The boundary conditions for the mean temperature model of a heat exchanger

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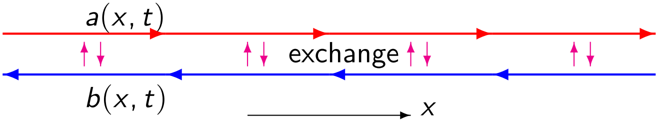
The heat exchanger



$$\begin{aligned}
\frac{\partial a}{\partial t} &= \tfrac{1}{2}(b-a) + \tfrac{1}{2}a^2 - \frac{\partial a}{\partial x} + 3\frac{\partial^2 a}{\partial x^2}\,, \\
\frac{\partial b}{\partial t} &= \tfrac{1}{2}(a-b) - \tfrac{1}{2}b^2 + \frac{\partial b}{\partial x} + 3\frac{\partial^2 b}{\partial x^2}\,,
\end{aligned}$$

A mean temperature model $C := (a+b)/2$ is (Roberts 2015)

$$\frac{\partial C}{\partial t} = \tfrac{1}{2}C^3 - 2C\frac{\partial C}{\partial x} + 4\frac{\partial^2 C}{\partial x^2} + \mathcal{O}(C^4 + \partial_x^4).$$

Boundary conditions

The microscale boundary conditions are

$$a(0,t) = a_0, \quad b(0,t) = b_0, \quad a(L,t) = a_L, \quad \text{and} \quad b(L,t) = b_L.$$

The heuristic boundary conditions for the mean temperature model are

$$C(0,t) = (a_0 + b_0)/2, \quad \text{and} \quad C(L,t) = (a_L + b_L)/2.$$

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Our derived boundary conditions

$$C - (0.5 - 2.8b_0 + 3.7a_0)\frac{\partial C}{\partial x} - 3\left(\frac{\partial C}{\partial x}\right)^2$$
$$\approx 0.25b_0 - 0.29b_0^2 + 0.75a_0 - 0.63a_0b_0 + 0.18a_0^2, \quad \text{at} \quad x = 0$$
$$C - (-0.5 - 2.8a_L + 3.7b_L)\frac{\partial C}{\partial x} + 3\left(\frac{\partial C}{\partial x}\right)^2$$
$$\approx 0.25a_L - 0.29a_L^2 + 0.75b_L - 0.63a_Lb_L + 0.18b_L^2. \quad \text{at} \quad x = L$$

Numerical result

The domains are $0 \leq x \leq 30$, $0 \leq t \leq 21$. The boundary values are $a_0 = 0.2 \tanh^2 t$, $b_0 = 0$, $a_L = 0$ and $b_L = 0.2 \tanh^2 t$. The initial values are zeros.

Assuming steady state

Assume steady state and rearrange the heat exchanger PDE into a dynamical system form

$$\frac{\partial}{\partial x}\begin{bmatrix} a \\ b \\ a' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{6} & 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} a \\ b \\ a' \\ b' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}a^2 \\ \frac{1}{2}b^2 \end{bmatrix}.$$

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This dynamical system has a generalised eigenvalue, apply perturbation method so that $\epsilon = 1$ gives original system

$$\frac{\partial}{\partial x}\begin{bmatrix} a \\ b \\ a' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{6} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} a \\ b \\ a' \\ b' \end{bmatrix} + \begin{bmatrix} \epsilon b' \\ \epsilon a' \\ -\frac{1}{2}a^2 + \frac{1}{2}\epsilon a' - \frac{1}{2}\epsilon b' \\ \frac{1}{2}b^2 + \frac{1}{2}\epsilon a' - \frac{1}{2}\epsilon b' \end{bmatrix}.$$

Centre manifold analysis

1. The eigenvalues of the perturbed system are $0, 0, \pm\frac{2}{3}$.
2. These correspond to a stable manifold, an unstable manifold, and a two dimensional slow manifold.
3. Apply a coordinate transform to separate the three manifolds

$$s_1 := \tfrac{1}{2}\left(a + b\right) = C, \qquad s_3 := \tfrac{1}{8}\left(3a - 3b - 3a' + 9b'\right),$$
$$s_2 := \tfrac{1}{2}\left(a' + b'\right) = \frac{\partial C}{\partial x}, \quad s_4 := \tfrac{1}{8}\left(3a - 3b + 9a' - 3b'\right),$$

Coordinate transform gives invariant manifolds

Asymptotic analysis gives

$$\begin{aligned}
a &\approx s_1 - s_2 + 0.25 s_3 + 1.5 s_1^2 + 6 s_2^2 - 1.1 s_1 s_3 - 3.4 s_2 s_3 - 0.035 s_3^2 \\
&\quad + 0.75 s_4 + 0.74 s_2 s_4 + 0.56 s_3 s_4 - 0.25 s_4^2\,, \\
b &\approx s_1 + s_2 - 0.75 s_3 - 1.5 s_1^2 - 6 s_2^2 - 0.74 s_2 s_3 + 0.25 s_3^2 \\
&\quad - 0.25 s_4 - 1.1 s_1 s_4 + 3.4 s_2 s_4 - 0.56 s_3 s_4 - 0.035 s_4^2\,, \\
a' &\approx s_2 + 1.5 s_1 s_2 - 0.17 s_3 + 0.56 s_1 s_3 + 0.91 s_2 s_3 + 0.47 s_3^2 \\
&\quad + 0.5 s_4 - 0.56 s_1 s_4 + 1.2 s_2 s_4 - 0.33 s_4^2\,, \\
b' &\approx s_2 - 1.5 s_1 s_2 + 0.5 s_3 + 0.56 s_1 s_3 + 1.2 s_2 s_3 - 0.33 s_3^2 \\
&\quad - 0.17 s_4 - 0.56 s_1 s_4 + 0.91 s_2 s_4 - 0.047 s_4^2\,,
\end{aligned}$$

and the norm forms are

$$\begin{aligned}
\frac{\partial s_1}{\partial x} &\approx s_2\,, & \frac{\partial s_3}{\partial x} &\approx -0.67 s_3 - 0.75 s_3 s_1 - 0.94 s_3 s_2\,, \\
\frac{\partial s_2}{\partial x} &\approx 1.5 s_1 s_2\,, & \frac{\partial s_4}{\partial x} &\approx +0.67 s_4 - 0.75 s_4 s_1 + 0.94 s_4 s_2\,.
\end{aligned}$$

Coordinate transform gives invariant manifolds

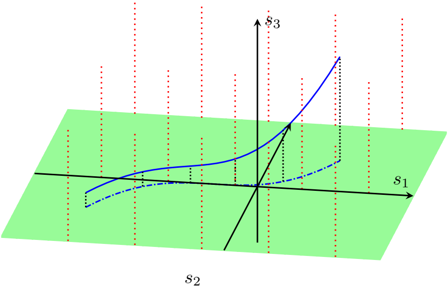
1. The centre-stable manifold is

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \approx \begin{bmatrix} s_1^0 - s_2^0 + 0.25 s_3^0 + 1.5 s_1^{0\,2} + 6 s_2^{0\,2} - 1.1 s_1^0 s_3^0 - 3.4 s_2^0 s_3^0 - 0.035 s_3^{0\,2} \\ s_1^0 + s_2^0 - 0.75 s_3^0 - 1.5 s_1^{0\,2} - 6 s_2^{0\,2} - 0.74 s_2^0 s_3^0 + 0.25 s_3^{0\,2} \end{bmatrix}.$$

2. Rearranging gives

$$\begin{aligned} s_1^0 &\approx \left(0.25 b_0 - 0.29 b_0^2 + 0.75 a_0 - 0.63 a_0 b_0 + 0.18 a_0^2\right) \\ &\quad + s_2^0 \left(0.5 - 2.8 b_0 + 3.7 a_0\right) + 3 s_2^{0\,2}. \\ s_3^0 &\approx \left(-b_0 - 0.19 b_0^2 + a_0 - 2.3 a_0 b_0 - 0.56 a_0^2\right) \\ &\quad + s_2^0 \left(2 - 4.6 b_0 + 3.8 a_0\right) - 5.2 s_2^{0\,2}. \end{aligned}$$

Coordinate transform gives invariant manifolds



$$s_1^0 \approx \left(0.25b_0 - 0.29b_0^2 + 0.75a_0 - 0.63a_0b_0 + 0.18a_0^2\right) + s_2^0\left(0.5 - 2.8b_0 + 3.7a_0\right) + 3{s_2^0}^2.$$

Hence the macroscale left boundary condition is

$$C - \left(0.5 - 2.8b_0 + 3.7a_0\right)\frac{\partial C}{\partial x} - 3\left(\frac{\partial C}{\partial x}\right)^2 \approx \left(0.25b_0 - 0.29b_0^2 + 0.75a_0 - 0.63a_0b_0 + 0.18a_0^2\right).$$

Conclusion

1. We derived a nonlinear boundary conditions for a mean model of a heat exchanger.
2. We plan to extend this method to homogenization of nonlinear periodic materials and homogenization of flow in porous media.

Roberts, A. J. (2015), 'Macroscale, slowly varying, models emerge from the microscale dynamics', *IMA Journal of Applied Mathematics* .