T-duality & noncommutative geometry

Type IIA $\Leftrightarrow$ Type IIB duality rephrased

Higher Structures in String Theory and Quantum Field Theory

Instructional workshop for students and junior researchers

Mathai Varghese

INSTITUTE FOR GEOMETRY AND ITS APPLICATIONS

THE UNIVERSITY of ADELAIDE

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[MR05]

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String theory in a background flux

• (Super) string theory is a candidate for the ***Theory of Everything***, in which strings are the fundamental objects.

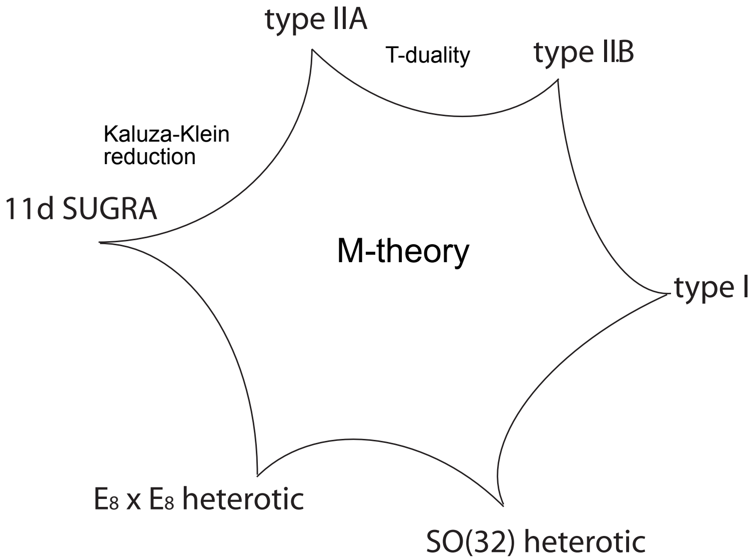
(Super) string theory does not currently have a complete definition. What we have instead are a set of partial definitions.

• $\exists$ five manifestations of (super) string theories + SUGRA: **type I, type II (A, B), heterotic (E8 $\times$ E8, SO(32)), SUGRA**; A question naturally arises given this state of affairs.

- Is each partial definition consistent with the others, via string dualities?

We will be concerned with 2 of the 6 known manifestations of (super) string theory, viz. type IIA and type IIB string theories.

string theory and dualities



The idea of T-duality

Symmetries are critically important to physical modelling because they relate the outcomes of experiments for different observers, and constrain the number of possible models one can write down.

The mathematical modelling of symmetries have led to many important advances, e.g. the theory of groups and algebras.

Apart from the familiar symmetries such as Lorentz invariance, which relates observers in different reference frames, string theory has some peculiar symmetries known as **dualities**.

These are less well understood and their description requires new mathematics to study global aspects of a particular duality, known as **Target space duality** or **T-duality**.

T-duality - The case of circle bundles

In [BEM], we isolated the geometry in the case when E is a principal $\mathbb{T}$-bundle over M

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & E \\ & & \Big\downarrow \pi \\ & & M \end{array} \tag{1}$$

classified by its first Chern class $c_1(E) \in H^2(M, \mathbb{Z})$, with H-flux $H \in H^3(E, \mathbb{Z})$.

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The **T-dual** is another principal $\mathbb{T}$-bundle over M, denoted by $\hat{E}$,

$$\begin{array}{ccc} \hat{\mathbb{T}} & \longrightarrow & \hat{E} \\ & & \downarrow{\scriptstyle \hat{\pi}} \\ & & M \end{array} \tag{2}$$

which has first Chern class $c_1(\hat{E}) = \pi_* H$.

T-duality in a background flux

The Gysin sequence for E enables us to define a T-dual H-flux $\hat{H} \in H^3(\hat{E}, \mathbb{Z})$, satisfying

$$c_1(E) = \hat{\pi}_* \hat{H}\,, \tag{3}$$

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N.B. $\hat{H}$ is not fixed by this data, since any integer degree 3 cohomology class on M that is pulled back to $\hat{E}$ also satisfies (3). However, $[\hat{H}]$ is determined uniquely upon imposing the condition $[H] = [\hat{H}]$ on the correspondence space $E \times_M \hat{E}$.

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Thus a slogan for T-duality for **circle bundles** is the exchange,

background H-flux $\Longleftrightarrow$ Chern class

The surprising **new** phenomenon that we discovered is that there is a **change in topology** when either the background H-flux, or the Chern class is topologically nontrivial.

T-duality in a background flux - isomorphism of charges

Remark

It turns out that T-duality gives rise to a map inducing degree-shifting isomorphisms between the H-twisted cohomology of E and $\widehat{H}$-twisted cohomology of $\widehat{E}$ and also between their twisted K-theories, where charges of RR-fields live.

It is a vast generalization of the smooth analog of the **Fourier-Mukai transform** = a geometric Fourier transform.

If the T-duality map is assumed to be an isometry, then it also takes radius R to radius $1/R$, a salient feature of T-duality.

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T-duality in a background flux - Examples

Lens space $L(p,1) = S^3/\mathbb{Z}_p$, where
$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ & $\mathbb{Z}_p$ acts on S^3 by

$$\exp(2\pi ik/p).(z_1, z_2) = (z_1, \exp(2\pi ik/p)z_2), \quad k = 0, 1, \ldots, p-1.$$

$L(p,1)$ is the total space of the circle bundle over S^2 with Chern class equal to p times the generator of $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$.
Then $L(p,1)$ is never homeomorphic to $L(q,1)$ whenever $p \neq q$.
Nevertheless

$$(L(p,1), H = q) \qquad \text{and} \qquad (L(q,1), H = p)\,.$$

are T-dual pairs! Thus T-duality is the interchange

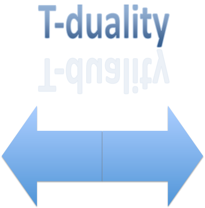
$$\boxed{p \Longleftrightarrow q}$$

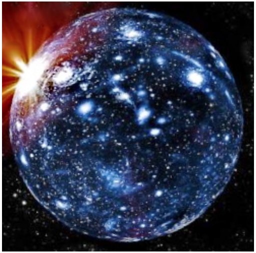
T-duality in a background flux - Examples

Since $L(1,1) = S^3$ & $L(0,1) = S^2 \times S^1$, we get the T-dual pairs:

$$(S^2 \times S^1, H = 1) \qquad \text{and} \qquad (S^3, H = 0)$$

A picture (suppressing one dimension) illustrating this is the *doughnut universe* $(H = 1)$ & the *spherical universe* $(H = 0)$





Preliminaries

Dixmier-Douady theory asserts that isomorphism classes of locally trivial algebra bundles $\mathcal{K}_P$ with fiber the algebra of compact operators $\mathcal{K}$ and structure group $PU = U/\mathbb{T}$ over a manifold X are in bijective correspondence with $H^3(X, \mathbb{Z})$. Moreover since $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$, such algebra bundles form a group the **infinite Brauer group**, $\mathrm{Br}(X)$ (isomorphic to $H^3(X, \mathbb{Z})$.)

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$$[X, BPU] = [X, K(\mathbb{Z}, 3)] = H^3(X, \mathbb{Z}).$$

Then the associated bundle $\mathcal{K}_P = (P \times \mathcal{K})/PU$ and $H = DD(\mathcal{K}_P) \in H^3(X, \mathbb{Z})$ is its **Dixmier-Douady invariant**.

Decomposable nontorsion example & the Heisenberg group.

$\alpha \in H^1(X, \mathbb{Z})$ can be viewed as a character $\chi_\alpha : \pi_1(X) \to \mathbb{Z}$ with associated principal $\mathbb{Z}$-covering space $\mathbb{Z} \to \hat{X} \to X$.

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Similarly, $\beta \in H^2(X, \mathbb{Z})$ can be viewed as the 1st Chern class of a principal circle bundle $U(1) \to P \to X$.

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Now $(\gamma', n) \in U(1) \times \mathbb{Z}$ acts on $L^2(U(1))$,

$$(\sigma(n)f)(\gamma) = \gamma^n f(\gamma), \quad (\sigma(\gamma')f)(\gamma) = f(\gamma'\gamma).$$

Since $[\sigma(\gamma), \sigma(n)] = \gamma^n I$, this is a projective action, i.e. $\sigma : U(1) \times \mathbb{Z} \to PU(L^2(U(1)))$ is a homomorphism.

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The associated bundle of compact operators over X is $P \times_X \hat{X} \times_{Ad(\sigma)} \mathcal{K}(L^2(U(1))$ with DD invariant $\alpha \cup \beta$.

Preliminaries

Twisted K-theory. Consider the C^*-algebra of continuous sections, $C(X, \mathcal{K}_P)$ - we will also denote this algebra by $CT(X, H)$, where $H = DD(\mathcal{K}_P)$. By fiat, this algebra is **locally** Morita equivalent to $C(X)$ (i.e. locally physically equivalent) but **not** globally Morita equivalent to it if $[H] \neq 0$.
Thus $CT(X, H)$ is a **mildly noncommutative spacetime** algebra in the presence of an H-flux.

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Twisted K-theory, denoted by $K^\bullet(X, H)$, was defined by J. Rosenberg as the K-theory of $CT(X, H)$. $K^\bullet(X, H)$ is a module over $K^0(X)$ and possesses many nice functorial properties.

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Charges of RR-fields in an H-flux lie in the K-theory of $CT(X, H)$, ie in twisted K-theory $K^\bullet(X, H)$.

Preliminaries on C^*-algebras

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If G is abelian, then on the crossed product $A \rtimes_\alpha G$, there is an action $\hat{\alpha}$ of the Pontryagin dual group $\hat{G}$ given by multiplication by $\hat{G}$ on functions on G, with formal relations:-

$$\gamma.a = a.\gamma, \; \gamma.g.\gamma^{-1} = \langle \gamma, g \rangle g \text{ for all } \gamma \in \hat{G}, g \in G, a \in A.$$

3 basic principles from C^*-algebras

1. Let G be a locally compact group and K, H are normal subgroups of G. Then the **Rieffel-Green theorem** states that the following algebras are Morita equivalent.

$$C(K\backslash G) \rtimes H, \quad \text{and } C(G/H) \rtimes K$$

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3. If G is an abelian group acting on A, then **Takai duality** says that there is a canonical isomorphism,

$$A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G} \cong A \otimes \mathcal{K}.$$

Rephrasing T-duality in terms of NCG

1. $C(M \times \mathbb{R}^n/\mathbb{Z}^n) \rtimes \mathbb{R}^n$ and $C(M \times \mathbb{R}^n \backslash \mathbb{R}^n) \rtimes \mathbb{Z}^n$ are strongly Morita equivalent, by the Rieffel-Green theorem.

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2. $K_j(C(M \times \mathbb{R}^n/\mathbb{Z}^n) \rtimes \mathbb{R}^n)$ and $K_{j+n}(C(M \times \mathbb{R}^n/\mathbb{Z}^n))$ are isomorphic as a consequence of the Connes-Thom isomorphism theorem.

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3. $C(M \times \mathbb{R}^n/\mathbb{Z}^n) \rtimes \mathbb{R}^n \rtimes \widehat{\mathbb{R}}^n$ and $C(M \times \mathbb{R}^n/\mathbb{Z}^n)$ are strongly Morita equivalent, by Takai duality.

Let A belong to some class $\mathfrak{C}$ of C^*-algebras, and $A \to \mathrm{T}(A)$ be a covariant functor on $\mathfrak{C}$ satisfying the following properties:

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Then we call $\mathrm{T}(A)$ an **abstract T-dual** of A.

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eg. Let A be a G-C^*-algebra, where $G = \mathbb{R}^n$. Set $\mathrm{T}(A) = A \rtimes G$. Then $\mathrm{T}(A)$ is an abstract T-dual of A.

The case of circle bundles

Comnsider the principal circle bundle

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and H a closed, integral 3-form on E.
Then there is a continuous trace C^*-algebra $CT(E, H)$ with spectrum equal to E and Dixmier-Douady invariant equal to $[H] \in H^3(E, \mathbb{Z})$.

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Then there is a continuous trace C^*-algebra $CT(E, H)$ with spectrum equal to E and Dixmier-Douady invariant equal to $[H] \in H^3(E, \mathbb{Z})$.

Using a connection on the associated principal PU bundle, the $\mathbb{R}$ action on E lifts to an $\mathbb{R}$ action on $CT(E, H)$ (uniquely up to exterior equivalence, cf. **Raeburn-Rosenberg**), and one has a commutative diagram,

$$\begin{array}{ccccc} & & \mathrm{spec}(CT(E,H)\rtimes\mathbb{Z}) & & \\ & \swarrow^{p} & & \searrow^{\hat{p}} & \\ \mathrm{spec}(CT(E,H)) & & & & \mathrm{spec}(CT(E,H)\rtimes\mathbb{R}) \\ & \searrow^{\pi} & & \swarrow^{\hat{\pi}} & \\ & & \mathrm{spec}(CT(E,H))/\mathbb{R} & & \end{array} \tag{4}$$

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That is, **Raeburn-Rosenberg** show that the C^*-algebras $CT(E,H) \rtimes \mathbb{Z}$ and $CT(E,H) \rtimes \mathbb{R}$ are also continuous trace C^*-algebras with $\mathrm{spec}(CT(E,H) \rtimes \mathbb{R}) = \widehat{E}$ a circle bundle over $M = \mathrm{spec}(CT(E,H))/\mathbb{R}$, such that $c_1(\widehat{E}) = \pi_*[H]$ and the Dixmier-Douady invariant of $CT(E,H) \rtimes \mathbb{R}$ is $[\hat{H}] \in H^3(\widehat{E}, \mathbb{Z})$, such that $c_1(E) = \hat{\pi}_*[\widehat{H}]$, and $\mathrm{spec}(CT(E,H) \rtimes \mathbb{Z}) = E \times_M \widehat{E}$ is the correspondence space.

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This reformulation turns out to be essential when considering T-duality of higher rank torus bundles with H-flux, as the T-dual in this case can be a purely noncommutative manifold, as will be discussed later i.e. it is possible that there can be **no** commutative spacetime with flux that is a T-dual in the higher rank case.

This recasts T-duality for principal circle bundles completely in terms of noncommutative geometry.

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However the tools of noncommutative geometry that were discussed earlier can however be used to determine the T-dual.