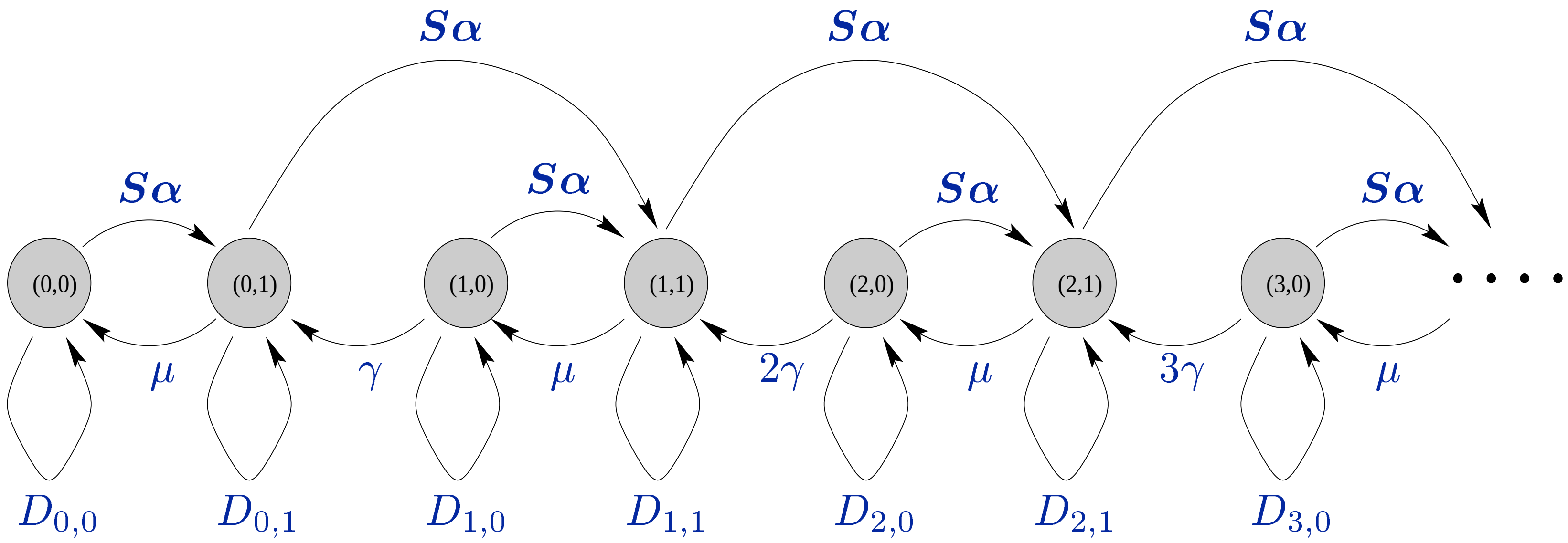
Consider now a *PH/M/1/1* retrial queue where arrivals finding the server busy, go away and retry after an exponentially distributed length of time with parameter γ. Let m be the number of phases of the arrival process given by $(\boldsymbol{\alpha}, S)$. The customers are persistent, in that they do not leave the system until they have received service. We can represent this diagrammatically as



Arranging the states in lexicographical order, and defining

$$D_{n,j} = \begin{cases} S - I_m\mu & \text{if } j = 1 \\ S - nI_m\gamma & \text{if } j = 0 \end{cases},$$

we can construct a state transition diagram for the retrial scheme as



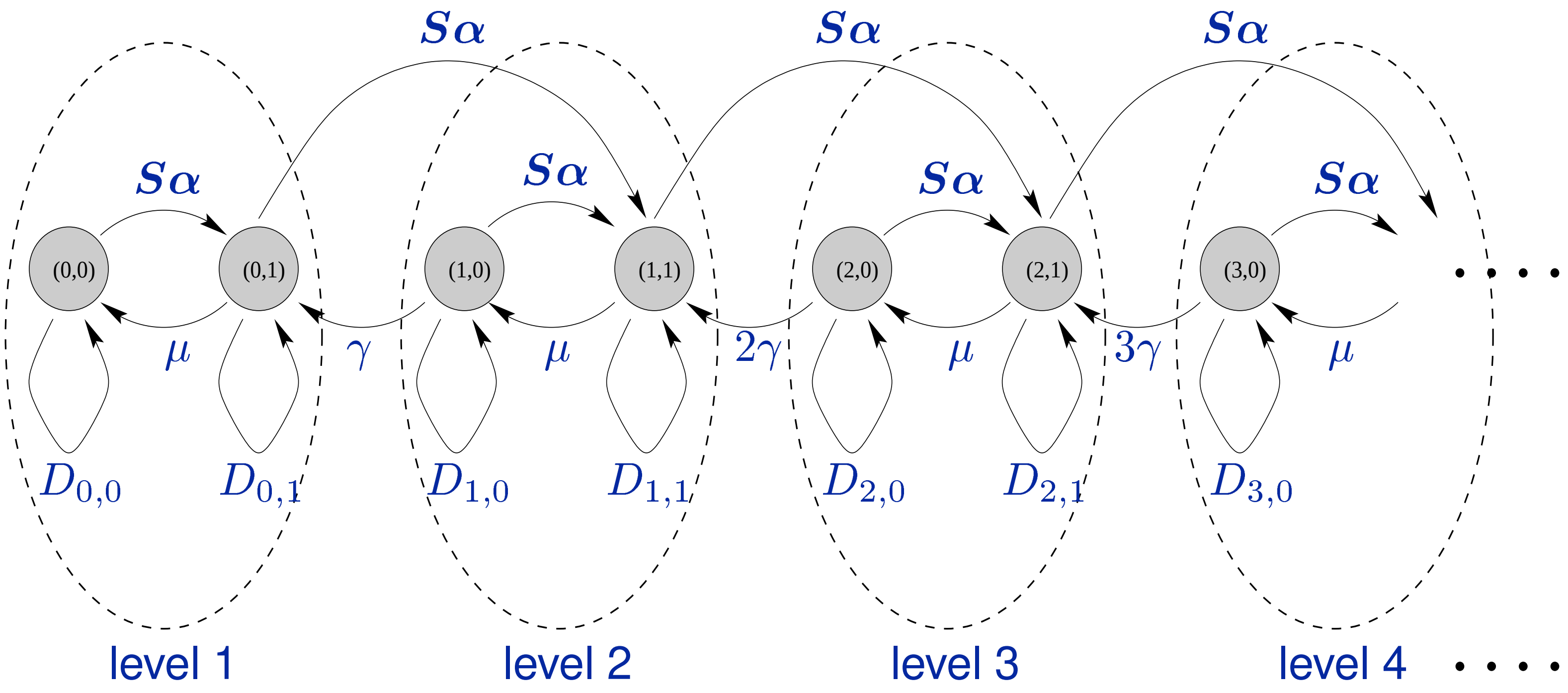
Here the n and j subscripts on the $D_{n,j}$ matrices represent the number of customers in retrial and the server occupancy respectively.

It is then a simple step to construct the following transition rate matrix.

$$Q = \begin{bmatrix} D_{0,0} & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ I_m\mu & D_{0,1} & 0 & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & 0 & 0 & \\ 0 & I_m\gamma & D_{1,0} & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & 0 & 0 & \\ 0 & 0 & I_m\mu & D_{1,1} & 0 & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & \\ 0 & 0 & 0 & 2I_m\gamma & D_{2,0} & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & \\ 0 & 0 & 0 & 0 & I_m\mu & D_{2,1} & 0 & \boldsymbol{S}^0\boldsymbol{\alpha} & \\ 0 & 0 & 0 & 0 & 0 & 3I_m\gamma & D_{3,0} & \boldsymbol{S}^0\boldsymbol{\alpha} & \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m\mu & D_{3,1} & \\ \vdots & & & & & & & & \ddots \end{bmatrix}, \qquad (1)$$

The state transition diagram and block matrix form are not skip free in the upward direction and hence does not obviously fall into either of the level-independent *QBD* or level-dependent *QBD* categories.

However, consider the regrouping of states as follows



Formally, we let n be the number of customers in retrial, i be the phase of the *PH*-type arrival process and j represent the server occupancy. Then we let the number of customers in retrial correspond to the level and let the description within each level include information about the arrival phase and the server occupancy as (i, j). Then we can re-block the rate matrix Q in the form of a level-dependent *QBD* as

$$
Q = \left[\begin{array}{cc|cc|cc|cc|c}
D_{0,0} & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
I_m\mu & D_{0,1} & 0 & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & 0 & 0 & \\
\hline
0 & I_m\gamma & D_{1,0} & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & 0 & 0 & \\
0 & 0 & I_m\mu & D_{1,1} & 0 & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & \\
\hline
0 & 0 & 0 & 2I_m\gamma & D_{2,0} & \boldsymbol{S}^0\boldsymbol{\alpha} & 0 & 0 & \\
0 & 0 & 0 & 0 & I_m\mu & D_{2,1} & 0 & \boldsymbol{S}^0\boldsymbol{\alpha} & \\
\hline
0 & 0 & 0 & 0 & 0 & 3I_m\gamma & D_{3,0} & \boldsymbol{S}^0\boldsymbol{\alpha} & \\
0 & 0 & 0 & 0 & 0 & 0 & I_m\mu & D_{3,1} & \\
\hline
\vdots & & & & & & & & \ddots
\end{array}\right]
$$

A level-dependent *QBD* cannot be considered in the same way as a level-independent *QBD*, since the level-independent *QBD* has a nice homogeneous structure above level 0. However, the level-dependent *QBD* has been addressed by Ramaswami and an algorithm for numerical calculation of the steady state distribution has both been given and demonstrated by Bright and Taylor. Their algorithm for the level-dependent *QBD* is based on the Logarithmic Reduction Algorithm of Latouche and Ramaswami.

We have seen already that the level-dependent *QBD* can be directly used to model queues in a random environment, re-trial queues with multiple servers, re-trial queues with *PH*-type arrival processes and can be easily extended to model infinite server queues in a random environment.

The general block partitioned form of the rate matrix for a level-dependent *QBD* is

$$Q = \begin{bmatrix} A_1^{(0)} & A_0^{(0)} & \mathbf{0} & \cdots & \cdots \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & \mathbf{0} & \cdots \\ \mathbf{0} & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & \\ \vdots & \mathbf{0} & A_2^{(3)} & A_1^{(3)} & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{bmatrix} \tag{2}$$

Assuming positive recurrence, Ramaswami showed that the equilibrium distribution $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots)$ for the level-dependent *QBD* satisfies the following relationship.

$$\boldsymbol{\pi}_{n+1} \quad = \quad \boldsymbol{\pi}_n R_n, \qquad \text{for } n \geq 0,$$

which may be re-written as

$$\boldsymbol{\pi}_{n+1} \quad = \quad \boldsymbol{\pi}_0 \prod_{k=o}^{n} R_k, \qquad \text{for } n \geq 0,$$

This family of matrices $\{R_n, n \geq 0\}$ are the minimal non-negative solutions to the system of equations

$$A_0^{(n)} + R_n A_1^{(n+1)} + R_n R_{n+1} A_2^{(n+2)} = 0, \text{ for } n \geq 0.$$

Note that $\boldsymbol{\pi}_0$ is the solution to

$$\boldsymbol{\pi}_0 \left(A_1^{(0)} + R_0 A_2^{(1)} \right) = 0,$$

subject to

$$\boldsymbol{\pi}_0 \sum_{n=0}^{\infty} \left(\prod_{k=o}^{n-1} R_k \right) \boldsymbol{e} = 1.$$

In order to find the equilibrium distribution of the level-dependent *QBD*, we need to know the matrices $\{R_n, n \geq 0\}$. As we no longer have a homogeneous form above level 0, this becomes a difficult task as the matrices $\{R_n, n \geq 0\}$ may generally only be numerically computed.

An “exact” truncation method used by Bright and Taylor has the following finite rate matrix for a given truncation level M.

$$Q = \begin{bmatrix} A_1^{(0)} & A_0^{(0)} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & \mathbf{0} & & \vdots \\ \mathbf{0} & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & \ddots & \vdots \\ \vdots & & A_2^{(3)} & A_1^{(3)} & \ddots & \mathbf{0} \\ \vdots & & & \ddots & \ddots & A_0^{(M-1)} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_2^{(M)} & \hat{A}_1^{(M)} \end{bmatrix},$$

with the matrix $\hat{A}_1^{(M)}$ given by

$$\hat{A}_1^{(M)} = A_1^{(M)} + R_M A_2^{(M+1)}.$$

The term “exact” refers to results by Ramaswami, where the generator matrix for transitions above level M is shown to be given by this matrix $A_1^{(M)}$. This process then has the interpretation of being the original level-dependent *QBD* on the infinite state space observed only at time points where it is in levels less than or equal to M. Hence the equilibrium distribution $\boldsymbol{\pi}(M)$ for the process has the following interpretation. The probability $(\boldsymbol{\pi}_k(M))_j$ is the stationary probability that the infinite process is in state (k, j), conditional on the process being in the set of states $\{(i, j)|0 \leq i \leq M, 1 \leq j \leq m_i\}$. That is, the invariant measure is exactly that of the original process, with $\boldsymbol{\pi}(M)$ being the normalisation over the truncated state space. Moreover, we have element wise that

$$\boldsymbol{\pi}_k \quad < \quad \boldsymbol{\pi}_k(M) \text{ for all } M \geq 0 \text{ and } \boldsymbol{\pi}_k = \lim_{M\to\infty} \boldsymbol{\pi}_k(M), \text{ for all } k \geq 0.$$

Therefore by choosing M large enough, we have $\boldsymbol{\pi}_k \approx \boldsymbol{\pi}_k(M)$ for $0 \leq k \leq M$.

Bright and Taylor advocate the direct computation of the rate matrix R_M for some truncation index M and then recursively calculating R_{M-1}, R_{M-2}, etc. The following closed form expression for R_k is the continuous time analogue of a Theorem given by Ramaswami and Taylor. If the level-dependent *QBD* is positive recurrent, then

$$R_k \quad = \quad \sum_{\ell=0}^{\infty} U_k^{\ell} \prod_{i=0}^{\ell-1} D_{k+2^{\ell-1}}^{\ell-i-1} \text{ for all } k \geq 0,$$

where U_k^{ℓ} and D_k^{ℓ} are $m_k \times m_{k+2^{\ell}}$ and $m_k \times m_{k-2^{\ell}}$ matrices respectively defined recursively by

$$\begin{aligned}
U_k^0 &= A_0^{(k)}(-A_1^{(k+1)})^{-1}, \\
D_k^0 &= A_2^{(k)}(-A_1^{(k-1)})^{-1}, \\
U_k^{\ell+1} &= U_k^{\ell} U_{k+2^{\ell}}^{\ell} \left[I - U_{k+2^{\ell+1}}^{\ell} D_{k+3.2^{\ell}}^{\ell} - D_{k+2^{\ell+1}}^{\ell} U_{k+2^{\ell}}^{\ell}\right]^{-1}, \\
D_k^{\ell+1} &= D_k^{\ell} D_{k-2^{\ell}}^{\ell} \left[I - U_{k-2^{\ell+1}}^{\ell} D_{k-2^{\ell}}^{\ell} - D_{k-2^{\ell+1}}^{\ell} U_{k-3.2^{\ell}}^{\ell}\right]^{-1}.
\end{aligned}$$

In order to calculate the matrix R_k, the infinite sum must be truncated (note that this truncation is not the same as M mentioned above). If we define $R_k(n)$ to be the sum of the first $n+1$ terms, then we truncate the sum when

$$\max\Big(R_k(L) - R_k(L-1)\Big) < \epsilon$$

for some ϵ and set $R_k = R_k(L)$.

Bright and Taylor point out that this does not generally guarantee that $\max\left(R_k(n) - R_k(n-1)\right) < \epsilon$ holds for all $n > L$, but is the case for most processes of potential interest.

Latouche and Ramaswami favour the calculation of the matrix G_{M+1} and provide a closed form and recursion for calculation of the same. The details and references of which are found in the notes.

Either by way of the matrix G_{M+1} or R_M, it is then a simple matter to use any one of the available methods for calculating the equilibrium distribution for the resultant finite state space.

Recurrence

It is important to consider whether the level-dependent *QBD* is positive recurrent or not. If it is, the distribution $\boldsymbol{\pi}(M)$ calculated is an upper bound for the equilibrium distribution for the infinite level-dependent *QBD*. Then choosing M large enough will mean that for each k, $\boldsymbol{\pi}_k(M)$ will be very close to $\boldsymbol{\pi}_k$ for the infinite level-dependent *QBD*.

The question of stability has been looked at by Latouche and Ramaswami, where recurrence is expressed in terms of the vector $\boldsymbol{\pi}_0$ or matrix $G^{(1)}$. For example, if the matrix $G^{(1)}$ is stochastic, then it can be proven that the level-dependent *QBD* is recurrent. However, for a sub-stochastic matrix $G^{(1)}$ which has been numerically computed and where $G^{(1)}\boldsymbol{e}$ is close to $\boldsymbol{e}$, it may be impossible to determine it to be sub-stochastic and hence to ascertain the non-recurrent nature of the level-dependent *QBD*. Bright and Taylor stated that to their knowledge, no general method for determining positive recurrence for a level-dependent *QBD* on an infinite state space existed and should therefore be addressed on an individual basis. To the authors knowledge, this is still the case.

A simple method for selection of an appropriate truncation M is given by Bright and Taylor, which will generally find an appropriate truncation level M for level-dependent *QBD*s of practical interest, although as they point out, a level-dependent *QBD* can easily be constructed where this will not be the case. For instance, the process can be transient because of transition rates beyond M. However this is unlikely to happen in realistic applications.

The level-dependent *QBD* under investigation should be carefully considered as there are sometimes specific characteristics which facilitate ease of selection of a truncation level M and the determination of positive recurrence. For example, in the case where we have $A_2^{(k)} = A_2$, $A_1^{(k)} = A_1$ and $A_0^{(k)} = A_0$ for all $k \geq \widehat{M}$, the level-dependent *QBD* can be considered as a level-independent *QBD* with a large number of boundary states in the levels $0 \leq k \leq \widehat{M}$. Positive recurrence in this instance has been shown by Neuts to correspond to the drift condition

$$\boldsymbol{\psi} A_0 \boldsymbol{e} < \boldsymbol{\psi} A_2 \boldsymbol{e},$$

where $\boldsymbol{\psi}$ is the solution to $\boldsymbol{\psi}\,(A_0 + A_1 + A_2) = \mathbf{0}$ such that $\boldsymbol{\psi}\ \boldsymbol{e} = 1$.

Numerical examples

Consider an *M/M/*∞ queue in a random environment, where both the service rate and arrival rate are dependent on the state (phase) of the underlying environment. The resultant continuous time level-dependent *QBD* form for this system has matrices $A_2^{(i)}$, $A_1^{(i)}$ and $A_0^{(i)}$ given by

$$
\begin{aligned}
A_2^{(i)} &= i\mathcal{S}, \\
A_1^{(i)} &= Q^* - \Lambda - i\mathcal{S}, \\
A_0^{(i)} &= \Lambda,
\end{aligned}
$$

Here we choose Q^* to be

$$Q^* = \begin{bmatrix} -8 & 8 \\ 5 & -5 \end{bmatrix}$$

and the arrival rate and service rate given respectively by the diagonal matrices of rates Λ and $\mathcal{S}$. That is

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad \mathcal{S} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix},$$

with $\mu_1 = 2$, $\mu_2 = 1$, $\lambda_1 = 40$ and $\lambda_2 = 5$.

In this case, a dominating *BD* process can be constructed to determine a truncation level M. Using a value $\epsilon = 10^{-10}$ yields a value of $M = 88$, which was subsequently verified as adequate by calculating the equilibrium distribution again using $M = 188$. The difference in the two results was less than 10^{-14} in all cases. The marginal probabilities of being in states $0 \leq k \leq M$ are given in the following table

k	ℓ_k	k	ℓ_k
0	2.43033×10^{-5}	18	4.36024×10^{-2}
1	2.12737×10^{-4}	19	3.34413×10^{-2}
2	9.61911×10^{-4}	20	2.47627×10^{-2}
3	2.99837×10^{-3}	21	1.77270×10^{-2}
4	7.25313×10^{-3}	22	1.22830×10^{-2}
5	1.45280×10^{-2}	23	8.24654×10^{-3}
6	2.50976×10^{-2}	24	5.36972×10^{-3}
7	3.84463×10^{-2}	25	3.39415×10^{-3}
8	5.32737×10^{-2}	26	2.08432×10^{-3}
9	6.77654×10^{-2}	27	1.24447×10^{-3}
10	8.00201×10^{-2}	28	7.22940×10^{-4}
11	8.84792×10^{-2}	29	4.08889×10^{-4}
12	9.22315×10^{-2}	30	2.25306×10^{-4}
13	9.11276×10^{-2}	31	1.21022×10^{-4}
14	8.57099×10^{-2}	32	6.34068×10^{-5}
15	7.70106×10^{-2}	33	3.24203×10^{-5}
16	6.62927×10^{-2}	34	1.61860×10^{-5}
17	5.48054×10^{-2}	$>$34	$< 10^{-5}$

Then choosing $\mu_2 = 0$ means that we no longer satisfy the conditions under which we can construct a dominating *BD* process. This corresponds to the situation where the servers are all idle when the underlying Markov modulating process given by Q^* is in state 2. Assuming that $M = 88$ would be at least required for this case, this was used as an initial guess and found to be sufficient. The level-dependent *QBD* was not shown to be positive-recurrent, but the results shown in the following table tend to indicate that this is the case.

k	ℓ_k	k	ℓ_k	k	ℓ_k
0 to 5	$< 10^{-5}$	22	7.01145×10^{-2}	39	3.22418×10^{-3}
6	1.54870×10^{-5}	23	7.35521×10^{-2}	40	2.22364×10^{-3}
7	4.99438×10^{-5}	24	7.43740×10^{-2}	41	1.51153×10^{-3}
8	1.41353×10^{-4}	25	7.26348×10^{-2}	42	1.01352×10^{-3}
9	3.56725×10^{-4}	26	6.86384×10^{-2}	43	6.70883×10^{-4}
10	8.12863×10^{-4}	27	6.28697×10^{-2}	44	4.38713×10^{-4}
11	1.68958×10^{-3}	28	5.59082×10^{-2}	45	2.83620×10^{-4}
12	3.23061×10^{-3}	29	4.83435×10^{-2}	46	1.81386×10^{-4}
13	5.72303×10^{-3}	30	4.07063×10^{-2}	47	1.14829×10^{-4}
14	9.45034×10^{-3}	31	3.34230×10^{-2}	48	7.20016×10^{-5}
15	1.46231×10^{-2}	32	2.67954×10^{-2}	49	4.47413×10^{-5}
16	2.13018×10^{-2}	33	2.10015×10^{-2}	50	2.75666×10^{-5}
17	2.93325×10^{-2}	34	1.61113×10^{-2}	51	1.68492×10^{-5}
18	3.83202×10^{-2}	35	1.21114×10^{-2}	52	1.02210×10^{-5}
19	4.76513×10^{-2}	36	8.93141×10^{-3}	$>$53	$< 10^{-5}$
20	5.65699×10^{-2}	37	6.46767×10^{-3}		
21	6.42890×10^{-2}	38	4.60375×10^{-3}		

Consider the *PH/M/*∞ queue with the mixture of two ten stage Erlangs given by distribution function

$$0.05841196\ E_{10}(1.168239125)\ +\ 0.94158804\ E_{10}(18.83176088)$$

With a constant service distribution rate of $\mu = 0.5$, a constructed dominating *BD* process yields a value of $M = 84$. The marginal probabilities are presented in the following table

k	ℓ_k	k	ℓ_k
0	2.94726×10^{-1}	10	4.00557×10^{-5}
1	1.68517×10^{-1}	11	4.09573×10^{-6}
2	1.57537×10^{-1}	12	3.30731×10^{-7}
3	1.55108×10^{-1}	13	2.12891×10^{-8}
4	1.19684×10^{-2}	14	1.10142×10^{-9}
5	6.70135×10^{-2}	15	4.61353×10^{-11}
6	2.71629×10^{-3}	16	1.57479×10^{-12}
7	8.09367×10^{-3}	17	4.40641×10^{-14}
8	1.80460×10^{-3}	$>$ 18	$< 10^{-14}$
9	3.06062×10^{-4}		

In all three of the examples given, the values of M are larger than necessary, so that the methods for determination of the truncation level all over estimate M. For example, in the case of the *PH/M/1* queue we have $M = 84$, yet the table of marginal probabilities indicate that the probability of being above level 18 is less than 10^{-14}.

Latouche and Ramaswami have observed that it is possible to construct level-independent *QBD* processes where an excessive number of iterations are required for their algorithm. This is also relevant to the level-dependent *QBD* case. The possibility of overflow or underflow errors also becomes a potential problem when recursively calculating the matrices R_k. Similar problems were observed by Gaver, Jacobs and Latouche in their computations. This is more likely when either $A_0^{(i)}$ or $A_2^{(i)}$ are very large or small compared to the other matrices in Q, where a numerical scaling technique may be appropriate.