Markovian Point Processes

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Poisson process
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• A point process where the distribution of time between points is a simple phase type distribution having a single phase.
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\[ X_1, X_2, X_3, X_4, X_5, \ldots \]

Time

what if the \( X_i \sim \text{I.I.D. phase type} \) with description \((\alpha, T)\)?
Phase renewal process
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- Infinitesimal generator or Q-matrix description.
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Consider first the Poisson process

\[
Q = \begin{bmatrix}
-\lambda & \lambda & 0 & \cdots \\
0 & -\lambda & \lambda & 0 \\
0 & 0 & -\lambda & \lambda & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}.
\]
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\end{bmatrix},
\]

where \( \lambda \) is the arrival rate from the single phase.
Phase renewal process

- Infinitesimal generator or $Q$-matrix description.

Then the more general phase type renewal process

$$Q = \begin{bmatrix}
T & T^0\alpha & 0 & 0 & \cdots \\
0 & T & T^0\alpha & 0 & \cdots \\
0 & 0 & T & T^0\alpha & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

where $T^0$ is a column of rates corresponding to the arrival rate out of each phase of the matrix $T$. 
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\end{bmatrix},
\]

where \( T^0 \) is a column of rates corresponding to the arrival rate out of each phase of the matrix \( T \). More formally \( T^0 = -Te \), where \( e \) is a column of ones of the appropriate dimension.
Markovian Arrival Process Notation

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This gives us a complete description from which we can write

$$Q = \begin{bmatrix}
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This is $MAP$ notation, where $D_0 = T$ and $D_1 = T^0 \alpha$. 

MAPs

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\textbf{MAPs}

- The phase renewal processes form an important sub-class of \textit{MAPs}.
- With a little thought, one can imagine much more general processes than renewal processes as having a \textit{MAP} description.
- The following properties of the \textit{MAP} are considered from this more general sense.
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$$
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$$
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[D_1]_{ij} \geq 0 \quad \text{for all } i, j
$$

and $De = (D_0 + D_1)e = 0$. 
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  - all of it’s eigenvalues have negative real parts: (Bellman)
  - inter-arrival times are finite with probability one: (Neuts)
  - the process does not terminate.
The generator matrix or Q-matrix $D$
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- Recalling that the matrix $D_1$ governs those transitions which correspond to arrivals,
  - in light of the information given by the vector $\pi$,
  - the process of arrivals has the following fundamental arrival rate

$$\lambda = \pi D_1 e.$$
The two dimensional representation
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This has state space
\[ \{(0, 1), (0, 2), \ldots, (0, m), (1, 1), (1, 2), \ldots, (1, m), \ldots\} \]
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Level zero
The two dimensional representation

• Consider an \( m \)-dimensional matrix pair \( D_0 \) and \( D_1 \)

• If we consider the Q-matrix for the evolution of the \( MAP \), essentially we have a \textit{two-dimensional} Markov process \( \{N_t, J_t\} \).
  • where, the \( \{N_t\} \) process keeps track of the number of arrivals (\textit{The level}); in particular, \( N_t \) denotes the number of arrivals during the interval \( (0, t] \).
  • and the \( \{J_t\} \) process keeps track of the phase of the \( MAP \). (\textit{The phase})

This has state space

\[
\{(0, 1), (0, 2), \ldots, (0, m), (1, 1), (1, 2), \ldots, (1, m), \ldots\}.
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Level one
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- \( F(t) = 1 - e^{-\lambda t} \), where

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T = -\lambda \quad \text{and} \quad \alpha = 1.
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Point process of 100 arrivals
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\begin{center}
\textit{Point process of 100 arrivals for $E_{50}$}
\end{center}
Hyper-exponentially distributed
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\end{bmatrix} \quad \text{and } \alpha = (\alpha_1, \ldots, \alpha_n).$$

• What does this look like?

Point process of 100 arrivals for $H_2$:

$$F(t) = \frac{10}{11}(1 - e^{-10t}) + \frac{1}{11}(1 - e^{-t})$$
Non-renewal processes
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• We alluded to more general MAPs while describing renewal processes as MAPs.
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• The feature of the renewal processes is that every time an arrival occurs, the process immediately restarts with the exact same distribution of phase.
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• The non-renewal MAPs will be introduced by way of an important example.
Non-renewal processes

- We alluded to more general MAPs while describing renewal processes as MAPs.
- The feature of the renewal processes is that every time an arrival occurs, the process immediately restarts with the exact same distribution of phase.
- The non-renewal MAPs will be introduced by way of an important example.
- Furthermore MAPs as we will also see are a sub-class of what are known as Batch Markovian Arrival Processes (BMAPs).
Markov modulated Poisson process.
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- Let’s now consider an $m$ state (phase) continuous time Markov process $\{J_t\}$, with Q-matrix $R$. 
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- Arrivals are “modulated” in such a way that during a time period in which the process is in state $k$, customers may arrive according to a Poisson process with rate $\lambda_k$, $k \in \{1, \ldots, m\}$, independent of everything else.
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- The corresponding arrival process \(\{N_t\}\) is called a Markov modulated Poisson process (MMPP).
- Immediately after an arrival in this case we do not restart the process with a fixed distribution of phase \(\alpha\), but remain in the same phase from which the arrival occurred.
- Hence in general we do not have a renewal process.
MAP notation for the MMPP.
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  • and $D_1 = \Lambda$,
• where $R$ is the Q-matrix of $\{J_t\}$, and

$$\Lambda = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
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\vdots & \ddots & \ddots & \\
0 & \cdots & \cdots & \lambda_m
\end{bmatrix}.$$
A special MMPP.
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- An Interrupted (or switched) Poisson Process (IPP) essentially has a switch which jumps between ON and OFF, staying ON (OFF) for a exponentially distributed time with parameter $\gamma(\omega)$. 
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$$R = \begin{bmatrix} -\omega & \omega \\ \gamma & -\gamma \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \tau \end{bmatrix}.$$
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since we only have arrivals from one state.
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Point process of 100 arrivals for the IPP

- In general, MMPPs are not renewal processes.
A bursty non-renewal $MMPP$. 
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- Consider the following non-renewal $MMPP$
A bursty non-renewal MMPP.

• Consider the following non-renewal MMPP

\[ R = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}. \]
A bursty non-renewal MMPP.

- Consider the following non-renewal MMPP

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R = \begin{bmatrix}
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1 & -1
\end{bmatrix}
\quad \text{and} \quad
\Lambda = \begin{bmatrix}
9 & 0 \\
0 & 1
\end{bmatrix}.
\]

- What does this look like?
A bursty non-renewal MMPP.

• Consider the following non-renewal MMPP

\[ R = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}. \]

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Point process of 100 arrivals for the MMPP
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Point process of 100 arrivals for the MMPP

- MMPPs have been used for modelling such things as packetised voice. (Heffes and Lucantoni)
A comparison of forms.

The Poisson process (random).
A comparison of forms.

Erlang inter-arrival time distribution (regular).
A comparison of forms.

hyper-exponential inter-arrival time distribution (bursty).
A comparison of forms.

$IPP$ renewal process (very bursty).
A comparison of forms.

$\textit{MMPP}$ non-renewal process (very bursty).
The last two bursty processes
The last two bursty processes

• The last two processes appear similar as would be expected by their \textit{MMPP} description.
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• The essential difference is that the IPP is a renewal process and so we have for example that the length of a burst can be shown to be geometrically distributed.
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- Hence the probability of a burst of length \( n \) is given by
  \[ p(n) = 0.1(0.9)^n \text{ for } n \in \{0, 1, 2, \ldots\}. \]
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  \[ p(n) = 0.1(0.9)^n \] for \( n \in \{0, 1, 2, \ldots\} \).

• The \textit{MMPP} does not have this property.
Another example
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- Consider three phase type distributions $(\alpha, T)$, $(\beta, S)$ and $(\gamma, R)$. 
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- Assume that we choose successive inter-arrival times according to a Markov chain $P$. 
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D_0 = \begin{bmatrix}
T & 0 & 0 \\
0 & S & 0 \\
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D_0 = \begin{bmatrix}
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0 & S & 0 \\
0 & 0 & R \\
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
p_{1,1}T^0\alpha & p_{1,2}T^0\beta & p_{1,3}T^0\gamma \\
p_{2,1}S^0\alpha & p_{2,2}S^0\beta & p_{2,3}S^0\gamma \\
p_{3,1}R^0\alpha & p_{3,2}R^0\beta & p_{3,3}R^0\gamma \\
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\end{bmatrix}
\]

- This could be a renewal process or otherwise depending on the form of the phase type distributions and the matrix \(P\).
What does this look like?
What does this look like?

• Consider

\[(\alpha, T) = (1, -100), (\beta, S) = \left((1, 0), \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}\right) \text{ and} \]

\[(\gamma, R) = \left((0.25, 0.75), \begin{bmatrix} -0.4 & 0.2 \\ 0.3 & -0.6 \end{bmatrix}\right).\]
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• \(P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}\),
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• We have demonstrated a $MAP$, which can clearly represent different behaviour over different time scales.
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• MAPs are a highly tractable modelling tool as will be shown in the following sessions.
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• They are therefore highly desirable for modelling.
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- They are therefore highly desirable for modelling.
- There exist some fitting mechanisms such as those talked about in the previous session, which fit phase type distributions to data sets that can be used as renewal approximations to the empirical data.
- The MAP however can enable much more than that, as it allows dependencies to exist between successive arrivals.
Even more capability: the $BMAP$
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- The \textit{BMAP} is essentially characterised by matrices $D_0, D_1, D_2, \ldots, D_k, \ldots$, with the following properties...
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  and $De = (\sum_k D_k) e = 0$. 

Even more capability: the BMAP

• The MAP is a sub-class of the batch Markovian arrival process or BMAP.

• However, it is convenient here to describe the BMAP as an extension of the MAP, which allows batch arrivals at an arrival epoch rather than just single arrivals.

• The BMAP is essentially characterised by matrices $D_0, D_1, D_2, \ldots D_k, \ldots$, with the following properties

\[
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\]

\[
[D_k]_{ij} \geq 0 \quad \text{for all } i, j, k
\]

and $De = \left(\sum_k D_k\right)e = 0$.

Note: the matrix $D_k$ governs those arrivals of batch size $k$. 
Q-matrix for the BMAP
Q-matrix for the \textit{BMAP}

- The evolution of a \textit{BMAP} can be modelled by the following Q-matrix
Q-matrix for the \textit{BMAP}

- The evolution of a \textit{BMAP} can be modelled by the following Q-matrix

\[
Q = \begin{bmatrix}
D_0 & D_1 & D_2 & D_3 & \cdots \\
0 & D_0 & D_1 & D_2 & \cdots \\
0 & 0 & D_0 & D_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
Q-matrix for the $BMAP$

• The evolution of a $BMAP$ can be modelled by the following Q-matrix

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0 & D_0 & D_1 & D_2 & \cdots \\
0 & 0 & D_0 & D_1 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}.$$

• The $MAP$ is then trivially a $BMAP$ with $D_k \equiv 0$ for all $k \geq 2$. 
The Batch Poisson process
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- Negative exponentially distributed inter-arrival times between batches as for the Poisson process.
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\[
Q = \begin{bmatrix}
-\lambda & p_1 \lambda & p_2 \lambda & p_3 \lambda & p_4 \lambda & \ldots \\
0 & -\lambda & p_1 \lambda & p_2 \lambda & p_3 \lambda & \ldots \\
0 & 0 & -\lambda & p_1 \lambda & p_2 \lambda & \ldots \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & &
\end{bmatrix}
\]
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\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

- where \( \lambda \) is the arrival rate of batches.
An \textit{MMPP} with i.i.d. batch arrivals
An **MMPP** with i.i.d. batch arrivals

- Consider
An \textit{MMPP} with i.i.d. batch arrivals

- Consider

\[
D_0 = \begin{bmatrix}
-4.0 & 1.0 & 0.2 \\
0.12 & -0.25 & 0.005 \\
0.7 & 0.3 & -5.0 \\
\end{bmatrix},
\quad D_1 = \begin{bmatrix}
2.8 & 0 & 0 \\
0 & 0.125 & 0 \\
0 & 0 & 4.0 \\
\end{bmatrix},
\]
An **MMPP** with i.i.d. batch arrivals

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with i.i.d. batch arrivals governed by probability vector

\( (0.1, 0.2, 0.1, 0.2, 0.15, 0.25) \)
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\end{bmatrix},
\]

with i.i.d. batch arrivals governed by probability vector

\[(0.1, 0.2, 0.1, 0.2, 0.15, 0.25)\]

Hence the Q-matrix looks like

\[
\begin{bmatrix}
D_0 & 0.1D_1 & 0.2D_1 & 0.1D_1 & 0.2D_1 & 0.15D_1 & 0.25D_1 & 0 & \cdots \\
0 & D_0 & 0.1D_1 & 0.2D_1 & 0.1D_1 & 0.2D_1 & 0.15D_1 & 0.25D_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]
What does it look like?
What does it look like?

The arrival epochs.
What does it look like?
What does it look like?
What does it look like?
What does it look like?
What does it look like?

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Some other notes
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• That is, the superposition of $n$ independent $BMAP$s is also a $BMAP$. 
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If $D_k(i)$ is the matrix governing the batches of size $k$ for the $i^{th}$ independent $BMAP$, for each $k \geq 0$. (Note that any number of these matrices could be identically zero.)
Some other notes

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If $D_k(i)$ is the matrix governing the batches of size $k$ for the $i^{th}$ independent BMAP, for each $k \geq 0$. (Note that any number of these matrices could be identically zero.)

Then the $D_k$ matrix for the superposition is given by

$$D_k = D_k(1) \oplus D_k(2) \oplus D_k(3) \oplus \cdots \oplus D_k(n), \quad \text{for } k \geq 0,$$
Some other notes

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where \( \oplus \) is the Kronecker sum.
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• The *BMAP* is closed under superposition.

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where \( \oplus \) is the Kronecker sum.

• This construction could for instance be used to model multiplexed traffic streams. (*Choudhury, Lucantoni and Whitt*)
Some different considerations
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That is, 

\[
Q = \begin{bmatrix}
-\lambda_1 & \lambda_1 & 0 & \cdots \\
0 & -\lambda_2 & \lambda_2 & 0 \\
0 & 0 & -\lambda_3 & \lambda_3 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix},
\]

where \(\lambda_i\) is the arrival rate at level \(i\).
The uniqueness of representation
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- Also as in the case of phase type distributions, MAPs can similarly have a variety of representations.
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• For example the following three MAPs are just complex representations of a Poisson process of rate 1 under stationary conditions.
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\[
D_0 = \begin{bmatrix}
-2 & \frac{1}{2} & \frac{1}{2} \\
1 & -4 & 1 \\
\frac{1}{2} & 1 & -2
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
1 & 1 & 0 \\
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\]
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- For example the following three MAPs are just complex representations of a Poisson process of rate 1 under stationary conditions.

\[
D_0 = \begin{bmatrix}
-4 & 2 & 1 \\
5 & -8 & 2 \\
1 & 2 & -4
\end{bmatrix},
D_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
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\[
D_0 = \begin{bmatrix}
-3 & 3 & 0 \\
0 & -6 & 4 \\
0 & 0 & -1
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 1 & 0 \\
\left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) & 0
\end{bmatrix}
\]
The uniqueness of representation

• Also as in the case of phase type distributions, MAPs can similarly have a variety of representations.

• For example the following three MAPs are just complex representations of a Poisson process of rate 1 under stationary conditions.

\[
D_0 = \begin{bmatrix}
-3 & 3 & 0 \\
0 & -6 & 4 \\
0 & 0 & -1
\end{bmatrix},
D_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 1 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix}
\]

• This adds another dimension to the fitting of MAPs, and adds to the reasons as to why it is a potentially difficult exercise. Particularly when it comes to minimising the order of representation.