Markovian Point Processes

David Green

University of Adelaide

Poisson process

Poisson process

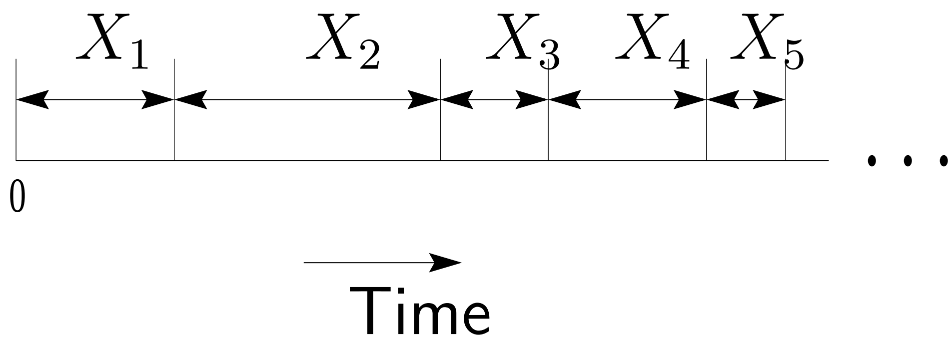
- Negative exponentially distributed inter-arrival times.

Poisson process

- Negative exponentially distributed inter-arrival times.
- A point process where the distribution of time between points is a simple phase type distribution having a single phase.

Poisson process

- Negative exponentially distributed inter-arrival times.
- A point process where the distribution of time between points is a simple phase type distribution having a single phase.

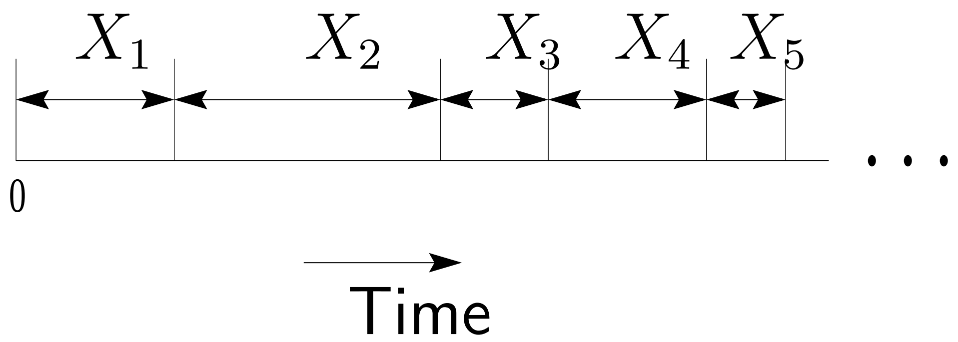


where, the $X_i \sim$ I.I.D. negative exponential

with some parameter λ.

Poisson process

- Negative exponentially distributed inter-arrival times.
- A point process where the distribution of time between points is a simple phase type distribution having a single phase.



what if the $X_i \sim$ I.I.D. phase type

with description $(\boldsymbol{\alpha}, T)$?

Phase renewal process

Phase renewal process

- Infinitesimal generator or Q-matrix description.

Phase renewal process

- Infinitesimal generator or Q-matrix description.

Consider first the Poisson process

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & & & \\ 0 & -\lambda & \lambda & 0 & & & \\ 0 & 0 & -\lambda & \lambda & 0 & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \end{bmatrix}.$$

Phase renewal process

- Infinitesimal generator or Q-matrix description.

Consider first the Poisson process

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & & & \\ 0 & -\lambda & \lambda & 0 & & & \\ 0 & 0 & -\lambda & \lambda & 0 & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \end{bmatrix},$$

where λ is the arrival rate from the single phase.

Phase renewal process

- Infinitesimal generator or Q-matrix description.

Then the more general phase type renewal process

$$Q = \begin{bmatrix} T & \boldsymbol{T}^0\boldsymbol{\alpha} & 0 & 0 & \cdots \\ 0 & T & \boldsymbol{T}^0\boldsymbol{\alpha} & 0 & \cdots \\ 0 & 0 & T & \boldsymbol{T}^0\boldsymbol{\alpha} & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix},$$

where $\boldsymbol{T}^0$ is a column of rates corresponding to the arrival rate out of each phase of the matrix T.

Phase renewal process

- Infinitesimal generator or Q-matrix description.

Then the more general phase type renewal process

$$Q = \begin{bmatrix} T & \boldsymbol{T}^0\boldsymbol{\alpha} & 0 & 0 & \cdots \\ 0 & T & \boldsymbol{T}^0\boldsymbol{\alpha} & 0 & \cdots \\ 0 & 0 & T & \boldsymbol{T}^0\boldsymbol{\alpha} & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix},$$

where $\boldsymbol{T}^0$ is a column of rates corresponding to the arrival rate out of each phase of the matrix T. More formally $\boldsymbol{T}^0 = -T\boldsymbol{e}$, where $\boldsymbol{e}$ is a column of ones of the appropriate dimension.

Markovian Arrival Process Notation

Markovian Arrival Process Notation

It is convenient to describe such an arrival process as follows.

Markovian Arrival Process Notation

It is convenient to describe such an arrival process as follows.

- Let D_0 be the $m \times m$ matrix which governs those transitions which do not correspond to an arrival.

Markovian Arrival Process Notation

It is convenient to describe such an arrival process as follows.

- Let D_0 be the $m \times m$ matrix which governs those transitions which do not correspond to an arrival.
- Let D_1 be the $m \times m$ matrix which governs those transitions which do correspond to an arrival.

Markovian Arrival Process Notation

It is convenient to describe such an arrival process as follows.

- Let D_0 be the $m \times m$ matrix which governs those transitions which do not correspond to an arrival.
- Let D_1 be the $m \times m$ matrix which governs those transitions which do correspond to an arrival.

This gives us a complete description from which we can write

$$Q = \begin{bmatrix} D_0 & D_1 & 0 & 0 & \cdots \\ 0 & D_0 & D_1 & 0 & \cdots \\ 0 & 0 & D_0 & D_1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Markovian Arrival Process Notation

It is convenient to describe such an arrival process as follows.

- Let D_0 be the $m \times m$ matrix which governs those transitions which do not correspond to an arrival.
- Let D_1 be the $m \times m$ matrix which governs those transitions which do correspond to an arrival.

This gives us a complete description from which we can write

$$Q = \begin{bmatrix} D_0 & D_1 & 0 & 0 & \cdots \\ 0 & D_0 & D_1 & 0 & \cdots \\ 0 & 0 & D_0 & D_1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}.$$

This is MAP notation, where $D_0 = T$ and $D_1 = \boldsymbol{T}^0 \boldsymbol{\alpha}$.

MAPs

$MAPs$

- The phase renewal processes form an important sub-class of MAPs.

$MAPs$

- The phase renewal processes form an important sub-class of MAPs.
- With a little thought, one can imagine much more general processes than renewal processes as having a MAP description.

$MAPs$

- The phase renewal processes form an important sub-class of MAPs.
- With a little thought, one can imagine much more general processes than renewal processes as having a MAP description.
- The following properties of the MAP are considered from this more general sense.

$MAPs$

- The phase renewal processes form an important sub-class of MAPs.
- With a little thought, one can imagine much more general processes than renewal processes as having a MAP description.
- The following properties of the MAP are considered from this more general sense.
- The matrices D_0 and D_1 have the following properties.

$MAPs$

- The phase renewal processes form an important sub-class of MAPs.
- With a little thought, one can imagine much more general processes than renewal processes as having a MAP description.
- The following properties of the MAP are considered from this more general sense.
- The matrices D_0 and D_1 have the following properties.

$$[D_0]_{ii} \; < \; 0 \quad \text{for all } i,$$

$MAPs$

- The phase renewal processes form an important sub-class of MAPs.
- With a little thought, one can imagine much more general processes than renewal processes as having a MAP description.
- The following properties of the MAP are considered from this more general sense.
- The matrices D_0 and D_1 have the following properties.

$$\begin{aligned} [D_0]_{ii} &< 0 \quad \text{for all } i, \\ [D_0]_{ij} &\geq 0 \quad \text{for all } i \neq j \end{aligned}$$

$MAPs$

- The phase renewal processes form an important sub-class of MAPs.
- With a little thought, one can imagine much more general processes than renewal processes as having a MAP description.
- The following properties of the MAP are considered from this more general sense.
- The matrices D_0 and D_1 have the following properties.

$$
\begin{aligned}
[D_0]_{ii} &< 0 \quad \text{for all } i, \\
[D_0]_{ij} &\geq 0 \quad \text{for all } i \neq j \ , \\
[D_1]_{ij} &\geq 0 \quad \text{for all } i, j
\end{aligned}
$$

$MAPs$

- The phase renewal processes form an important sub-class of MAPs.
- With a little thought, one can imagine much more general processes than renewal processes as having a MAP description.
- The following properties of the MAP are considered from this more general sense.
- The matrices D_0 and D_1 have the following properties.

$$\begin{aligned} [D_0]_{ii} &< 0 \quad \text{for all } i, \\ [D_0]_{ij} &\geq 0 \quad \text{for all } i \neq j \ , \\ [D_1]_{ij} &\geq 0 \quad \text{for all } i, j \\ \text{and } D\boldsymbol{e} &= (D_0 + D_1)\boldsymbol{e} = \mathbf{0} \ . \end{aligned}$$

The matrix D_0

The matrix D_0

- has strictly negative diagonal entries

The matrix D_0

- has strictly negative diagonal entries
- has non-negative off-diagonal entries

The matrix D_0

- has strictly negative diagonal entries
- has non-negative off-diagonal entries
- Row sums less than or equal to 0

The matrix D_0

- has strictly negative diagonal entries
- has non-negative off-diagonal entries
- Row sums less than or equal to 0
- is non-singular

The matrix D_0

- has strictly negative diagonal entries
- has non-negative off-diagonal entries
- Row sums less than or equal to 0
- is non-singular
- so that

The matrix D_0

- has strictly negative diagonal entries
- has non-negative off-diagonal entries
- Row sums less than or equal to 0
- is non-singular
- so that
  - all of it's eigenvalues have negative real parts : (Bellman)

The matrix D_0

- has strictly negative diagonal entries
- has non-negative off-diagonal entries
- Row sums less than or equal to 0
- is non-singular
- so that
  - all of it's eigenvalues have negative real parts : (Bellman)
  - inter-arrival times are finite with probability one : (Neuts)

The matrix D_0

- has strictly negative diagonal entries
- has non-negative off-diagonal entries
- Row sums less than or equal to 0
- is non-singular
- so that
  - all of it's eigenvalues have negative real parts : (Bellman)
  - inter-arrival times are finite with probability one : (Neuts)
  - the process does not terminate.

The generator matrix or Q-matrix D

The generator matrix or Q-matrix D

- The matrix D is the generator matrix of the MAP phase process.

The generator matrix or Q-matrix D

- The matrix D is the generator matrix of the MAP phase process.
- It has an associated vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} D = \mathbf{0}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$ with $[\boldsymbol{\pi}]_i \geq 0$ for all i.

The generator matrix or Q-matrix D

- The matrix D is the generator matrix of the MAP phase process.
- It has an associated vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} D = \mathbf{0}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$ with $[\boldsymbol{\pi}]_i \geq 0$ for all i.
- This vector is known as the stationary distribution of phase of the MAP .

The generator matrix or Q-matrix D

- The matrix D is the generator matrix of the MAP phase process.
- It has an associated vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} D = \mathbf{0}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$ with $[\boldsymbol{\pi}]_i \geq 0$ for all i.
- This vector is known as the stationary distribution of phase of the MAP .
- Recalling that the matrix D_1 governs those transitions which correspond to arrivals,

The generator matrix or Q-matrix D

- The matrix D is the generator matrix of the MAP phase process.
- It has an associated vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} D = \mathbf{0}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$ with $[\boldsymbol{\pi}]_i \geq 0$ for all i.
- This vector is known as the stationary distribution of phase of the MAP .
- Recalling that the matrix D_1 governs those transitions which correspond to arrivals,
  - in light of the information given by the vector $\boldsymbol{\pi}$,

The generator matrix or Q-matrix D

- The matrix D is the generator matrix of the MAP phase process.
- It has an associated vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} D = \mathbf{0}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$ with $[\boldsymbol{\pi}]_i \geq 0$ for all i.
- This vector is known as the stationary distribution of phase of the MAP .
- Recalling that the matrix D_1 governs those transitions which correspond to arrivals,
  - in light of the information given by the vector $\boldsymbol{\pi}$,
  - the process of arrivals has the following fundamental arrival rate

$$\lambda = \boldsymbol{\pi} D_1 \boldsymbol{e} .$$

The two dimensional representation

The two dimensional representation

- Consider an m-dimensional matrix pair D_0 and D_1

The two dimensional representation

- Consider an m-dimensional matrix pair D_0 and D_1
- If we consider the Q-matrix for the evolution of the MAP, essentially we have a *two-dimensional* Markov process $\{N_t, J_t\}$.

The two dimensional representation

- Consider an m-dimensional matrix pair D_0 and D_1
- If we consider the Q-matrix for the evolution of the MAP, essentially we have a *two-dimensional* Markov process $\{N_t, J_t\}$.
  - where, the $\{N_t\}$ process keeps track of the number of arrivals (The level); in particular, N_t denotes the number of arrivals during the interval $(0, t]$.

The two dimensional representation

- Consider an m-dimensional matrix pair D_0 and D_1
- If we consider the Q-matrix for the evolution of the MAP, essentially we have a *two-dimensional* Markov process $\{N_t, J_t\}$.
  - where, the $\{N_t\}$ process keeps track of the number of arrivals (The level); in particular, N_t denotes the number of arrivals during the interval $(0, t]$.
  - and the $\{J_t\}$ process keeps track of the phase of the MAP. (The phase)

The two dimensional representation

- Consider an m-dimensional matrix pair D_0 and D_1
- If we consider the Q-matrix for the evolution of the MAP, essentially we have a *two-dimensional* Markov process $\{N_t, J_t\}$.
  - where, the $\{N_t\}$ process keeps track of the number of arrivals (The level); in particular, N_t denotes the number of arrivals during the interval $(0, t]$.
  - and the $\{J_t\}$ process keeps track of the phase of the MAP. (The phase)

This has state space

$$\{(0,1),(0,2),\ldots,(0,m),(1,1),(1,2),\ldots,(1,m),\ldots\}.$$

The two dimensional representation

- Consider an m-dimensional matrix pair D_0 and D_1
- If we consider the Q-matrix for the evolution of the MAP, essentially we have a *two-dimensional* Markov process $\{N_t, J_t\}$.
  - where, the $\{N_t\}$ process keeps track of the number of arrivals (The level); in particular, N_t denotes the number of arrivals during the interval $(0, t]$.
  - and the $\{J_t\}$ process keeps track of the phase of the MAP. (The phase)

This has state space

$$\{\underbrace{(0,1),(0,2),\ldots,(0,m)}_{\text{Level zero}},(1,1),(1,2),\ldots,(1,m),\ldots\}.$$

The two dimensional representation

- Consider an m-dimensional matrix pair D_0 and D_1
- If we consider the Q-matrix for the evolution of the MAP, essentially we have a *two-dimensional* Markov process $\{N_t, J_t\}$.
  - where, the $\{N_t\}$ process keeps track of the number of arrivals (The level); in particular, N_t denotes the number of arrivals during the interval $(0, t]$.
  - and the $\{J_t\}$ process keeps track of the phase of the MAP. (The phase)

This has state space

$$\{(0,1),(0,2),\ldots,(0,m),\underbrace{(1,1),(1,2),\ldots,(1,m)}_{\text{Level one}},\ldots\}.$$

Poisson process

Poisson process

- Negative exponentially distributed inter-arrival times.

Poisson process

- Negative exponentially distributed inter-arrival times.
- $F(t) = 1 - e^{-\lambda t}$, where

$$T = -\lambda \quad \text{and } \boldsymbol{\alpha} = 1.$$

Poisson process

- Negative exponentially distributed inter-arrival times.
- $F(t) = 1 - e^{-\lambda t}$, where

$$T = -\lambda \quad \text{and } \boldsymbol{\alpha} = 1.$$

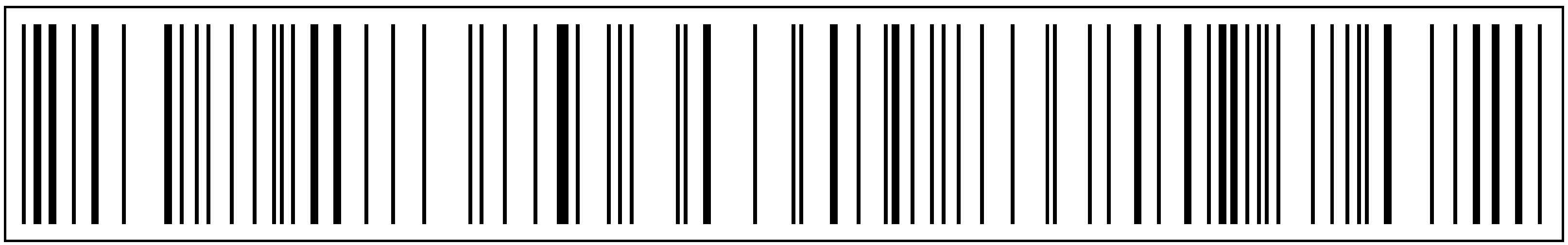
- What does it look like?

Poisson process

- Negative exponentially distributed inter-arrival times.
- $F(t) = 1 - e^{-\lambda t}$, where

$$T = -\lambda \quad \text{and } \boldsymbol{\alpha} = 1.$$

- What does it look like?



Point process of 100 arrivals

Erlang distributed inter-arrival times

Erlang distributed inter-arrival times

- E_n distributed inter-arrival times.

Erlang distributed inter-arrival times

- E_n distributed inter-arrival times.
- $F(t) = \int_{x=0}^{t} \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} dx$.

Erlang distributed inter-arrival times

- E_n distributed inter-arrival times.
- $F(t) = \int_{x=0}^{t} \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} dx$.

$$T = \begin{bmatrix} -\lambda & \lambda & & 0 \\ 0 & -\lambda & \ddots & \\ \vdots & & \ddots & \lambda \\ 0 & & 0 & -\lambda \end{bmatrix} \quad \text{and } \boldsymbol{\alpha} = (1, 0, \ldots, 0) \ .$$

Erlang distributed inter-arrival times

- E_n distributed inter-arrival times.
- $F(t) = \int_{x=0}^{t} \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} dx$.

$$T = \begin{bmatrix} -\lambda & \lambda & & 0 \\ 0 & -\lambda & \ddots & \\ \vdots & & \ddots & \lambda \\ 0 & & 0 & -\lambda \end{bmatrix} \quad \text{and } \boldsymbol{\alpha} = (1, 0, \ldots, 0) \ .$$

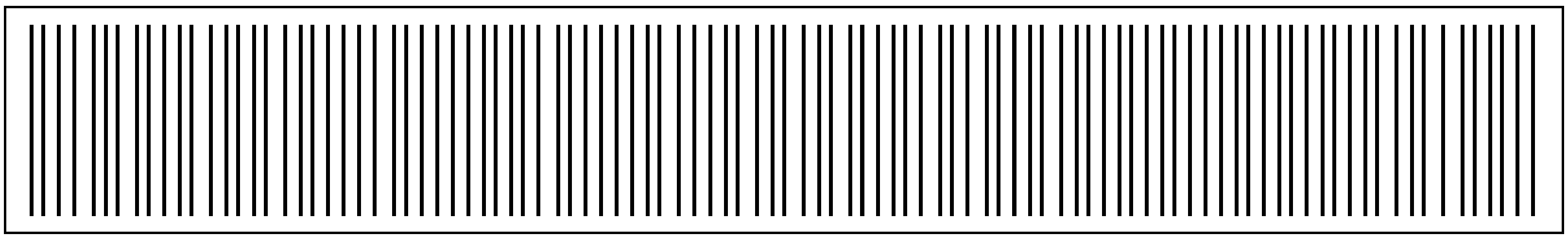
- What does this look like?

Erlang distributed inter-arrival times

- E_n distributed inter-arrival times.
- $F(t) = \int_{x=0}^{t} \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} dx$.

$$T = \begin{bmatrix} -\lambda & \lambda & & 0 \\ 0 & -\lambda & \ddots & \\ \vdots & & \ddots & \lambda \\ 0 & & 0 & -\lambda \end{bmatrix} \quad \text{and } \boldsymbol{\alpha} = (1, 0, \ldots, 0)\ .$$

- What does this look like?



Point process of 100 arrivals for E_{50}

Hyper-exponentially distributed

Hyper-exponentially distributed

- Inter-arrival times $\sim F(t) = \sum_{j=1}^{n} \alpha_j \left(1 - e^{-\lambda_j t}\right)$

Hyper-exponentially distributed

- Inter-arrival times $\sim F(t) = \sum_{j=1}^{n} \alpha_j \left(1 - e^{-\lambda_j t}\right)$

$$T = \begin{bmatrix} -\lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & & -\lambda_n \end{bmatrix} \quad \text{and } \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) .$$

Hyper-exponentially distributed

- Inter-arrival times $\sim F(t) = \sum_{j=1}^{n} \alpha_j \left(1 - e^{-\lambda_j t}\right)$

$$T = \begin{bmatrix} -\lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & & -\lambda_n \end{bmatrix} \quad \text{and } \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \ .$$

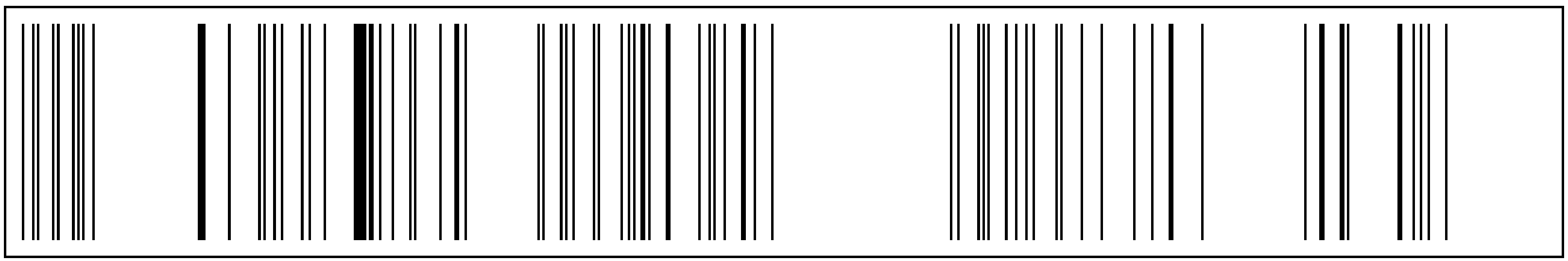
- What does this look like?

Hyper-exponentially distributed

- Inter-arrival times $\sim F(t) = \sum_{j=1}^{n} \alpha_j \left(1 - e^{-\lambda_j t}\right)$

$$T = \begin{bmatrix} -\lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & & -\lambda_n \end{bmatrix} \quad \text{and } \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \ .$$

- What does this look like?



Point process of 100 arrivals for H_2 *:*

$$F(t) = \frac{10}{11}(1 - e^{-10t}) + \frac{1}{11}(1 - e^{-t})$$

Non-renewal processes

Non-renewal processes

- We alluded to more general $MAPs$ while describing renewal processes as $MAPs$.

Non-renewal processes

- We alluded to more general $MAPs$ while describing renewal processes as $MAPs$.
- The feature of the renewal processes is that every time an arrival occurs, the process immediately restarts with the exact same distribution of phase.

Non-renewal processes

- We alluded to more general $MAPs$ while describing renewal processes as $MAPs$.
- The feature of the renewal processes is that every time an arrival occurs, the process immediately restarts with the exact same distribution of phase.
- The non-renewal $MAPs$ will be introduced by way of an important example.

Non-renewal processes

- We alluded to more general $MAPs$ while describing renewal processes as $MAPs$.
- The feature of the renewal processes is that every time an arrival occurs, the process immediately restarts with the exact same distribution of phase.
- The non-renewal $MAPs$ will be introduced by way of an important example.
- Furthermore $MAPs$ as we will also see are a sub-class of what are known as Batch Markovian Arrival Processes ($BMAPs$).

Markov modulated Poisson process.

Markov modulated Poisson process.

- Let's now consider an m state (phase) continuous time Markov process $\{J_t\}$, with Q-matrix R.

Markov modulated Poisson process.

- Let's now consider an m state (phase) continuous time Markov process $\{J_t\}$, with Q-matrix R.
- Arrivals are "modulated" in such a way that during a time period in which the process is in state k, customers may arrive according to a Poisson process with rate λ_k, $k \in \{1, \ldots, m\}$, independent of everything else.

Markov modulated Poisson process.

- Let's now consider an m state (phase) continuous time Markov process $\{J_t\}$, with Q-matrix R.
- Arrivals are "modulated" in such a way that during a time period in which the process is in state k, customers may arrive according to a Poisson process with rate λ_k, $k \in \{1, \ldots, m\}$, independent of everything else.
- The corresponding arrival process $\{N_t\}$ is called a Markov modulated Poisson process ($MMPP$).

Markov modulated Poisson process.

- Let's now consider an m state (phase) continuous time Markov process $\{J_t\}$, with Q-matrix R.
- Arrivals are "modulated" in such a way that during a time period in which the process is in state k, customers may arrive according to a Poisson process with rate λ_k, $k \in \{1, \ldots, m\}$, independent of everything else.
- The corresponding arrival process $\{N_t\}$ is called a Markov modulated Poisson process ($MMPP$).
- Immediately after an arrival in this case we do not restart the process with a fixed distribution of phase $\boldsymbol{\alpha}$, but remain in the same phase from which the arrival occurred.

Markov modulated Poisson process.

- Let's now consider an m state (phase) continuous time Markov process $\{J_t\}$, with Q-matrix R.
- Arrivals are "modulated" in such a way that during a time period in which the process is in state k, customers may arrive according to a Poisson process with rate λ_k, $k \in \{1, \ldots, m\}$, independent of everything else.
- The corresponding arrival process $\{N_t\}$ is called a Markov modulated Poisson process $(MMPP)$.
- Immediately after an arrival in this case we do not restart the process with a fixed distribution of phase $\boldsymbol{\alpha}$, but remain in the same phase from which the arrival occurred.
- Hence in general we do not have a renewal process.

MAP notation for the $MMPP$.

MAP notation for the $MMPP$.

- The corresponding Q-matrix for the arrival process may be written using the matrices

MAP notation for the $MMPP$.

- The corresponding Q-matrix for the arrival process may be written using the matrices
  - $D_0 = \boldsymbol{R} - \Lambda$

MAP notation for the $MMPP$.

- The corresponding Q-matrix for the arrival process may be written using the matrices
  - $D_0 = \boldsymbol{R} - \Lambda$
  - and $D_1 = \Lambda$,

MAP notation for the $MMPP$.

- The corresponding Q-matrix for the arrival process may be written using the matrices
  - $D_0 = \boldsymbol{R} - \Lambda$
  - and $D_1 = \Lambda$,
- where R is the Q-matrix of $\{J_t\}$, and

MAP notation for the $MMPP$.

- The corresponding Q-matrix for the arrival process may be written using the matrices
  - $D_0 = \boldsymbol{R} - \Lambda$
  - and $D_1 = \Lambda$,
- where R is the Q-matrix of $\{J_t\}$, and

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \\ \vdots & \ddots & \ddots & \\ 0 & & & \lambda_m \end{bmatrix}.$$

A special $MMPP$.

A special $MMPP$.

- An Interrupted (or switched) Poisson Process (IPP) essentially has a switch which jumps between ON and OFF, staying ON (OFF) for a exponentially distributed time with parameter $\gamma(\omega)$.

A special $MMPP$.

- An Interrupted (or switched) Poisson Process (IPP) essentially has a switch which jumps between ON and OFF, staying ON (OFF) for a exponentially distributed time with parameter $\gamma(\omega)$.
- While the switch is ON, a Poisson process of arrivals (rate τ) occurs, but this stream is interrupted when the switch moves to OFF.

A special *MMPP*.

- An Interrupted (or switched) Poisson Process (IPP) essentially has a switch which jumps between ON and OFF, staying ON (OFF) for a exponentially distributed time with parameter $\gamma(\omega)$.
- While the switch is ON, a Poisson process of arrivals (rate τ) occurs, but this stream is interrupted when the switch moves to OFF.
- If N_t denotes the number of arrivals in $(0, t]$, then $\{N_t\}$ is an $MMPP$ of the form described before with

A special $MMPP$.

- An Interrupted (or switched) Poisson Process (IPP) essentially has a switch which jumps between ON and OFF, staying ON (OFF) for a exponentially distributed time with parameter $\gamma(\omega)$.
- While the switch is ON, a Poisson process of arrivals (rate τ) occurs, but this stream is interrupted when the switch moves to OFF.
- If N_t denotes the number of arrivals in $(0, t]$, then $\{N_t\}$ is an $MMPP$ of the form described before with

$$\boldsymbol{R} = \begin{bmatrix} -\omega & \omega \\ \gamma & -\gamma \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \tau \end{bmatrix}.$$

A special renewal process.

A special renewal process.

- The previous interrupted (or switched) Poisson Process (IPP) is also a phase type renewal process, with

A special renewal process.

- The previous interrupted (or switched) Poisson Process (IPP) is also a phase type renewal process, with

$$T = \begin{bmatrix} -\omega & \omega \\ \gamma & -\gamma - \tau \end{bmatrix} \quad \text{and} \quad \boldsymbol{\alpha} = (0, 1),$$

A special renewal process.

- The previous interrupted (or switched) Poisson Process (IPP) is also a phase type renewal process, with

$$T = \begin{bmatrix} -\omega & \omega \\ \gamma & -\gamma - \tau \end{bmatrix} \quad \text{and} \quad \boldsymbol{\alpha} = (0, 1),$$

since we only have arrivals from one state.

A special renewal process.

- The previous interrupted (or switched) Poisson Process (IPP) is also a phase type renewal process, with

$$T = \begin{bmatrix} -\omega & \omega \\ \gamma & -\gamma - \tau \end{bmatrix} \quad \text{and} \quad \boldsymbol{\alpha} = (0, 1),$$

  since we only have arrivals from one state.

- What does it look like with $\omega = 1, \quad \gamma = 1$ and $\tau = 9$?

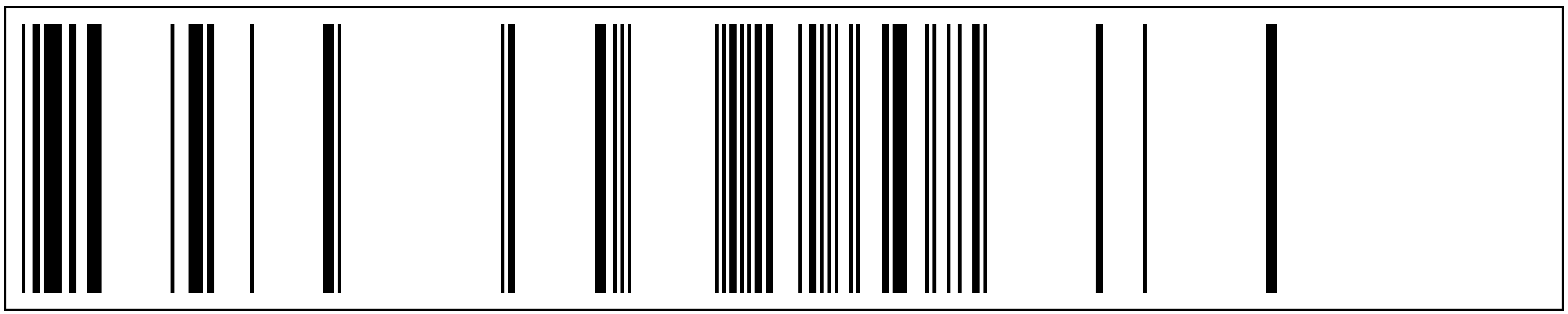
A special renewal process.

- The previous interrupted (or switched) Poisson Process (IPP) is also a phase type renewal process, with

$$T = \begin{bmatrix} -\omega & \omega \\ \gamma & -\gamma - \tau \end{bmatrix} \quad \text{and} \quad \boldsymbol{\alpha} = (0, 1),$$

  since we only have arrivals from one state.

- What does it look like with $\omega = 1, \quad \gamma = 1$ and $\tau = 9$?



Point process of 100 arrivals for the IPP

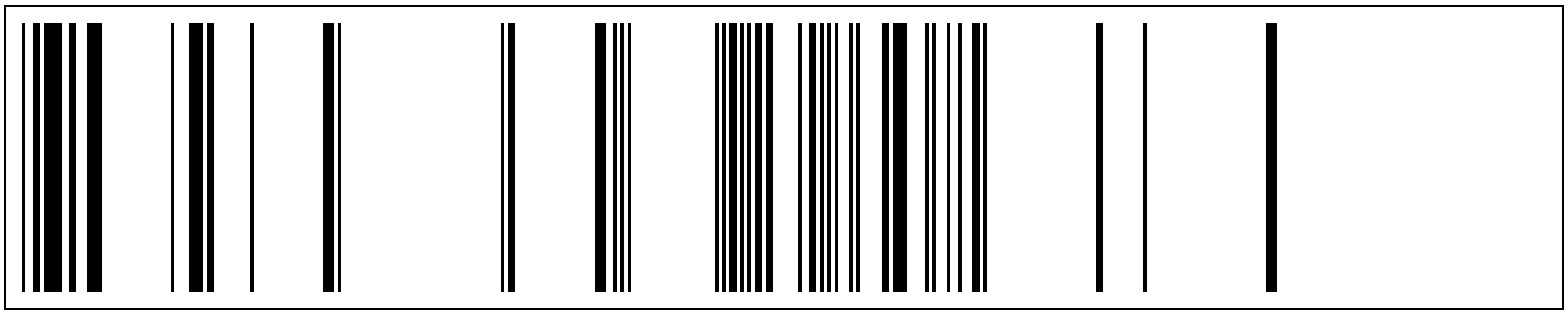
A special renewal process.

- The previous interrupted (or switched) Poisson Process (IPP) is also a phase type renewal process, with

$$T = \begin{bmatrix} -\omega & \omega \\ \gamma & -\gamma - \tau \end{bmatrix} \quad \text{and} \quad \boldsymbol{\alpha} = (0, 1),$$

since we only have arrivals from one state.

- What does it look like with $\omega = 1, \quad \gamma = 1$ and $\tau = 9$?



Point process of 100 arrivals for the IPP

- In general $MMPPs$ are not renewal processes.

A bursty non-renewal $MMPP$.

A bursty non-renewal $MMPP$.

- Consider the following non-renewal $MMPP$

A bursty non-renewal $MMPP$.

- Consider the following non-renewal $MMPP$

$$\boldsymbol{R} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}.$$

A bursty non-renewal $MMPP$.

- Consider the following non-renewal $MMPP$

$$\boldsymbol{R} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}.$$

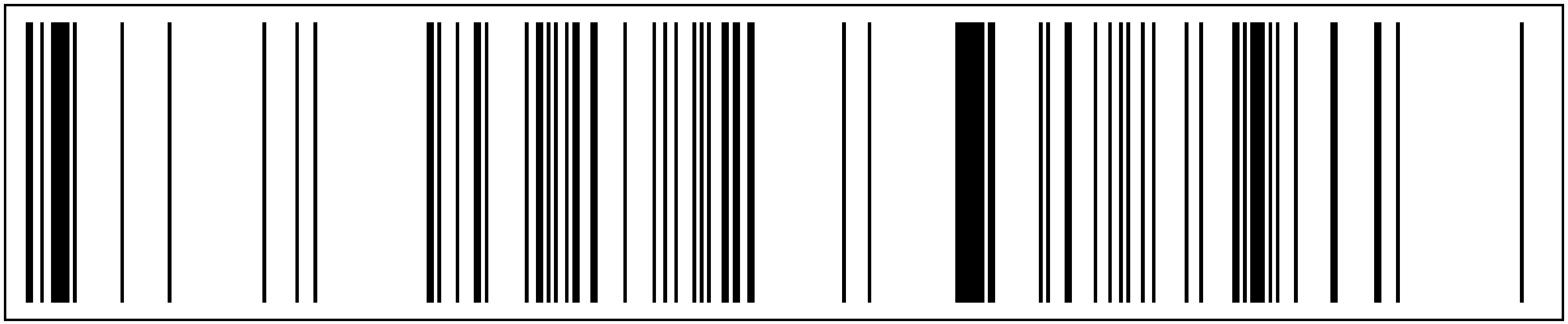
- What does this look like?

A bursty non-renewal $MMPP$.

- Consider the following non-renewal $MMPP$

$$\boldsymbol{R} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}.$$

- What does this look like?



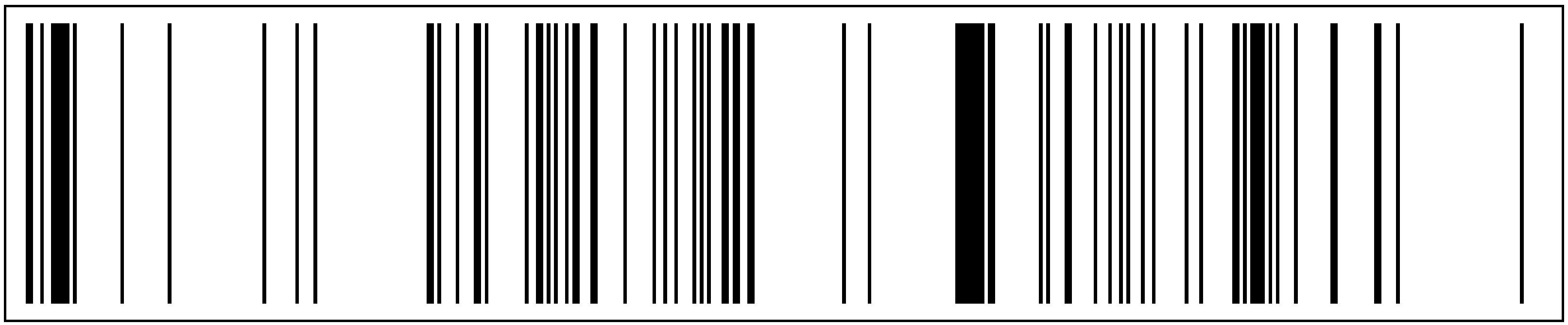
Point process of 100 arrivals for the $MMPP$

A bursty non-renewal *MMPP*.

- Consider the following non-renewal *MMPP*

$$\boldsymbol{R} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}.$$

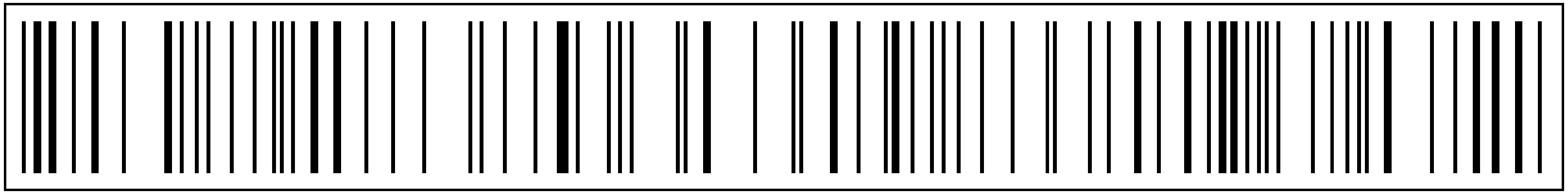
- What does this look like?



Point process of 100 arrivals for the MMPP

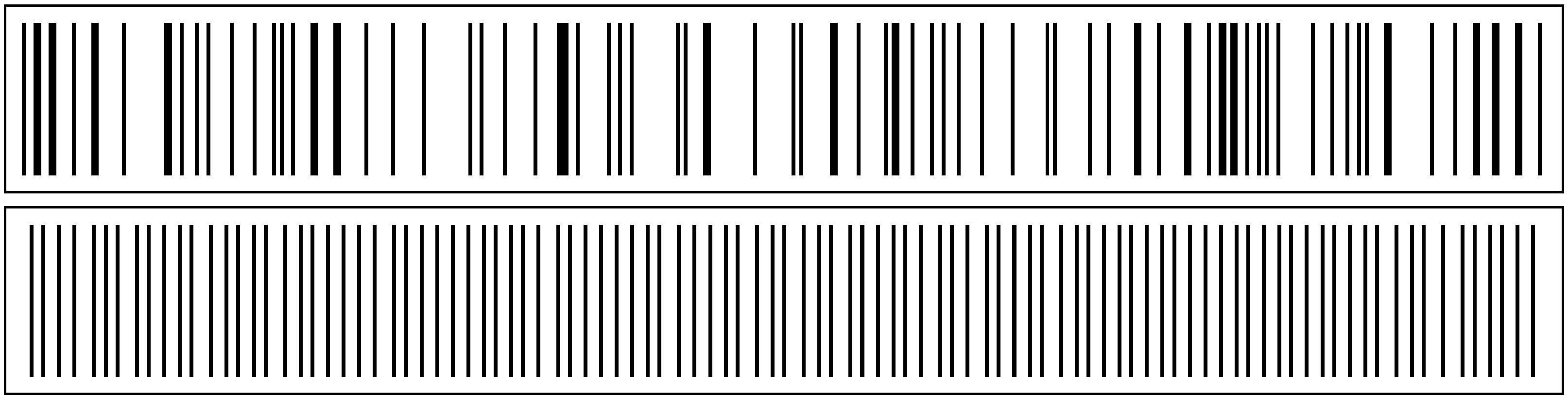
- *MMPPs* have been used for modelling such things as packetised voice. (Heffes and Lucantoni)

A comparison of forms.



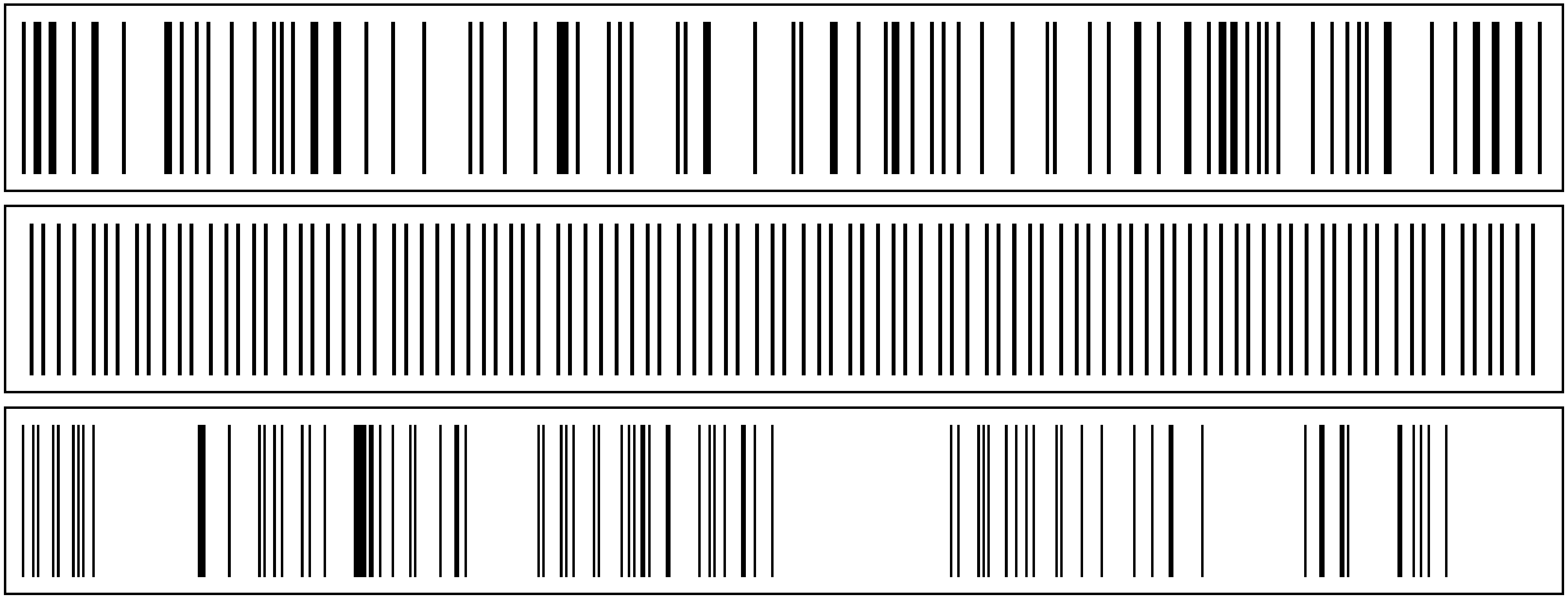
The Poisson process (random).

A comparison of forms.



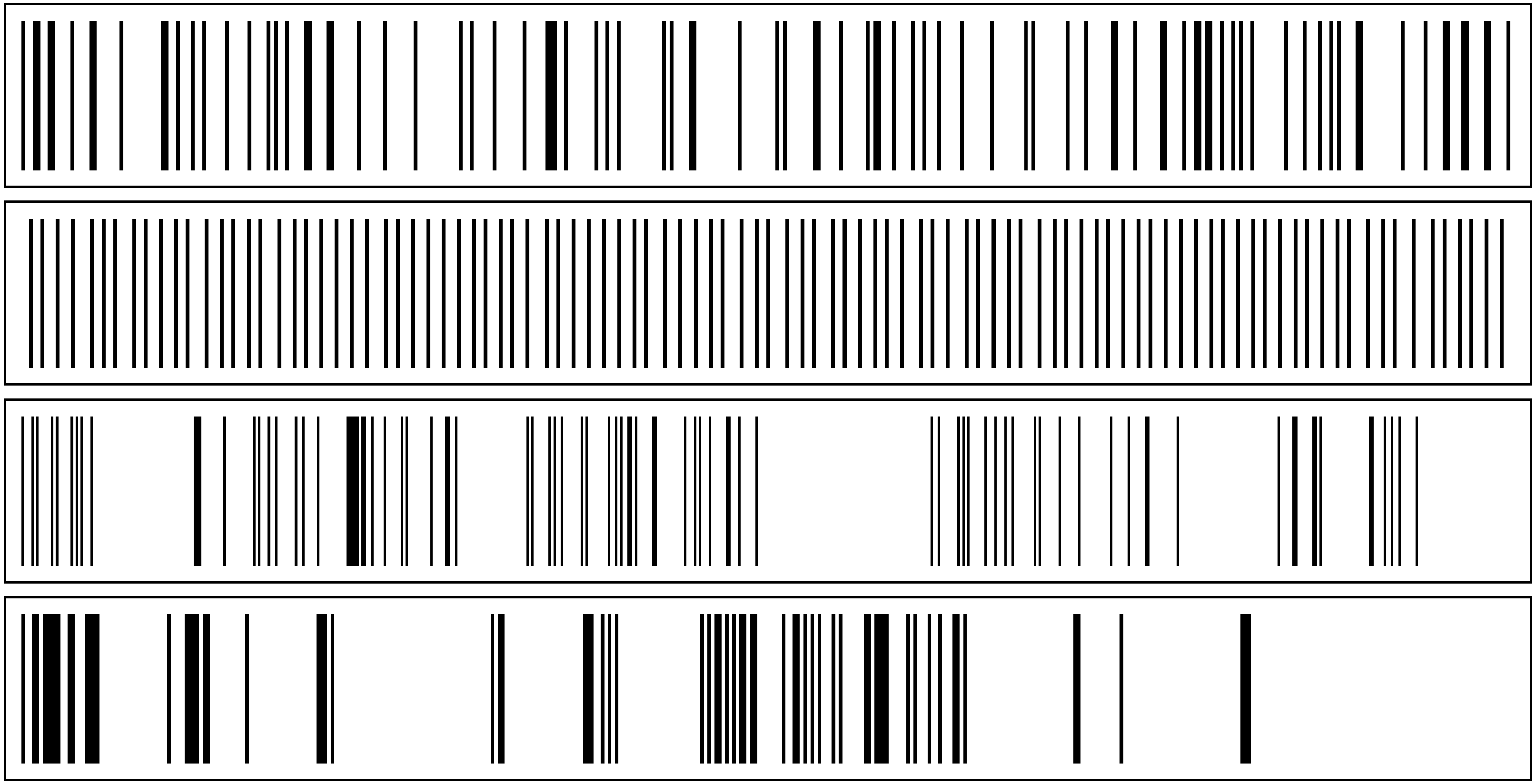
Erlang inter-arrival time distribution (regular).

A comparison of forms.



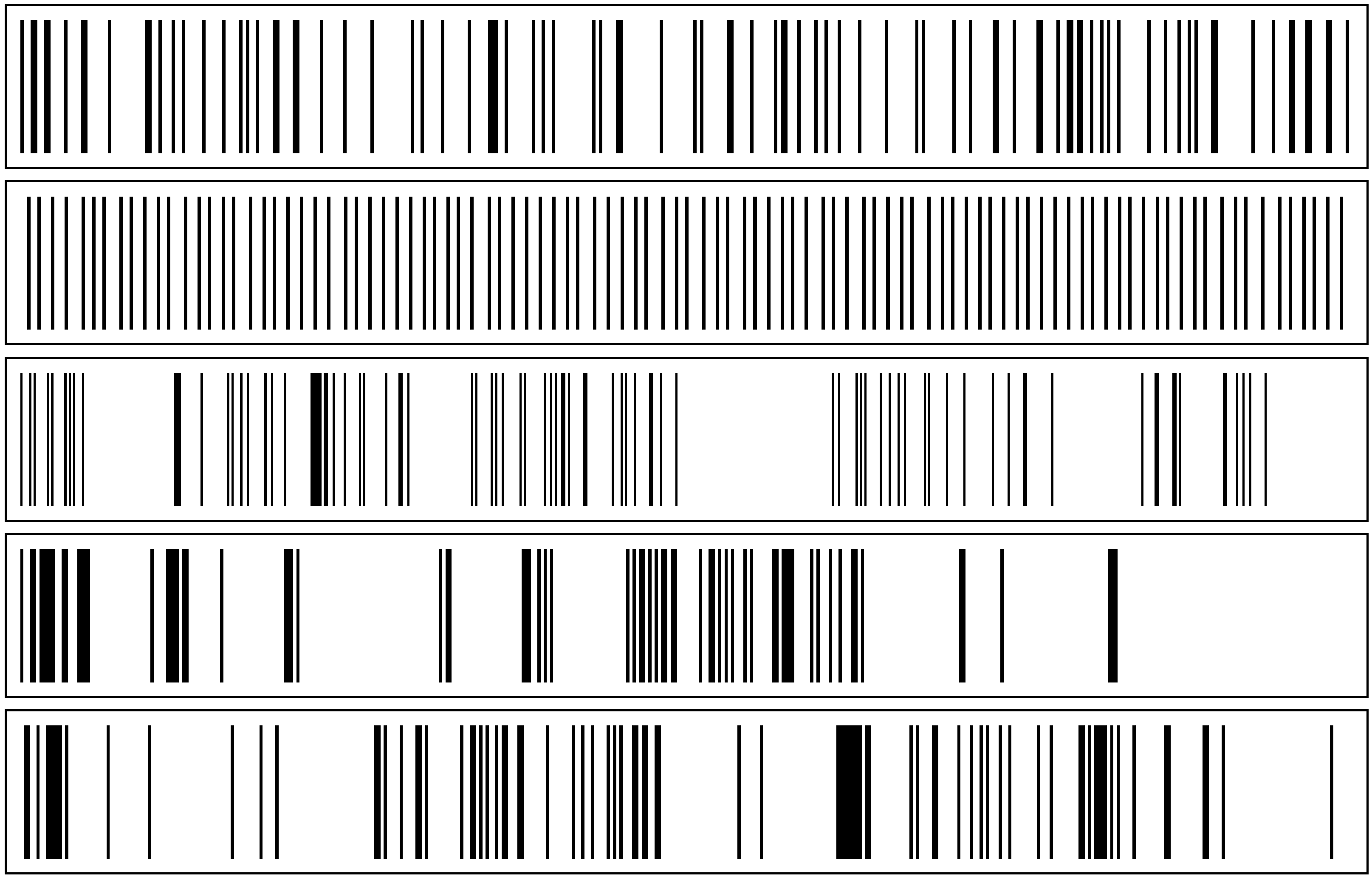
hyper-exponential inter-arrival time distribution (bursty).

A comparison of forms.



IPP renewal process (very bursty).

A comparison of forms.



MMPP non-renewal process (very bursty).

The last two bursty processes

The last two bursty processes

- The last two processes appear similar as would be expected by their $MMPP$ description.

The last two bursty processes

- The last two processes appear similar as would be expected by their $MMPP$ description.
- The essential difference is that the IPP is a renewal process and so we have for example that the length of a burst can be shown to be geometrically distributed.

The last two bursty processes

- The last two processes appear similar as would be expected by their $MMPP$ description.
- The essential difference is that the IPP is a renewal process and so we have for example that the length of a burst can be shown to be geometrically distributed.
- That is, upon entering the ON state from whence arrivals occur,

The last two bursty processes

- The last two processes appear similar as would be expected by their $MMPP$ description.
- The essential difference is that the IPP is a renewal process and so we have for example that the length of a burst can be shown to be geometrically distributed.
- That is, upon entering the ON state from whence arrivals occur,
    - we then can have an arrival with probability 0.9 and immediately re-enter the ON state

The last two bursty processes

- The last two processes appear similar as would be expected by their $MMPP$ description.
- The essential difference is that the IPP is a renewal process and so we have for example that the length of a burst can be shown to be geometrically distributed.
- That is, upon entering the ON state from whence arrivals occur,
  - we then can have an arrival with probability 0.9 and immediately re-enter the ON state
  - or the burst concludes with probability 0.1 and we enter the OFF state.

The last two bursty processes

- The last two processes appear similar as would be expected by their $MMPP$ description.
- The essential difference is that the IPP is a renewal process and so we have for example that the length of a burst can be shown to be geometrically distributed.
- That is, upon entering the ON state from whence arrivals occur,
  - we then can have an arrival with probability 0.9 and immediately re-enter the ON state
  - or the burst concludes with probability 0.1 and we enter the OFF state.
- Hence the probability of a burst of length n is given by $p(n) = 0.1(0.9)^n$ for $n \in \{0, 1, 2, \ldots\}$.

The last two bursty processes

- The last two processes appear similar as would be expected by their $MMPP$ description.
- The essential difference is that the IPP is a renewal process and so we have for example that the length of a burst can be shown to be geometrically distributed.
- That is, upon entering the ON state from whence arrivals occur,
  - we then can have an arrival with probability 0.9 and immediately re-enter the ON state
  - or the burst concludes with probability 0.1 and we enter the OFF state.
- Hence the probability of a burst of length n is given by $p(n) = 0.1(0.9)^n$ for $n \in \{0, 1, 2, \ldots\}$.
- The $MMPP$ does not have this property.

Another example

Another example

- Consider three phase type distributions $(\boldsymbol{\alpha}, T), (\boldsymbol{\beta}, S)$ and $(\boldsymbol{\gamma}, R)$.

Another example

- Consider three phase type distributions $(\boldsymbol{\alpha}, T), (\boldsymbol{\beta}, S)$ and $(\boldsymbol{\gamma}, R)$.
- Assume that we choose successive inter-arrival times according to a Markov chain P,

Another example

- Consider three phase type distributions $(\boldsymbol{\alpha}, T), (\boldsymbol{\beta}, S)$ and $(\boldsymbol{\gamma}, R)$.
- Assume that we choose successive inter-arrival times according to a Markov chain P,
- Then in MAP notation we have

Another example

- Consider three phase type distributions $(\boldsymbol{\alpha}, T), (\boldsymbol{\beta}, S)$ and $(\boldsymbol{\gamma}, R)$.
- Assume that we choose successive inter-arrival times according to a Markov chain P,
- Then in MAP notation we have

$$D_0 = \begin{bmatrix} T & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & R \end{bmatrix}$$

Another example

- Consider three phase type distributions $(\boldsymbol{\alpha}, T), (\boldsymbol{\beta}, S)$ and $(\boldsymbol{\gamma}, R)$.
- Assume that we choose successive inter-arrival times according to a Markov chain P,
- Then in MAP notation we have

$$D_0 = \begin{bmatrix} T & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & R \end{bmatrix}, D_1 = \begin{bmatrix} p_{1,1}\boldsymbol{T}^0\boldsymbol{\alpha} & p_{1,2}\boldsymbol{T}^0\boldsymbol{\beta} & p_{1,3}\boldsymbol{T}^0\boldsymbol{\gamma} \\ p_{2,1}\boldsymbol{S}^0\boldsymbol{\alpha} & p_{2,2}\boldsymbol{S}^0\boldsymbol{\beta} & p_{2,3}\boldsymbol{S}^0\boldsymbol{\gamma} \\ p_{3,1}\boldsymbol{R}^0\boldsymbol{\alpha} & p_{3,2}\boldsymbol{R}^0\boldsymbol{\beta} & p_{3,3}\boldsymbol{R}^0\boldsymbol{\gamma} \end{bmatrix}$$

Another example

- Consider three phase type distributions $(\boldsymbol{\alpha}, T), (\boldsymbol{\beta}, S)$ and $(\boldsymbol{\gamma}, R)$.
- Assume that we choose successive inter-arrival times according to a Markov chain P,
- Then in MAP notation we have

$$D_0 = \begin{bmatrix} T & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & R \end{bmatrix}, D_1 = \begin{bmatrix} p_{1,1}\boldsymbol{T}^0\boldsymbol{\alpha} & p_{1,2}\boldsymbol{T}^0\boldsymbol{\beta} & p_{1,3}\boldsymbol{T}^0\boldsymbol{\gamma} \\ p_{2,1}\boldsymbol{S}^0\boldsymbol{\alpha} & p_{2,2}\boldsymbol{S}^0\boldsymbol{\beta} & p_{2,3}\boldsymbol{S}^0\boldsymbol{\gamma} \\ p_{3,1}\boldsymbol{R}^0\boldsymbol{\alpha} & p_{3,2}\boldsymbol{R}^0\boldsymbol{\beta} & p_{3,3}\boldsymbol{R}^0\boldsymbol{\gamma} \end{bmatrix}$$

- This could be a renewal process or otherwise depending on the form of the phase type distributions and the matrix P.

What does this look like?

What does this look like?

- Consider
$$(\boldsymbol{\alpha}, T) = (1, -100), (\boldsymbol{\beta}, S) = \left((1,0), \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} \right)$$ and
$$(\boldsymbol{\gamma}, R) = \left((0.25, 0.75), \begin{bmatrix} -0.4 & 0.2 \\ 0.3 & -0.6 \end{bmatrix} \right).$$

What does this look like?

- Consider $(\boldsymbol{\alpha}, T) = (1, -100), (\boldsymbol{\beta}, S) = \left((1,0), \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} \right)$ and $(\boldsymbol{\gamma}, R) = \left((0.25, 0.75), \begin{bmatrix} -0.4 & 0.2 \\ 0.3 & -0.6 \end{bmatrix} \right)$.
- $P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$,

What does this look like?

- Consider

  $(\boldsymbol{\alpha}, T) = (1, -100), (\boldsymbol{\beta}, S) = \left((1,0), \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} \right)$ and

  $(\boldsymbol{\gamma}, R) = \left((0.25, 0.75), \begin{bmatrix} -0.4 & 0.2 \\ 0.3 & -0.6 \end{bmatrix} \right)$.

- $P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$,

- Then we have

What does this look like?

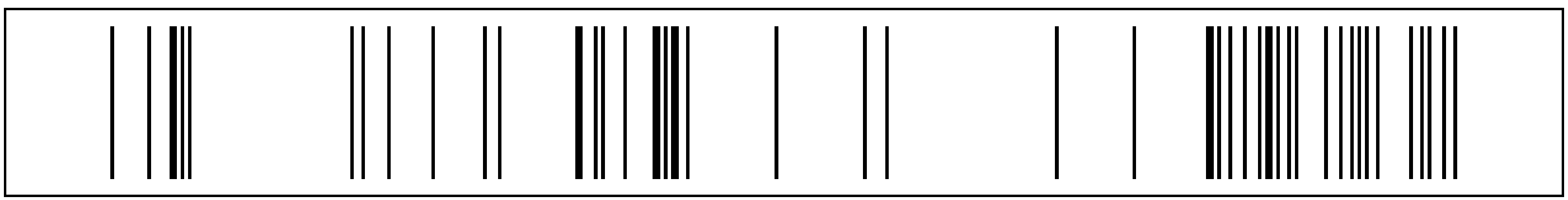
- Consider

$$(\boldsymbol{\alpha}, T) = (1, -100), (\boldsymbol{\beta}, S) = \left((1, 0), \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} \right) \text{ and}$$

$$(\boldsymbol{\gamma}, R) = \left((0.25, 0.75), \begin{bmatrix} -0.4 & 0.2 \\ 0.3 & -0.6 \end{bmatrix} \right).$$

- $P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix},$

- Then we have



What does this tell us?

What does this tell us?

- We have used simple intrinsically non-bursty processes to yield what appears to be a very bursty process.

What does this tell us?

- We have used simple intrinsically non-bursty processes to yield what appears to be a very bursty process.
- We have demonstrated a MAP, which can clearly represent different behaviour over different time scales.

What does this tell us?

- We have used simple intrinsically non-bursty processes to yield what appears to be a very bursty process.
- We have demonstrated a MAP, which can clearly represent different behaviour over different time scales.
- $MAPs$ are a highly tractable modelling tool as will be shown in the following sessions.

What does this tell us?

- We have used simple intrinsically non-bursty processes to yield what appears to be a very bursty process.
- We have demonstrated a MAP, which can clearly represent different behaviour over different time scales.
- $MAPs$ are a highly tractable modelling tool as will be shown in the following sessions.
- They are therefore highly desirable for modelling.

What does this tell us?

- We have used simple intrinsically non-bursty processes to yield what appears to be a very bursty process.
- We have demonstrated a MAP, which can clearly represent different behaviour over different time scales.
- $MAPs$ are a highly tractable modelling tool as will be shown in the following sessions.
- They are therefore highly desirable for modelling.
- There exist some fitting mechanisms such as those talked about in the previous session, which fit phase type distributions to data sets that can be used as renewal approximations to the empirical data.

What does this tell us?

- We have used simple intrinsically non-bursty processes to yield what appears to be a very bursty process.
- We have demonstrated a MAP, which can clearly represent different behaviour over different time scales.
- $MAPs$ are a highly tractable modelling tool as will be shown in the following sessions.
- They are therefore highly desirable for modelling.
- There exist some fitting mechanisms such as those talked about in the previous session, which fit phase type distributions to data sets that can be used as renewal approximations to the empirical data.
- The MAP however can enable much more than that, as it allows dependencies to exist between successive arrivals.

Even more capability: the $BMAP$

Even more capability: the $BMAP$

- The MAP is a sub-class of the batch Markovian arrival process or $BMAP$.

Even more capability: the $BMAP$

- The MAP is a sub-class of the batch Markovian arrival process or $BMAP$.
- However, it is convenient here to describe the $BMAP$ as an extension of the MAP, which allows batch arrivals at an arrival epoch rather than just single arrivals.

Even more capability: the $BMAP$

- The MAP is a sub-class of the batch Markovian arrival process or $BMAP$.
- However, it is convenient here to describe the $BMAP$ as an extension of the MAP, which allows batch arrivals at an arrival epoch rather than just single arrivals.
- The $BMAP$ is essentially characterised by matrices $D_0, D_1, D_2, \ldots D_k, \ldots$, with the following properties

Even more capability: the $BMAP$

- The MAP is a sub-class of the batch Markovian arrival process or $BMAP$.
- However, it is convenient here to describe the $BMAP$ as an extension of the MAP, which allows batch arrivals at an arrival epoch rather than just single arrivals.
- The $BMAP$ is essentially characterised by matrices $D_0, D_1, D_2, \ldots D_k, \ldots$, with the following properties

$$[D_0]_{ii} \; < \; 0 \quad \text{for all } i,$$

Even more capability: the $BMAP$

- The MAP is a sub-class of the batch Markovian arrival process or $BMAP$.
- However, it is convenient here to describe the $BMAP$ as an extension of the MAP, which allows batch arrivals at an arrival epoch rather than just single arrivals.
- The $BMAP$ is essentially characterised by matrices $D_0, D_1, D_2, \ldots D_k, \ldots$, with the following properties

$$\begin{aligned} [D_0]_{ii} &< 0 \quad \text{for all } i, \\ [D_0]_{ij} &\geq 0 \quad \text{for all } i \neq j \end{aligned}$$

Even more capability: the $BMAP$

- The MAP is a sub-class of the batch Markovian arrival process or $BMAP$.
- However, it is convenient here to describe the $BMAP$ as an extension of the MAP, which allows batch arrivals at an arrival epoch rather than just single arrivals.
- The $BMAP$ is essentially characterised by matrices $D_0, D_1, D_2, \ldots D_k, \ldots$, with the following properties

$$\begin{aligned} [D_0]_{ii} &< 0 \quad \text{for all } i, \\ [D_0]_{ij} &\geq 0 \quad \text{for all } i \neq j \ , \\ [D_k]_{ij} &\geq 0 \quad \text{for all } i, j, k \end{aligned}$$

Even more capability: the $BMAP$

- The MAP is a sub-class of the batch Markovian arrival process or $BMAP$.
- However, it is convenient here to describe the $BMAP$ as an extension of the MAP, which allows batch arrivals at an arrival epoch rather than just single arrivals.
- The $BMAP$ is essentially characterised by matrices $D_0, D_1, D_2, \ldots D_k, \ldots$, with the following properties

$$\begin{aligned} [D_0]_{ii} &< 0 \quad \text{for all } i, \\ [D_0]_{ij} &\geq 0 \quad \text{for all } i \neq j \ , \\ [D_k]_{ij} &\geq 0 \quad \text{for all } i, j, k \\ \text{and } D\boldsymbol{e} &= (\sum_k D_k)\boldsymbol{e} = \mathbf{0} \ . \end{aligned}$$

Even more capability: the $BMAP$

- The MAP is a sub-class of the batch Markovian arrival process or $BMAP$.
- However, it is convenient here to describe the $BMAP$ as an extension of the MAP, which allows batch arrivals at an arrival epoch rather than just single arrivals.
- The $BMAP$ is essentially characterised by matrices $D_0, D_1, D_2, \ldots D_k, \ldots$, with the following properties

$$\begin{aligned} [D_0]_{ii} &< 0 \quad \text{for all } i, \\ [D_0]_{ij} &\geq 0 \quad \text{for all } i \neq j\ , \\ [D_k]_{ij} &\geq 0 \quad \text{for all } i, j, k \\ \text{and } D\boldsymbol{e} &= (\sum_k D_k)\boldsymbol{e} = \mathbf{0}\ . \end{aligned}$$

Note: the matrix D_k governs those arrivals of batch size k.

Q-matrix for the $BMAP$

Q-matrix for the $BMAP$

- The evolution of a $BMAP$ can be modelled by the following Q-matrix

Q-matrix for the $BMAP$

- The evolution of a $BMAP$ can be modelled by the following Q-matrix

$$Q = \begin{bmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ 0 & D_0 & D_1 & D_2 & \cdots \\ 0 & 0 & D_0 & D_1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Q-matrix for the $BMAP$

- The evolution of a $BMAP$ can be modelled by the following Q-matrix

$$Q = \begin{bmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ 0 & D_0 & D_1 & D_2 & \cdots \\ 0 & 0 & D_0 & D_1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}.$$

- The MAP is then trivially a $BMAP$ with $D_k \equiv 0$ for all $k \geq 2$.

The Batch Poisson process

The Batch Poisson process

- Negative exponentially distributed inter-arrival times between batches as for the Poisson process.

The Batch Poisson process

- Negative exponentially distributed inter-arrival times between batches as for the Poisson process.
- Successive batch sizes have probability mass function $\{p_k, k \geq 1\}$

The Batch Poisson process

- Negative exponentially distributed inter-arrival times between batches as for the Poisson process.
- Successive batch sizes have probability mass function $\{p_k, k \geq 1\}$
- Hence we have the following Q-matrix

The Batch Poisson process

- Negative exponentially distributed inter-arrival times between batches as for the Poisson process.
- Successive batch sizes have probability mass function $\{p_k, k \geq 1\}$
- Hence we have the following Q-matrix

$$Q = \begin{bmatrix} -\lambda & p_1\lambda & p_2\lambda & p_3\lambda & p_4\lambda & \dots \\ 0 & -\lambda & p_1\lambda & p_2\lambda & p_3\lambda & \dots \\ 0 & 0 & -\lambda & p_1\lambda & p_2\lambda & \dots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

The Batch Poisson process

- Negative exponentially distributed inter-arrival times between batches as for the Poisson process.
- Successive batch sizes have probability mass function $\{p_k, k \geq 1\}$
- Hence we have the following Q-matrix

$$Q = \begin{bmatrix} -\lambda & p_1\lambda & p_2\lambda & p_3\lambda & p_4\lambda & \dots \\ 0 & -\lambda & p_1\lambda & p_2\lambda & p_3\lambda & \dots \\ 0 & 0 & -\lambda & p_1\lambda & p_2\lambda & \dots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

- where λ is the arrival rate of batches.

An $MMPP$ with i.i.d. batch arrivals

An $MMPP$ with i.i.d. batch arrivals

- Consider

An $MMPP$ with i.i.d. batch arrivals

- Consider

$$D_0 = \begin{bmatrix} -4.0 & 1.0 & 0.2 \\ 0.12 & -0.25 & 0.005 \\ 0.7 & 0.3 & -5.0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 2.8 & 0 & 0 \\ 0 & 0.125 & 0 \\ 0 & 0 & 4.0 \end{bmatrix},$$

An $MMPP$ with i.i.d. batch arrivals

- Consider

$$D_0 = \begin{bmatrix} -4.0 & 1.0 & 0.2 \\ 0.12 & -0.25 & 0.005 \\ 0.7 & 0.3 & -5.0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 2.8 & 0 & 0 \\ 0 & 0.125 & 0 \\ 0 & 0 & 4.0 \end{bmatrix},$$

with $i.i.d.$ batch arrivals governed by probability vector

$$(0.1, 0.2, 0.1, 0.2, 0.15, 0.25)$$

An $MMPP$ with i.i.d. batch arrivals

- Consider

$$D_0 = \begin{bmatrix} -4.0 & 1.0 & 0.2 \\ 0.12 & -0.25 & 0.005 \\ 0.7 & 0.3 & -5.0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 2.8 & 0 & 0 \\ 0 & 0.125 & 0 \\ 0 & 0 & 4.0 \end{bmatrix},$$

  with $i.i.d.$ batch arrivals governed by probability vector

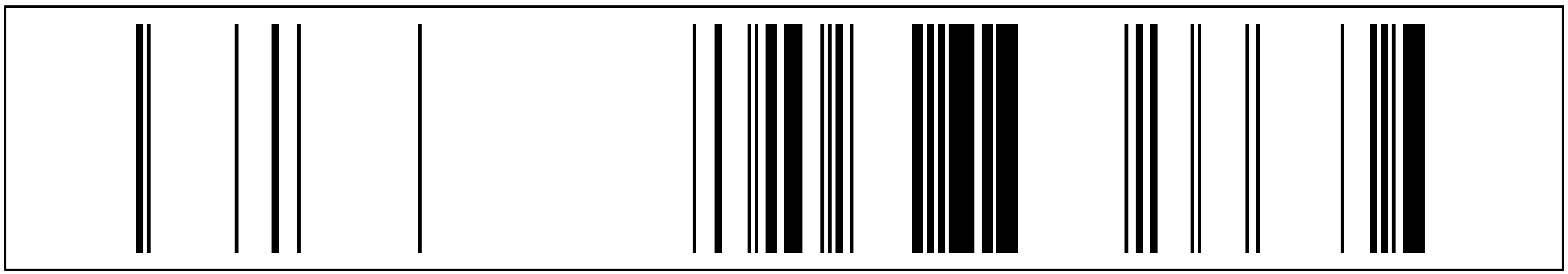
$$(0.1, 0.2, 0.1, 0.2, 0.15, 0.25)$$

  Hence the Q-matrix looks like

$$\begin{bmatrix} D_0 & 0.1D_1 & 0.2D_1 & 0.1D_1 & 0.2D_1 & 0.15D_1 & 0.25D_1 & 0 & \dots \\ 0 & D_0 & 0.1D_1 & 0.2D_1 & 0.1D_1 & 0.2D_1 & 0.15D_1 & 0.25D_1 & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

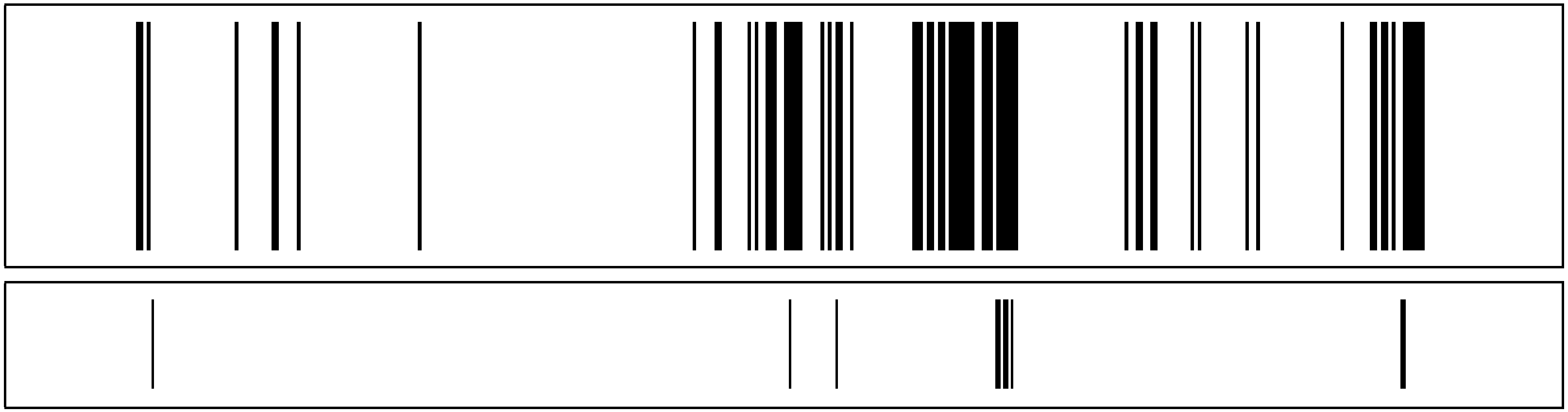
What does it look like?

What does it look like?



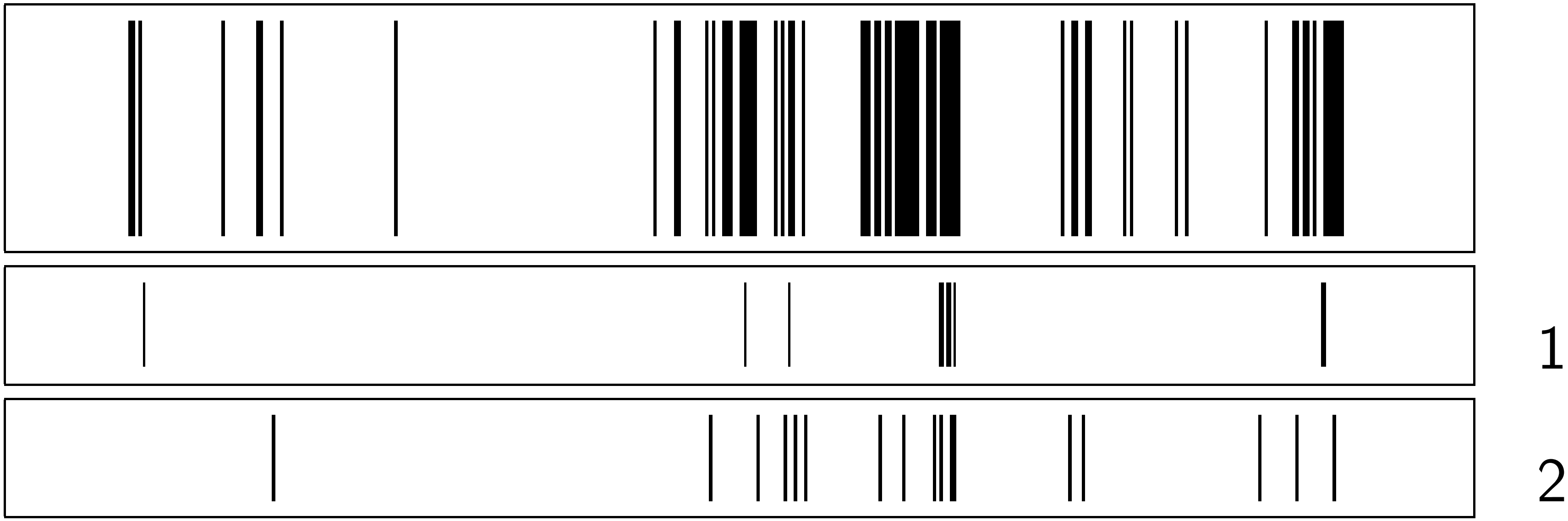
The arrival epochs.

What does it look like?

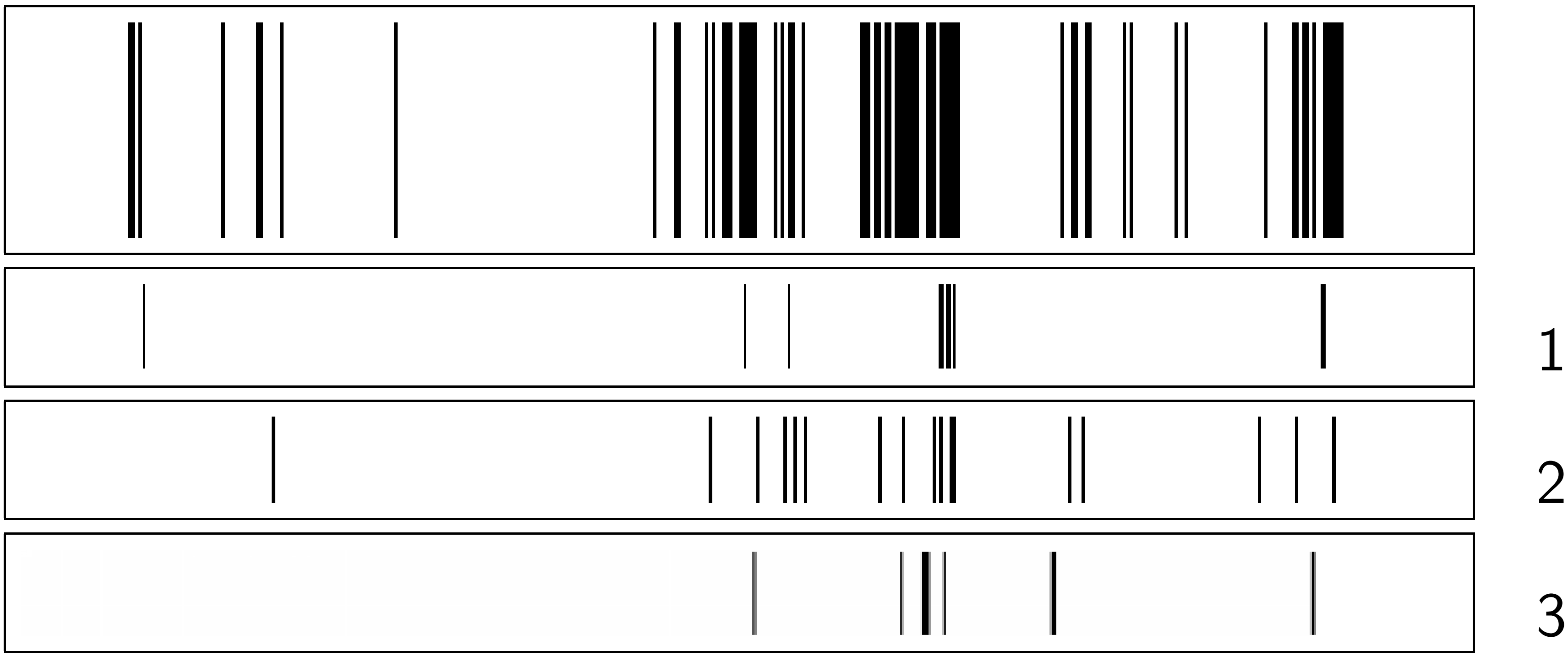


1

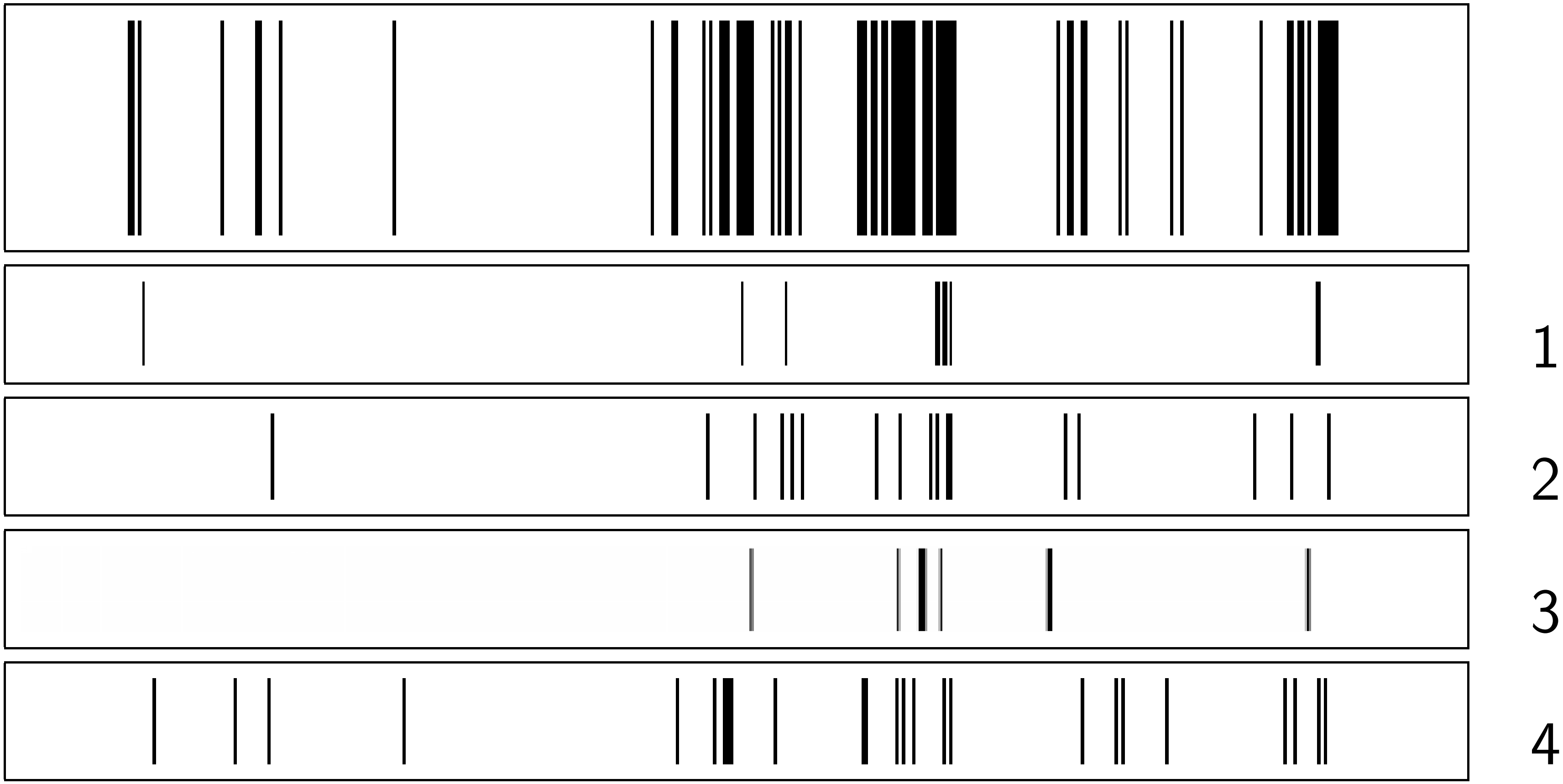
What does it look like?



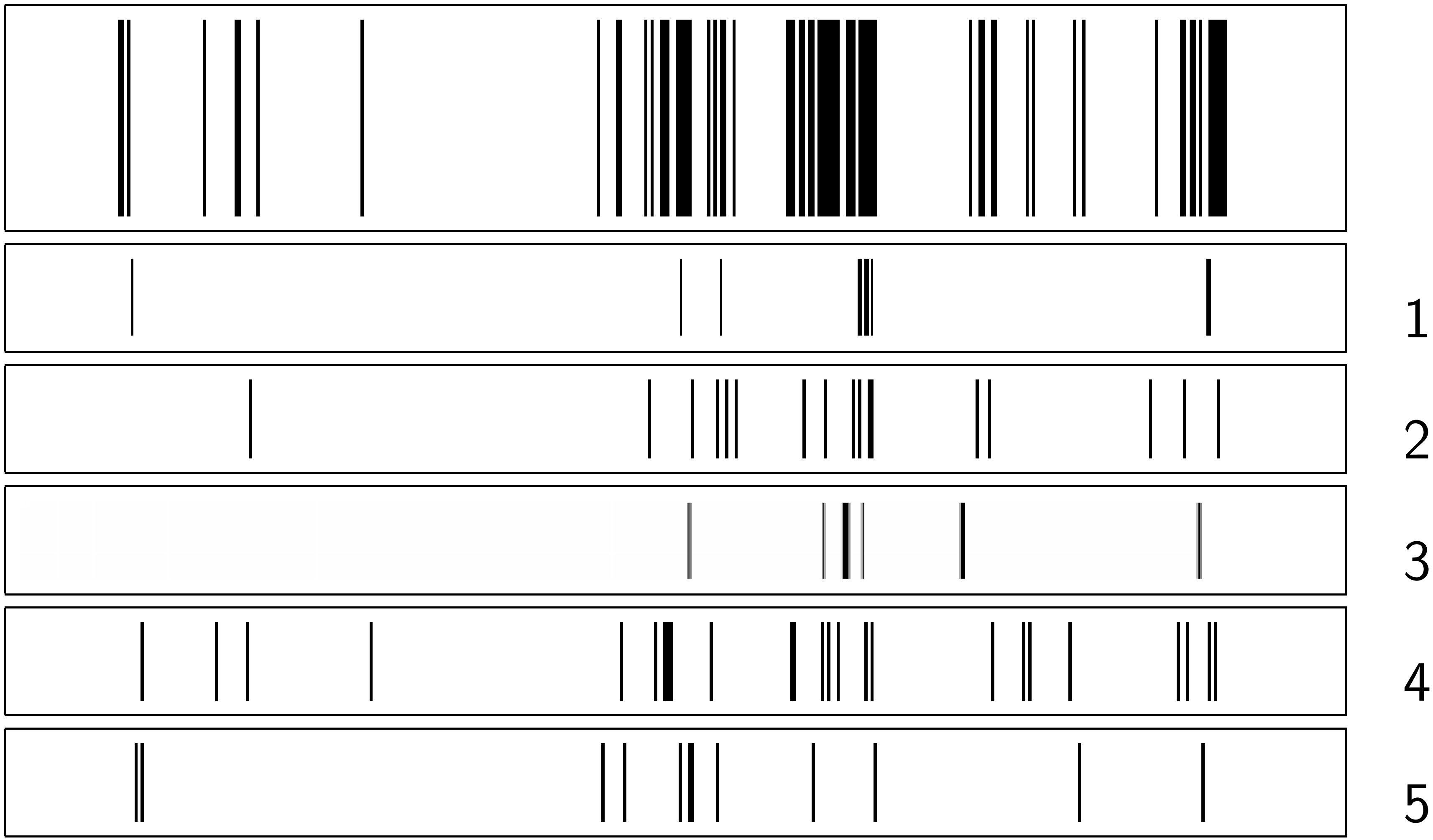
What does it look like?



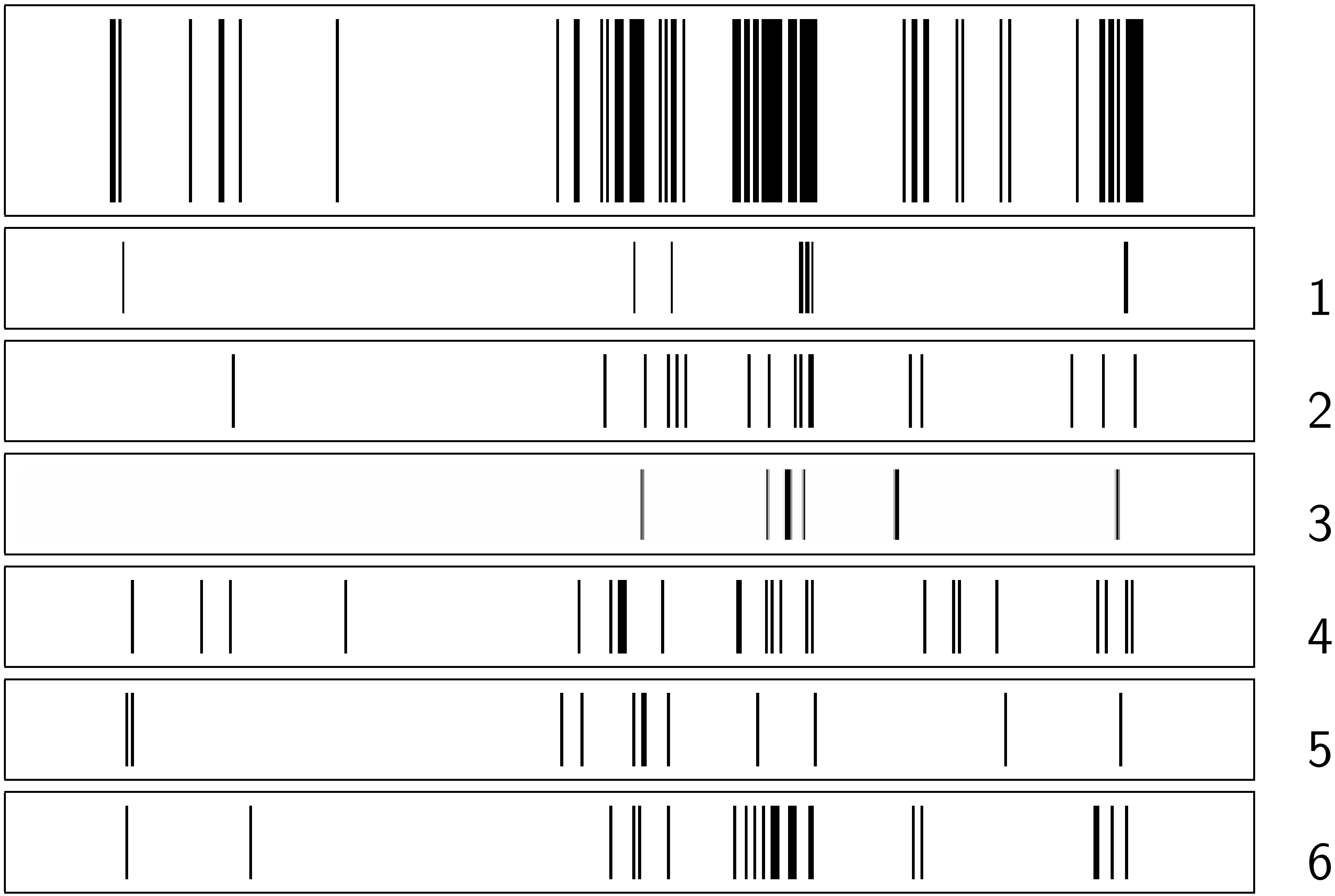
What does it look like?



What does it look like?



What does it look like?



Some other notes

Some other notes

- The $BMAP$ is closed under superposition.

Some other notes

- The $BMAP$ is closed under superposition.
- That is, the superposition of n independent $BMAPs$ is also a $BMAP$.

Some other notes

- The $BMAP$ is closed under superposition.
- That is, the superposition of n independent $BMAPs$ is also a $BMAP$.

  If $D_k(i)$ is the matrix governing the batches of size k for the i^{th} independent $BMAP$, for each $k \geq 0$. (Note that any number of these matrices could be identically zero.)

Some other notes

- The $BMAP$ is closed under superposition.
- That is, the superposition of n independent $BMAPs$ is also a $BMAP$.

  If $D_k(i)$ is the matrix governing the batches of size k for the i^{th} independent $BMAP$, for each $k \geq 0$. (Note that any number of these matrices could be identically zero.)

  Then the D_k matrix for the superposition is given by

$$D_k = D_k(1) \oplus D_k(2) \oplus D_k(3) \oplus \cdots \oplus D_k(n), \quad \text{for } k \geq 0 ,$$

Some other notes

- The $BMAP$ is closed under superposition.
- That is, the superposition of n independent $BMAPs$ is also a $BMAP$.

  If $D_k(i)$ is the matrix governing the batches of size k for the i^{th} independent $BMAP$, for each $k \geq 0$. (Note that any number of these matrices could be identically zero.)

  Then the D_k matrix for the superposition is given by

$$D_k = D_k(1) \oplus D_k(2) \oplus D_k(3) \oplus \cdots \oplus D_k(n), \quad \text{for } k \geq 0 \, ,$$

where $\oplus$ is the Kronecker sum.

Some other notes

- The $BMAP$ is closed under superposition.
- That is, the superposition of n independent $BMAPs$ is also a $BMAP$.

  If $D_k(i)$ is the matrix governing the batches of size k for the i^{th} independent $BMAP$, for each $k \geq 0$. (Note that any number of these matrices could be identically zero.)

  Then the D_k matrix for the superposition is given by

$$D_k = D_k(1) \oplus D_k(2) \oplus D_k(3) \oplus \cdots \oplus D_k(n), \quad \text{for } k \geq 0 \,,$$

  where $\oplus$ is the Kronecker sum.
- This construction could for instance be used to model multiplexed traffic streams.(Choudhury,Lucantoni and Whitt)

Some different considerations

Some different considerations

- We have only considered processes homogeneous in the level.

Some different considerations

- We have only considered processes homogeneous in the level.
- That is, the MAP descriptors are unchanged as $\{N(t)\}$ changes.

Some different considerations

- We have only considered processes homogeneous in the level.
- That is, the MAP descriptors are unchanged as $\{N(t)\}$ changes.
- It is also possible to have a non-homogeneous process, where there exists a dependency on the level.

Some different considerations

- We have only considered processes homogeneous in the level.
- That is, the MAP descriptors are unchanged as $\{N(t)\}$ changes.
- It is also possible to have a non-homogeneous process, where there exists a dependency on the level.
- The simplest case here would be the non-homogeneous Poisson process.

Some different considerations

- We have only considered processes homogeneous in the level.
- That is, the MAP descriptors are unchanged as $\{N(t)\}$ changes.
- It is also possible to have a non-homogeneous process, where there exists a dependency on the level.
- The simplest case here would be the non-homogeneous Poisson process.

  That is,

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & & & \\ 0 & -\lambda_2 & \lambda_2 & 0 & & & \\ 0 & 0 & -\lambda_3 & \lambda_3 & 0 & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \end{bmatrix},$$

  where λ_i is the arrival rate at level i.

The uniqueness of representation

The uniqueness of representation

- Also as in the case of phase type distributions, $MAPs$ can similarly have a variety of representations.

The uniqueness of representation

- Also as in the case of phase type distributions, $MAPs$ can similarly have a variety of representations.
- For example the following three $MAPs$ are just complex representations of a Poisson process of rate 1 under stationary conditions.

The uniqueness of representation

- Also as in the case of phase type distributions, $MAPs$ can similarly have a variety of representations.
- For example the following three $MAPs$ are just complex representations of a Poisson process of rate 1 under stationary conditions.

$$D_0 = \begin{bmatrix} -2 & \frac{1}{2} & \frac{1}{2} \\ 1 & -4 & 1 \\ \frac{1}{2} & 1 & -2 \end{bmatrix},\ D_1 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The uniqueness of representation

- Also as in the case of phase type distributions, $MAPs$ can similarly have a variety of representations.
- For example the following three $MAPs$ are just complex representations of a Poisson process of rate 1 under stationary conditions.

$$D_0 = \begin{bmatrix} -4 & 2 & 1 \\ 5 & -8 & 2 \\ 1 & 2 & -4 \end{bmatrix}, D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The uniqueness of representation

- Also as in the case of phase type distributions, $MAPs$ can similarly have a variety of representations.
- For example the following three $MAPs$ are just complex representations of a Poisson process of rate 1 under stationary conditions.

$$D_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -6 & 4 \\ 0 & 0 & -1 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ (\frac{1}{2}) & (\frac{1}{2}) & 0 \end{bmatrix}$$

The uniqueness of representation

- Also as in the case of phase type distributions, $MAPs$ can similarly have a variety of representations.
- For example the following three $MAPs$ are just complex representations of a Poisson process of rate 1 under stationary conditions.

$$D_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -6 & 4 \\ 0 & 0 & -1 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

- This adds another dimension to the fitting of $MAPs$, and adds to the reasons as to why it is a potentially difficult exercise. Particularly when it comes to minimising the order of representation.