The boundary conditions for a wave equation with microscopically varying density and elasticity

Chen Chen
Supervisors: Tony Roberts, Judith Bunder and Tony Miller

The University of Adelaide
The School of Mathematical Sciences

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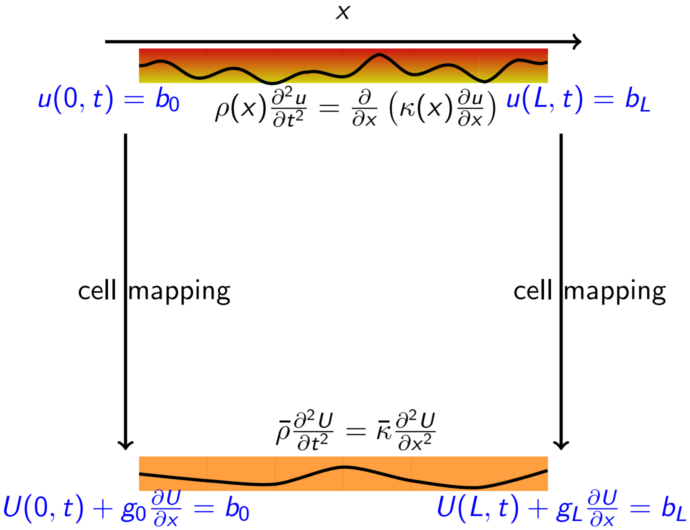
Overview


x
$u(0,t) = b_0$
$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right)$
$u(L,t) = b_L$
cell mapping
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Results for linear systems

Find appropriate boundary conditions for macroscale model

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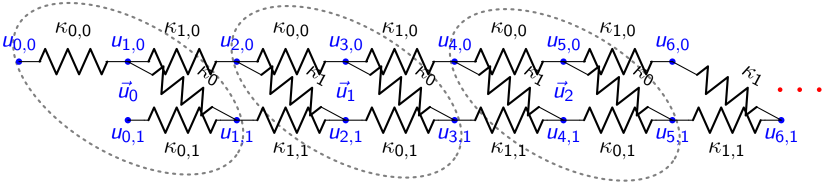
microscale boundary conditions	macroscale boundary conditions
Dirichlet, $u = 0$	Robin, $U + g\frac{\partial U}{\partial x} = 0$
Neumann, $\frac{\partial u}{\partial x} = 0$	the same Neumann, $\frac{\partial U}{\partial x} = 0$
Robin, $u + g_1\frac{\partial u}{\partial x} = 0$	a different Robin, $U + g_2\frac{\partial U}{\partial x} = 0$

Wave equation in a two-strand medium



$$\begin{aligned}
\rho_{m,0}\frac{\partial^2 u_{m,0}}{\partial t^2} &= \kappa_{m-1,0}(u_{m-1,0}-u_{m,0})+\kappa_{m,0}(u_{m+1,0}-u_{m,0})+\kappa_m(u_{m,1}-u_{m,0}),\\
\rho_{m,1}\frac{\partial^2 u_{m,1}}{\partial t^2} &= \kappa_{m-1,1}(u_{m-1,1}-u_{m,1})+\kappa_{m,1}(u_{m+1,1}-u_{m,1})+\kappa_m(u_{m,0}-u_{m,1}).
\end{aligned}$$

- Spatial domain $m = 1, 2, \dots, N-1$,
- ρ and κ are two periodic horizontally,
- with Dirichlet boundary conditions $u_{0,i} = b_{0,i}$ and $u_{N,i} = b_{N,i}$.

Homogenized theory derives macroscale model

$$\bar{\rho}\frac{\partial^2 U(x,t)}{\partial t^2} = \bar{\kappa}\frac{\partial^2 U(x,t)}{\partial x^2}.$$

- Spatial domain $[0, L]$, equivalent density is the arithmetic mean

$$\bar{\rho} = \frac{1}{4}\sum_{i=0}^{1}\sum_{j=0}^{1}\rho_{i,j}$$

and equivalent spring constant is a kind of harmonic mean

$$\bar{\kappa} = \left[\kappa_0\kappa_1\left(\kappa_{0,1}+\kappa_{0,0}\right)\left(\kappa_{1,1}+\kappa_{1,0}\right)+\left(\kappa_0+\kappa_1\right)\sum_{j=0}^{1}\sum_{i=0}^{1}\kappa_{0,0}\kappa_{0,1}\kappa_{1,0}\kappa_{1,1}/\kappa_{i,j}\right]/D$$

where

$$D = \kappa_1\kappa_0(\kappa_{0,0}+\kappa_{0,1}+\kappa_{1,0}+\kappa_{1,1})+(\kappa_1+\kappa_0)(\kappa_{1,1}+\kappa_{0,1})(\kappa_{1,0}+\kappa_{0,0})$$

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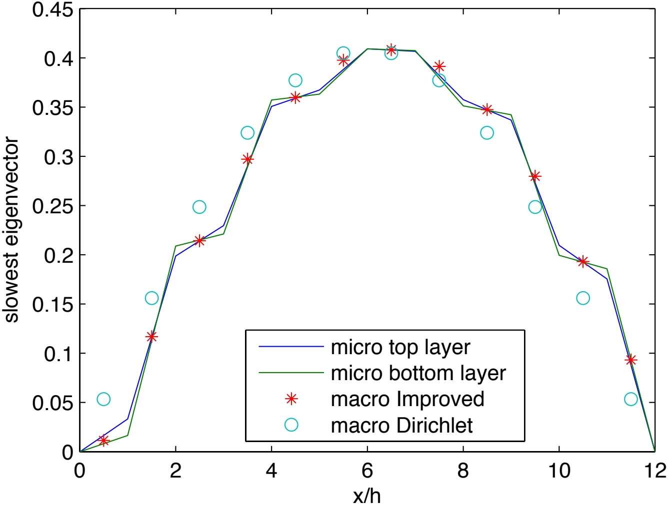
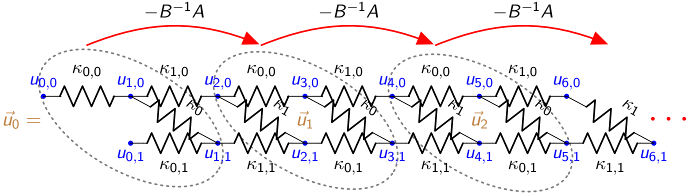
- with Robin boundary conditions
$U(0,t) + g_0\left.\frac{\partial U}{\partial x}\right|_{x=0} = b_0$ and $U(L,t) + g_L\left.\frac{\partial U}{\partial x}\right|_{x=L} = b_L$.

Slowest eigenvector of micro and macro system



Assuming steady state gives cell mapping



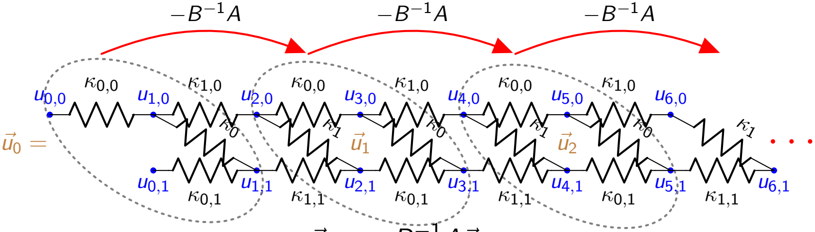
$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \vec{u}_0 \\ \vec{u}_1 \end{bmatrix} = \vec{0},$$

where

$$A = \begin{bmatrix} \kappa_{0,0} & 0 & -\kappa_{0,0}-\kappa_{1,0}-\kappa_1 & \kappa_1 \\ 0 & \kappa_{0,1} & \kappa_1 & -\kappa_{0,1}-\kappa_{1,1}-\kappa_1 \\ 0 & 0 & \kappa_{1,0} & 0 \\ 0 & 0 & 0 & \kappa_{1,1} \end{bmatrix},$$

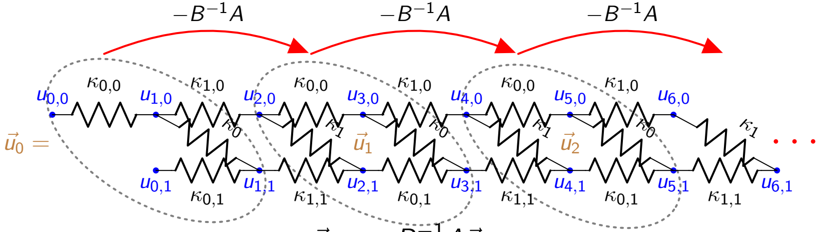
$$B = \begin{bmatrix} \kappa_{1,0} & 0 & 0 & 0 \\ 0 & \kappa_{1,1} & 0 & 0 \\ -\kappa_{1,0}-\kappa_{0,0}-\kappa_0 & \kappa_0 & \kappa_{0,0} & 0 \\ \kappa_0 & -\kappa_{1,1}-\kappa_{0,1}-\kappa_0 & 0 & \kappa_{0,1} \end{bmatrix}.$$

Assuming steady state gives cell mapping



$$\vec{u}_1 = -B^{-1}A\vec{u}_0,$$

Assuming steady state gives cell mapping



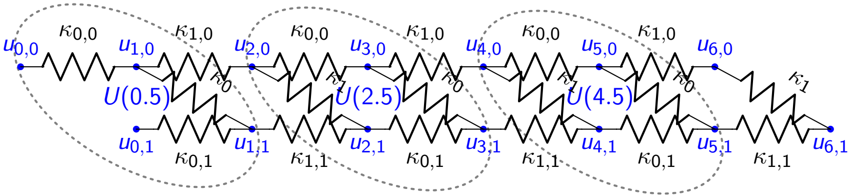
$$\vec{u}_1 = -B^{-1}A\vec{u}_0,$$

$$\vec{u}_{\nu+1} = -B^{-1}A\vec{u}_\nu,$$

$$\vec{u}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4,$$

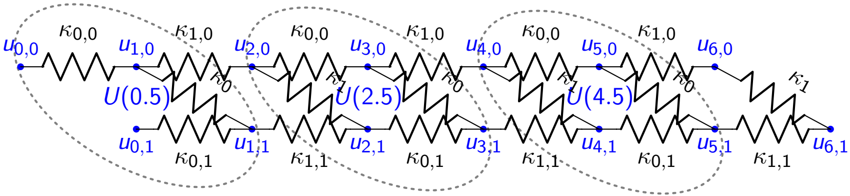
where $\vec{v}_i$ is the eigenvectors of mapping matrix T.

Cell mapping derives macroscale model and boundary conditions



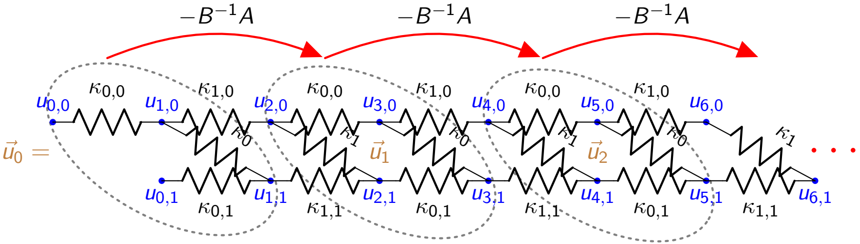
- Write U and $\frac{\partial U}{\partial x}$ as a function of $\vec{u}_\nu$.
- $$\begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ 0 & \bar{v}_2 & (\bar{v}_3 - \frac{1}{4}\bar{v}_2) \\ 0 & 0 & \frac{1}{2h}\bar{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_{00} \\ b_{01} \\ U(x=0) \\ \frac{\partial U}{\partial x}\Big|_{x=0} \end{bmatrix}.$$
- Boundary condition comes from considering the basis vector of the null space.

Use Levenberg–Marquardt algorithm check the improved boundary conditions



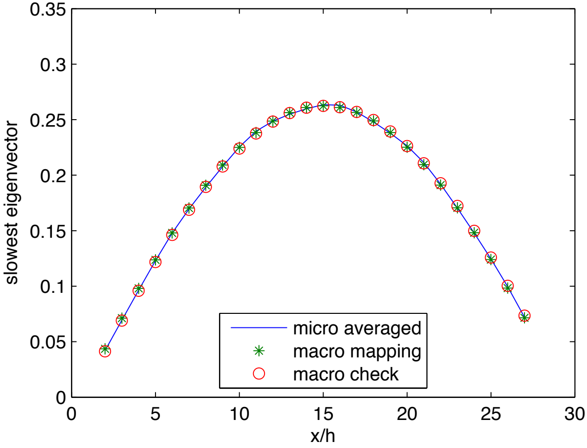
- This method captures out of equilibrium.
- Require solve the full microscale problem once and solve the macroscale problem many times.
- $$\min_{g_0, g_L} \left|\vec{v}_{\text{opt}} - \vec{v}_{\text{micro}}\right|$$
- The theory of Rayleigh quotient justifies eigenvectors are more sensitive.
- This algorithm verifies the derived boundary conditions.

Mapping methods generalise to more complicated problems



- Extend to any number of strands.
- Any periodicity.
- The method can be applied to non-linear problems.

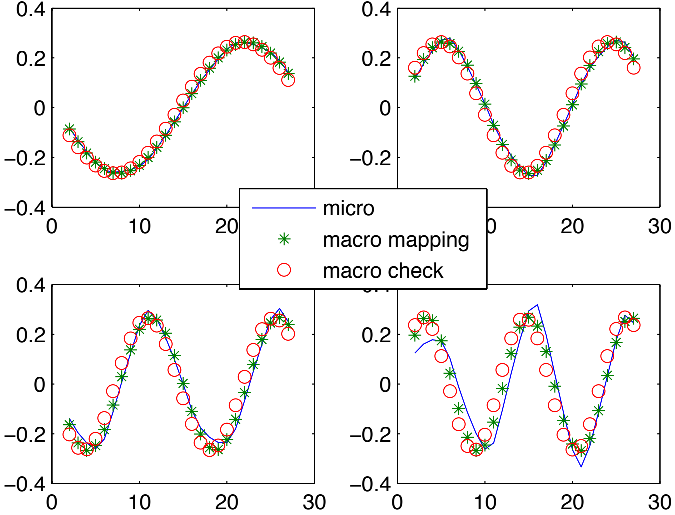
Micro vs Levenberg vs Cell mapping



$$\lambda_{\text{micro}} = 1.18, 4.70, 10.4, 18.0, 26.4,$$

$$\lambda_{\text{dirichlet}} \approx \lambda_{\text{mapping}} = 1.18, 4.72, 10.6, 18.8, 29.1.$$

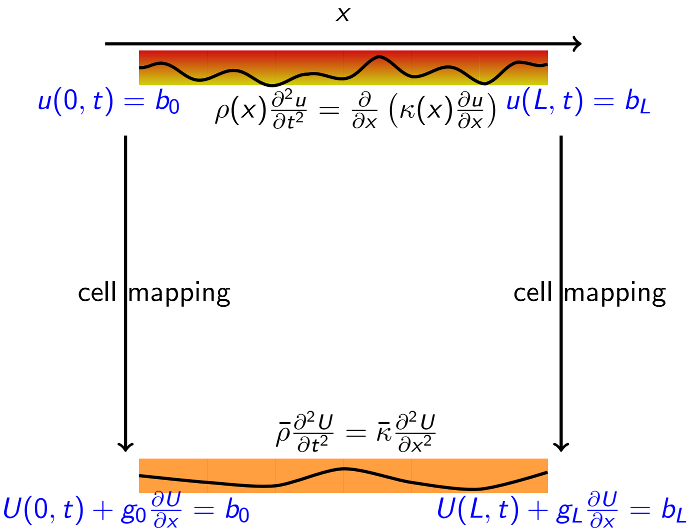
Micro vs Levenberg vs Cell mapping



Non-linear problems

- Assume Quasi-equilibrium and regard the problem as spatial evolution.
- Deduce centre, stable and unstable manifold.
- Set the coefficients of unstable model to zero.
- Project the boundary conditions from centre stable manifold to centre manifold of the spatial evolution.

Conclusion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